Online Appendix to "A Structural Theorem for Rationalizability in the Normal Form of Dynamic Games"

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Abstract

Omitted proofs for results in "A Structural Theorem for Rationalizability in the Normal Form of Dynamic Games" Chen (2011) are presented.

A Online Appendix

Lemmas 1 and 2 are the counterparts of Lemmas 6 and 7 of WY under RURA. Before we present their proofs, we provide some sketches for readers who are familiar with WY's arguments. WY prove their Lemma 7 by induction on k. Their Richness assumption guarantees that when $k = 0$, i.e., when it is vacuously true that $\widetilde{t}_i^{k'} = t_i^{k'}$ for all $k' \leq k$, choosing $\widetilde{t}_i$ with $\widetilde{t}_i [\theta^{s_i}] = 1$ proves the claim. Here when $k = 0$, we use RURA to set $\widetilde{t}_i$ to be a finite type with $S_i^{\infty} [t_i] = \{s_i\}$. This is possible because finite types are dense in T_i^* (see (Mertens and Zamir, 1985, Theorem 3.1)) and $S_i^{\infty} [\cdot]$ is a nonempty and upper hemicontinuous correspondence (see Dekel et al. (2006)). In words, WY start the "infection argument" from the

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dominance regions, while we start it from the "types with unique ICR actions" defined by RURA. The proof of the induction step is similar to WY.

The modification of Lemma 6 is as follows. Let t_i be a finite type contained in the model $(\Theta \times T, \kappa)$ and $s_i \in S_i^{\infty}[t_i]$. Suppose that s_i is a best reply to $\text{marg}_{\Theta \times S_{-i}} \pi^{t_i, s_i}$ for some π^{t_i, s_i} which is valid for t_i. WY make s_i a strict best reply $t_i(m)$ by setting the belief of $t_i(m)$ to be the $\left(1 - \frac{1}{m}, \frac{1}{m}\right)$-mixture between π^{t_i, s_i} and some belief π which assigns probability 1 to θ^{s_i}. In our case, since s_i is the unique rationalizable action for some type by RURA, s_i is also a strict best reply to some belief π^{s_i} whose support contains only uniquely rationalizable actions which by RURA are also strict best replies to some other beliefs, and so on.

A.1 Proof of Lemma 1

To prove Lemma 1, we need the following lemma which is a straightforward consequence of RURA.

Lemma 1 *Under RURA, for each i and each s_i such that $s_i \in S_i^{\infty}[t_i']$ for some $t_i' \in T_i^*$, there is some $\pi^{s_i} \in \Delta(\Theta \times S_{-i})$ such that $\{s_i\} = BR_i(\pi^{s_i})$, and moreover, $\pi^{s_i}(s_{-i}) > 0$ only if $s_{-i} \in S_{-i}^{\infty}[t_{-i}]$ for some $t_{-i} \in T_{-i}^*$.*

Proof. Since $s_i \in S_i^{\infty}[t_i']$ for some t_i', by RURA, $S_i^{\infty}[t_i] = \{s_i\}$ for some type t_i. Hence, $\{s_i\} = BR_i\left(\text{marg}_{\Theta \times S_{-i}} \pi^{t_i, s_i}\right)$ for some valid π^{t_i, s_i} for t_i. Let $\pi^{s_i} = \text{marg}_{\Theta \times S_{-i}} \pi^{t_i, s_i}$ and hence $\{s_i\} = BR_i(\pi^{s_i})$. Since $\pi^{t_i, s_i}\left(\left\{(\theta, t_{-i}, s_{-i}) : s_{-i} \in S_{-i}^{\infty}[t_{-i}]\right\}\right) = 1$, $\pi^{s_i}(s_{-i}) > 0$ only if $s_{-i} \in S_{-i}^{\infty}[t_{-i}]$ for some $t_{-i} \in T_{-i}^*$.■

We now prove Lemma 1.

Lemma 1 *Under RURA, for any finite type $t_i \in T_i^*$ and any action $s_i \in S_i^{\infty}[t_i]$, there exists a sequence of finite models $((\Theta \times T^m, \kappa^m))_{m=1}^{\infty}$ and a sequence of finite types $(t_i(m))_{m=1}^{\infty}$ such that $t_i(m) \in T_i^m$ and $s_i \in V_i^m[t_i(m)]$ for some profile of correspondences $\left(V_j^m\right)_{j \in N}$ with $V_j^m : T_j^m \rightrightarrows S_j$ which satisfies the strict best reply property, for all m, and $\lim_{m \to \infty} t_i(m) = t_i$.*

Proof. Consider any $s_j \in S_j^{\infty}[t_j]$, $s_j \in BR_j\left(\text{marg}_{\Theta \times S_{-j}} \pi^{t_j, s_j}\right)$ for some valid π^{t_j, s_j} for t_j. Moreover, by Lemma 1, there is some $\pi^{s_j} \in \Delta(\Theta \times S_{-j})$ such that $\{s_j\} = BR_j(\pi^{s_j})$, and moreover, $\pi^{s_j}(s_{-j}) > 0$ only if $s_{-j} \in S_{-j}^{\infty}[t_{-j}]$ for some $t_{-j} \in T_{-j}^{*}$.

We now define $(\Theta \times T^m, \kappa^m)$ as follows.[1]

$$\begin{aligned} T_j^m &= \left\{\overline{\tau}_j(t_j, s_j, m) : t_j \in T_j, s_j \in S_j^{\infty}[t_j]\right\} \bigcup \\ &\quad \left\{\overline{\tau}_j(\theta, s_j) : \theta \in \Theta, s_j \in S_j^{\infty}[t_j] \text{ for some } t_j \in T_j^{*}\right\}. \end{aligned}$$

$\kappa^m_{\overline{\tau}_j(t_j,s_j,m)}$ and $\kappa^m_{\overline{\tau}_j(\theta,s_j)}$ are defined respectively by

$$\begin{aligned} \kappa^m_{\overline{\tau}_j(t_j,s_j,m)} &= \left(\frac{1}{m}\right)\pi^{s_j} \circ \widehat{\eta}_{-j}^{-1} + \left(1 - \frac{1}{m}\right)\pi^{t_j,s_j} \circ \widehat{\tau}_{-j,m}^{-1}; \\ \kappa^m_{\overline{\tau}_j(\theta,s_j)} &= \pi^{s_j} \circ \widehat{\eta}_{-j}^{-1}, \end{aligned}$$

where

$$\begin{aligned} \widehat{\tau}_{-j,m} &: (\theta, t_{-j}, s_{-j}) \mapsto (\theta, \overline{\tau}_{-j}(t_{-j}, s_{-j}, m)), \forall (\theta, t_{-j}, s_{-j}) \text{ s.t. } t_{-j} \in T_{-j}, s_{-j} \in S_{-j}^{\infty}[t_{-j}], \\ \widehat{\eta}_{-j} &: (\theta, s_{-j}) \mapsto (\theta, \overline{\tau}_{-j}(\theta, s_{-j})), \forall (\theta, s_{-j}) \text{ s.t. } s_{-j} \in S_{-j}^{\infty}[t_{-j}] \text{ for some } t_{-j} \in T_{-j}^{*}. \end{aligned}$$

For each $\overline{\tau}_j(t_j, s_j, m)$, define the belief

$$\begin{aligned} \widehat{\pi} &= \left(\frac{1}{m}\right)\pi^{s_j} \circ \widehat{\eta}_{-j}^{-1} \circ \xi^{-1} + \left(1 - \frac{1}{m}\right)\pi^{t_j,s_j} \circ \widehat{\tau}_{-j,m}^{-1} \circ \gamma^{-1} \in \Delta\left(\Theta \times T_{-j}^m \times S_{-j}\right) \text{ where} \\ \gamma &: (\theta, \overline{\tau}_{-j}(t_{-j}, s_{-j}, m)) \mapsto (\theta, \overline{\tau}_{-j}(t_{-j}, s_{-j}, m), s_{-j}); \\ \xi &: (\theta, \overline{\tau}_{-j}(\theta, s_{-j})) \mapsto (\theta, \overline{\tau}_{-j}(\theta, s_{-j}), s_{-j}). \end{aligned}$$

That is, $\overline{\tau}_j(t_j, s_j, m)$ believes that s_{-j} is played at each $(\theta, \overline{\tau}_{-j}(t_{-j}, s_{-j}, m))$ and s'_{-j} is played at each $\left(\theta, \overline{\tau}_{-j}\left(\theta, s'_{-j}\right)\right)$. Then, by construction,

$$\mathrm{marg}_{\Theta \times S_{-j}}\widehat{\pi} = \left(\frac{1}{m}\right)\pi^{s_j} + \left(1 - \frac{1}{m}\right)\mathrm{marg}_{\Theta \times S_{-j}}\pi^{t_j,s_j}.$$

Since $s_j \in BR_j\left(\mathrm{marg}_{\Theta \times S_{-j}}\pi^{t_j,s_j}\right)$, $\{s_j\} = BR_j(\pi^{s_j})$, and $\frac{1}{m} \in (0,1]$, $\{s_j\} = BR_j\left(\mathrm{marg}_{\Theta \times S_{-j}}\widehat{\pi}\right)$. Similarly, for each $\overline{\tau}_j(\theta, s_j)$ define the belief

$$\widetilde{\pi} = \pi^{s_j} \circ \widehat{\eta}_{-j}^{-1} \circ \xi^{-1} \in \Delta\left(\Theta \times T_{-j}^m \times S_{-j}\right).$$

Then, by construction, $\mathrm{marg}_{\Theta \times S_{-j}}\widetilde{\pi} = \pi^{s_j}$. Hence, $\{s_j\} = BR_j\left(\mathrm{marg}_{\Theta \times S_{-j}}\widetilde{\pi}\right)$. Thus, if we define

$$\begin{aligned} V_j^m[\overline{\tau}_j(t_j, s_j, m)] &= \{s_j\}, \forall \overline{\tau}_j(t_j, s_j, m), \forall j; \\ V_j^m[\overline{\tau}_j(\theta, s_j)] &= \{s_j\}, \forall \overline{\tau}_j(\theta, s_j), \forall j, \end{aligned}$$

[1] If Θ is an infinite compact metric space and t_i is in a finite model $(\Theta' \times T, \kappa)$, we replace $(\Theta \times T^m, \kappa^m)$ in the proof by $(\Theta'' \times T^m, \kappa^m)$ where $\Theta'' = \overline{\Theta} \cup \left\{\theta \in \Theta' : \kappa_{t_j}[\theta] > 0 \text{ for some } t_j \in T_j \text{ and } j \in N\right\}$ and $\overline{\Theta}$ is defined in RURA′ in (Chen, 2011, Section A.1).

then $V_j^m[\cdot]$ has the strict best reply property stated in the model $(\Theta \times T^m, \kappa^m)$.

It remains to show that $\lim_{m\to\infty} \overline{\tau}_j(t_j, s_j, m) = t_j$. By construction, each probability distribution is continuous in (t_j, s_j, m). Hence, by Lemma 4 of WY, $h_j(\overline{\tau}_j(t_j, s_j, m)) \to h_j(\overline{\tau}_j(t_j, s_j, 0))$ (in product topology) as $m \to \infty$. The proof that $h_j(\overline{\tau}_j(t_j, s_j, 0)) = h_j(t_j)$ for each t_j and j is exactly the same as that in (Weinstein and Yildiz, 2007, Lemma 6).■

A.2 Proof of Lemma 2

Lemma 2 *Let $(\Theta \times T, \kappa)$ be a finite model. Under RURA, for any type $t_i \in T_i$, any action $s_i \in V_i[t_i]$ for some profile of correspondences $(V_j)_{j\in N}$* with $V_j : T_j \rightrightarrows S_j$ *which satisfies the strict best reply property, and any integer $k \geq 1$, there exists a finite type $\widetilde{t}_i$ such that $\widetilde{t}_i^{k'} = t_i^{k'}$ for all $k' \leq k$ and $S_i^\infty[\widetilde{t}_i] = \{s_i\}$.*

Proof. We prove this claim by induction on k. First, suppose that $s_i \in V_i[t_i]$ for some profile of correspondences $(V_j)_{j\in N}$ with $V_j : T_j \rightrightarrows S_j$ which satisfies the strict best reply property. A correspondence which has the strict best reply property clearly has the best reply property (as defined in Dekel et al. (2007)). Hence, $s_i \in S_i^\infty[t_i]$. By RURA, there is a finite type $\widetilde{t}_i$ such that $S_i^\infty[\widetilde{t}_i] = \{s_i\}$.

Now fix any $k > 0$ and any $i \in N$. Write each t_{-i} as $t_{-i} = (l, h)$, where

$$l = \left(t_{-i}^1, t_{-i}^2, ..., t_{-i}^{k-1}\right) \text{ and } h = \left(t_{-i}^k, t_{-i}^{k+1}, ...\right)$$

are the lower- and higher-order beliefs, respectively, Let $L = \left\{l | \exists h : (l, h) \in T_{-i}^*\right\}$. The induction hypothesis is that for each finite $t_{-i} = (l, h)$ and each $s_{-i} \in V_{-i}[t_{-i}]$, there exists finite type $\widetilde{t}_{-i}[s_{-i}] = \left(l, \widetilde{h}[l, s_{-i}]\right)$ such that

$$S_{-i}^\infty\left[\widetilde{t}_{-i}[s_{-i}]\right] = \{s_{-i}\}. \tag{IH}$$

Take any $s_i \in V_i[t_i]$ for some finite type $t_i \in T_i^*$. We will construct a finite type $\widetilde{t}_i$ as in the lemma. Since $(V_j)_{j\in N}$ satisfies the strict best reply property, $BR_i\left(\text{marg}_{\Theta\times S_{-i}}\pi\right) = \{s_i\}$ for some $\pi \in \Delta(\Theta \times T_{-i} \times S_{-i})$ such that $\text{marg}_{\Theta\times T_{-i}^*}\pi = \kappa_{t_i}$ and $\pi(s_{-i} \in V_{-i}[t_{-i}]) = 1$.

Using the induction hypothesis, define the mapping μ :support$\left(\text{marg}_{\Theta\times L\times S_{-i}}\pi\right) \to \Theta \times T_{-i}^*$, by

$$\mu : (\theta, l, s_{-i}) \to \left(\theta, \widetilde{t}_{-i}[s_{-i}]\right),$$

where type $\widetilde{t}_{-i}[s_{-i}] = \left(l, \widetilde{h}[l, s_{-i}]\right)$ is as in (IH). Define $\widetilde{t}_i$ by

$$\kappa_{\widetilde{t}_i} \equiv \left(\text{marg}_{\Theta\times L\times S_{-i}}\pi\right)\circ\mu^{-1}.$$

As (Weinstein and Yildiz, 2007, pp.395-396), we can verify that

$$\begin{aligned}
\text{marg}_{\Theta\times L}\kappa_{\widetilde{t}_i} &= \pi\circ\text{proj}^{-1}_{\Theta\times L\times S_{-i}}\circ\mu^{-1}\circ\text{proj}^{-1}_{\Theta\times L} \\
&= \pi\circ\text{proj}^{-1}_{\Theta\times L} = \pi\circ\text{proj}^{-1}_{\Theta\times T^*_{-i}}\circ\text{proj}^{-1}_{\Theta\times L} \\
&= \text{marg}_{\Theta\times L}\kappa_{t_i}.
\end{aligned}$$

Moreover, by (IH), each (θ, t_{-i}) on the support of $\kappa_{\widetilde{t}_i}$ which is of the form $\left(\theta, \widetilde{t}_{-i}[s_{-i}]\right)$ and $\widetilde{t}_{-i}[s_{-i}]$ has the unique rationalizable action s_{-i}. Thus, there exists a unique $\widetilde{\pi}$ which is valid for $\widetilde{t}_i$. This belief is $\widetilde{\pi} = \kappa_{\widetilde{t}_i}\circ\gamma^{-1}$ where $\gamma : \left(\theta, \widetilde{t}_{-i}[s_{-i}]\right) \mapsto \left(\theta, \widetilde{t}_{-i}[s_{-i}], s_{-i}\right)$. By construction,

$$\begin{aligned}
\text{marg}_{\Theta\times L\times S_{-i}}\widetilde{\pi} &= \pi\circ\text{proj}^{-1}_{\Theta\times L\times S_{-i}}\circ\mu^{-1}\circ\gamma^{-1}\circ\text{proj}^{-1}_{\Theta\times L\times S_{-i}} \\
&= \pi\circ\text{proj}^{-1}_{\Theta\times L\times S_{-i}} \\
&= \text{marg}_{\Theta\times L\times S_{-i}}\pi
\end{aligned}$$

where the second equality follows because $\text{proj}_{\Theta\times L\times S_{-i}}\circ\gamma\circ\mu$ is the identity mapping on $\text{support}\left(\text{marg}_{\Theta\times L\times S_{-i}}\pi\right)$. However, s_i is the only best reply to the belief $\text{marg}_{\Theta\times S_{-i}}\pi$ which is the same as $\text{marg}_{\Theta\times S_{-i}}\widetilde{\pi}$. Hence, $S^\infty_i\left[\widetilde{t}_i\right] = \{s_i\}$. Finally, $\widetilde{t}_i$ is indeed a finite type since $\text{support}\kappa_{\widetilde{t}_i}$ is a finite set and consists entirely of finite types.■

A.3 Proof of Lemma 3

In this proof, we only require that Θ is a compact metric space equipped with metric d^0. Let $j \in \{1, 2, ..., n\}$ denote a generic player. Recall that the universal type space T^*_j endowed with the product topology is a compact metrizable space. The compatible metric d_j on T^*_j used in the proof is the one obtained from the Prohorov distance between beliefs of the same order.[2] Specifically, for any $t_j, t'_j \in T^*_j$, let $d^1_j\left(t^1_j, t'^1_j\right)$ be the Prohorov distance between t^1_j and

[2]Let Y be an arbitrary compact metric space endowed with metric ρ and the Borel σ-algebra. For any two $\mu, \mu' \in \Delta(Y)$, the Prohorov distance between μ and μ' is defined as

$$d(\mu, \mu') = \inf\{\varepsilon > 0 : \mu(E) \leq \mu'(E^\varepsilon) + \varepsilon \text{ for all Borel set } E \subseteq Y\}$$

$t_j'^1$ (recall $t_j^1, t_j'^1 \in \Delta(\Theta)$). Recursively, for any integer $k \geq 2$, and $t_j, t_j' \in T_j^*$, let $d_j^k\left(t_j^k, t_j'^k\right)$ be the Prohorov distance between t_j^k and $t_j'^k$ where $t_j^k, t_j'^k \in \Delta\left(\Theta \times T_{-j}^{k-1}\right)$ in which T_{-j}^{k-1} is the space of all $(k-1)^{th}$-order beliefs of player j's opponents and $\Theta \times T_{-j}^{k-1}$ is equipped with the metric ρ_{-j}^{k-1} defined as $\rho_{-j}^{k-1}\left(\left(\theta, t_{-j}^{k-1}\right), \left(\theta', t_{-j}'^{k-1}\right)\right) \equiv \max\left(d^0(\theta, \theta'), \max_{j' \neq j} d_{j'}^{k-1}\left(t_{j'}', t_{j'}\right)\right)$. Let $d_j\left(t_j, t_j'\right) \equiv \sum_{k=1}^{\infty} 2^{-k} d_j^k\left(t_j^k, t_j'^k\right)$, i.e., d_j is the product metric which metrizes the product topology on T_j^*.

Lemma 3 *For any type $\bar{t}_i \in T_i^*$, there is a sequence of finite types $\left(t_i(m)\right)_{m=1}^{\infty}$ such that $S_i^{\infty}\left[t_i(m)\right] = S_i^{\infty}\left[\bar{t}_i\right]$ for all m and $\lim_{m \to \infty} t_i(m) = \bar{t}_i$.*

Proof. We divide the proof into three steps.

Step 1. Construct the sequence of finite types.

Since T_j^* is a compact metric space, for each natural number m, T_j^* can be covered by finitely many open balls with radius $1/2m$. Let $\mathcal{T}_{j,m}$ be the finite measurable partition of T_j^* induced from these open balls and thus for any $T_j \in \mathcal{T}_{j,m}$, and t_j and t_j' in T_j, $d_j\left(t_j, t_j'\right) < 1/m$. Second, let $\mathcal{T}_{j,0}$ be the finite measurable partition induced by rationalizable sets, i.e., for any $T_j \in \mathcal{T}_{j,0}$, $t_j, t_j' \in T_j$ iff $S_j^{\infty}\left[t_j\right] = S_j^{\infty}\left[t_j'\right]$.[3] Let $\widetilde{\mathcal{T}}_{j,m}$ be the join (coarsest common refinement) of $\mathcal{T}_{j,0}$ and $\mathcal{T}_{j,m}$. Let $f_{j,m} : T_j^* \to \widetilde{\mathcal{T}}_{j,m}$ be the mapping such that $f_{j,m}\left(t_j\right) = \widetilde{t}_{j,m}$ iff $t_j \in \widetilde{t}_{j,m}$. Moreover, for each $\widetilde{t}_{j,m} \in \widetilde{\mathcal{T}}_{j,m}$, select arbitrarily a type $t_{j,m} \in \widetilde{t}_{j,m}$. It follows that

$$d_j\left(t_j, t_{j,m}\right) < 1/m, \forall t_j \in \widetilde{t}_{j,m}. \tag{1}$$

Define a sequence of finite models $\left(\left(\Theta \times \widetilde{T}^m, \widetilde{\kappa}^m\right)\right)_{m=1}^{\infty}$ by letting $\widetilde{T}_j^m \equiv \widetilde{\mathcal{T}}_{j,m}$, and for each $\widetilde{t}_{j,m} \in \widetilde{T}_j^m$,

$$\widetilde{\kappa}_{\widetilde{t}_{j,m}}^m\left[\left(\theta, \widetilde{t}_{-j,m}\right)\right] \equiv \kappa_{t_{j,m}}^*\left[\left\{\left(\theta, t_{-j}\right) : f_{-j,m}\left(t_{-j}\right) = \widetilde{t}_{-j,m}\right\}\right], \forall\left(\theta, \widetilde{t}_{-j,m}\right) \in \Theta \times \widetilde{T}_{-j}^m. \tag{2}$$

Note $\widetilde{t}_{j,m}$ denotes both a subset of T_j^* and a type in the model $\left(\Theta \times \widetilde{T}^m, \widetilde{\kappa}^m\right)$. We will write $\widetilde{t}_{j,m} \in \widetilde{\mathcal{T}}_{j,m}$ for the former and $\widetilde{t}_{j,m} \in \widetilde{T}_j^m$ for the latter when necessary. Let $\bar{t}_i(m) \equiv f_{i,m}\left(\bar{t}_i\right)$

where $A^{\varepsilon} \equiv \{y \in Y : \inf_{y' \in E} \rho(y, y') < \varepsilon\}$. It is known that the Prohorov metric metrizes the weak*-topology on $\Delta(Y)$ (see (Dudley, 2002, 11.3.3. Theorem)).

[3] Measurability follows from upper hemicontinuity (u.h.c.) of $S_j^{\infty}[\cdot]$: If $A_j' \subseteq A_j$ is $1-$minimal in the sense that there is no type t_j with $S_j^{\infty}\left[t_j\right] \subsetneq A_j'$, then u.h.c. implies $\left\{t_j : S_j^{\infty}\left[t_j\right] = A_j'\right\} = \left\{t_j : S_j^{\infty}\left[t_j\right] \subseteq A_j'\right\}$ is open and hence measurable; if $A_j' \subseteq A_j$ is $2-$minimal in the sense that $S_j^{\infty}\left[t_j\right] \subsetneq A_i'$ iff $S_j^{\infty}\left[t_j\right]$ is $1-$minimal then $\left\{t_j : S_j^{\infty}\left[t_j\right] = A_j'\right\} = \left\{t_j : S_j^{\infty}\left[t_j\right] \subseteq A_j'\right\} \setminus \left\{t_j : S_j^{\infty}\left[t_j\right] \text{ is } 1-\text{minimal}\right\}$ is measurable, and so on. Since A_j is a finite set, every $S_j^{\infty}\left[t_j\right]$ is $k-$minimal for some k and thus $\left\{t_j : S_j^{\infty}\left[t_j\right] = A_j'\right\}$ is measurable,$\forall A_j' \subseteq A_j$.

for every m. Step 2 and Step 3 below show that $\lim_{m\to\infty}\bar{t}_i(m)=\bar{t}_i$ and $S_i^{\infty}\left[\bar{t}_i(m)\right]\supseteq S_i^{\infty}\left[\bar{t}_i\right]$ for all m. Since $S_i^{\infty}[\cdot]$ is upper hemicontinuous and $\lim_{m\to\infty}\bar{t}_i(m)=\bar{t}_i$, it follows that $S_i^{\infty}\left[\bar{t}_i(m)\right]=S_i^{\infty}\left[\bar{t}_i\right]$ for sufficiently large m, say $m\geq\overline{m}$. We then define $t_i(m)=\bar{t}_i(\overline{m}+m)$,$\forall m$ and $(t_i(m))_{m=1}^{\infty}$ is the desired sequence.

Step 2. For each m and each $t_j\in T_j^*$, $S_j^{\infty}[f_{j,m}(t_j)]\supseteq S_j^{\infty}[t_j]$.

First, for each $\widetilde{t}_{j,m}\in\widetilde{T}_j^m$, we define $\overline{S}_j\left[\widetilde{t}_{j,m}\right]=S_j^{\infty}[t_{j,m}]$. We show that $\overline{S}_j[\cdot]$ satisfies the best-reply property on the model $\left(\Theta\times\widetilde{T}^m,\widetilde{\kappa}^m\right)$ (see (Dekel et al., 2007, Definition 1)). To see this, suppose that $s_j\in\overline{S}_j\left[\widetilde{t}_{j,m}\right]$. Since $\overline{S}_j\left[\widetilde{t}_{j,m}\right]=S_j^{\infty}[t_{j,m}]$, $s_j\in S_j^{\infty}[t_{j,m}]$. Thus, $s_j\in BR_j\left(\text{marg}_{\Theta\times S_{-j}}\pi\right)$ for some $\pi\in\Delta\left(\Theta\times T_{-j}^*\times S_{-j}\right)$ which is valid for $t_{j,m}$.

Define $\widetilde{\pi}\in\Delta\left(\Theta\times\widetilde{T}_{-j}^m\times S_{-j}\right)$ such that

$$\widetilde{\pi}\left[\left(\theta,\widetilde{t}_{-j,m},s_{-j}\right)\right]\equiv\pi\left(\left\{(\theta,t_{-j},s_{-j}):f_{-j,m}(t_{-j})=\widetilde{t}_{-j,m}\right\}\right),\forall\left(\theta,\widetilde{t}_{-j,m},s_{-j}\right)\tag{3}$$

Since π is valid for $t_{j,m}$, $\text{marg}_{\Theta\times T_{-j}^*}\pi=\kappa_{t_{j,m}}^*$. Hence, by (2), $\text{marg}_{\Theta\times\widetilde{T}_{-j}^m}\widetilde{\pi}=\widetilde{\kappa}_{\widetilde{t}_{j,m}}^m$. Moreover,

$$\begin{aligned}
&\widetilde{\pi}\left(\left\{\left(\theta,\widetilde{t}_{-j,m},s_{-j}\right):s_{-j}\in\overline{S}_{-j}\left[\widetilde{t}_{-j,m}\right]\right\}\right)\\
=\;&\pi\left\{(\theta,t_{-j,},s_{-j}):f_{-j,m}(t_{-j})=\widetilde{t}_{-j,m}\text{ and }s_{-j}\in\overline{S}_{-j}\left[\widetilde{t}_{-j,m}\right]\right\}\\
=\;&\pi\left\{(\theta,t_{-j,},s_{-j}):f_{-j,m}(t_{-j})=\widetilde{t}_{-j,m}\text{ and }s_{-j}\in S_{-j}^{\infty}[t_{-j,m}]\right\}\\
=\;&\pi\left(\left\{(\theta,t_{-j},s_{-j}):s_{-j}\in S_{-j}^{\infty}[t_{-j}]\right\}\right)\\
=\;&1
\end{aligned}$$

where the first equality follows from (3), the second follows because $\overline{S}_{-j}\left[\widetilde{t}_{-j,m}\right]=S_{-j}^{\infty}[t_{-j,m}]$, the third follows because every $t_{-j}\in\widetilde{t}_{-j,m}$ has the same rationalizable set as $t_{-j,m}$, and the last is because π is valid for $t_{j,m}$. Finally, since $s_j\in BR_j\left(\text{marg}_{\Theta\times S_{-j}}\pi\right)$ and $\widetilde{\pi}$ and π have the same marginal distribution on $\Theta\times S_{-j}$, it follows that $s_j\in BR_j\left(\text{marg}_{\Theta\times S_{-j}}\widetilde{\pi}\right)$. Hence, $\overline{S}_j[\cdot]$ satisfies the best-reply property on $\left(\Theta\times\widetilde{T}^m,\widetilde{\kappa}^m\right)$. Thus, by (Dekel et al., 2007, Proposition 4), $\overline{S}_j\left[\widetilde{t}_{j,m}\right]\subseteq S_j^{\infty}\left[\widetilde{t}_{j,m}\right]$, and moreover, since $\overline{S}_j\left[\widetilde{t}_{j,m}\right]=S_j^{\infty}[t_{j,m}]$, we obtain $S_j^{\infty}[t_{j,m}]\subseteq S_j^{\infty}\left[\widetilde{t}_{j,m}\right]$ and because every $t_j\in\widetilde{t}_{j,m}$ has the same rationalizable set as $t_{j,m}$, we get $S_j^{\infty}[t_j]\subseteq S_j^{\infty}[f_{j,m}(t_j)]$.

Step 3. $\lim_{m\to\infty}\sup_{t_j\in T_j^*}d_j(f_{j,m}(t_j),t_j)=0$.

For each $t_j\in T_j^*$, let $\widetilde{t}_{j,m}=f_{j,m}(t_j)$. We show that the k^{th}-order belief of $\widetilde{t}_{j,m}$ (viewed as a type in the model $\left(\Theta\times\widetilde{T}^m,\widetilde{\kappa}^m\right)$) converges to t_j^k and the convergence is uniform in t_j. We prove this by induction on k. For $k=1$, observe that by (2) $\widetilde{t}^1_{j,m}=t^1_{j,m}$. By (1), $d_j\left(t_j,\widetilde{t}_{j,m}\right)=d_j\left(t_j,t_{j,m}\right)<1/m$, and since $d^1_j\left(\widetilde{t}^1_{j,m},t^1_j\right)\leq d_j\left(\widetilde{t}_{j,m},t_j\right)$, $\lim_{m\to\infty}\sup_{t_j\in T^*_j}d^1_j\left(\widetilde{t}^1_{j,m},t^1_j\right)=0$.

Now consider $k>1$. Let $\varepsilon\in(0,1)$ and we show that for sufficiently large m, $d^k_j\left(\widetilde{t}^k_{j,m},t^k_j\right)<\varepsilon$ for all $t_j\in T^*_j$. By the induction hypothesis, there is some $\overline{m}(\varepsilon)$ such that for any $m>\overline{m}(\varepsilon)$, $\max_{j'\neq j}d^{k-1}_{j'}\left(f_{j',m}\left(t_{j'}\right)^{k-1},t^{k-1}_{j'}\right)<\varepsilon/2$ for all $t_{-j}=\left(t_{j'}\right)_{j'\neq j}\in T^*_{-j}$. Consider $m>\{2/\varepsilon,\overline{m}(\varepsilon)\}$. Recall that d^k_j is the Prohorov metric on the space of all k^{th}-order beliefs. Since $\widetilde{t}_{j,m}$ is a finite type, it suffices to verify that for each $\left(\theta,\widetilde{t}^{k-1}_{-j,m}\right)$ in the support of $\widetilde{t}_{j,m}$, we have

$$\begin{aligned}\widetilde{t}^k_{j,m}\left[\left(\theta,\widetilde{t}^{k-1}_{-j,m}\right)\right] &= \kappa^*_{t_{j,m}}\left(\left\{\left(\theta,t_{-j}\right):f_{-j,m}\left(t_{-j}\right)^{k-1}=\widetilde{t}^{k-1}_{-j,m}\right\}\right)\\ &\leq \kappa^*_{t_{j,m}}\left(\left(\theta,\widetilde{t}^{k-1}_{-j,m}\right)^{\varepsilon/2}\right)\\ &< t^k_j\left(\left(\theta,\widetilde{t}^{k-1}_{-j,m}\right)^{\varepsilon}\right)+\varepsilon\end{aligned}$$

where the first equality follows from (2); the first inequality follows because $f_{-j,m}\left(t_{-j}\right)^{k-1}=\widetilde{t}^{k-1}_{-j,m}$ implies $\max_{j'\neq j}d^{k-1}_{j'}\left(t^{k-1}_{j'},\widetilde{t}^{k-1}_{j',m}\right)<\varepsilon/2$ (since $m>\overline{m}(\varepsilon)$); the second follows because by (1), $d^k_j\left(t^k_{j,m},t^k_j\right)<1/m<\varepsilon/2$ (since $m>2/\varepsilon$). Thus, for $m>\{2/\varepsilon,\overline{m}(\varepsilon)\}$, $d^k_j\left(f_{j,m}\left(t_j\right)^k,t^k_j\right)<\varepsilon$ for all $t_j\in T^*_j$. Since $\varepsilon>0$ is arbitrary, the induction step follows.■

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