A structure theorem for rationalizability in the normal form of dynamic games*

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Abstract

We prove that the structure theorem for rationalizability originally from Weinstein and Yildiz (2007) applies to any finite extensive-form game with perfect recall and suitably rich payoffs. We demonstrate that the ties induced by the extensive form do not change the result of Weinstein and Yildiz (2007). Specifically, like Weinstein and Yildiz (2007), we adopt the normal-form concept of interim correlated rationalizability and we assume that players have no relevant knowledge of the extensive-form payoff structure. The extensive-form result is weaker in the sense that while the result of Weinstein and Yildiz (2007) does not depend on the latter assumption, our result does. Our result implies that without restrictions on players' knowledge of payoffs, the dynamic structure of extensive-form games offers no force for robust refinements of rationalizability. We also strengthen the main selection result of Weinstein and Yildiz (2007) by showing that the result holds for any (not necessarily finite) type.

Keywords: Rationalizability, Incomplete information, Robustness, Universal type space, Higher-order beliefs, Extensive-form games

JEL classification: C72, D80

1 Introduction

Rationalizability is one of the major solution concepts in game theory, and economic models typically have a large number of rationalizable outcomes. Yet, Weinstein and Yildiz (2007) (hereafter, WY) prove a striking generic uniqueness result for the structure of the (interim correlated) rationalizable (ICR) correspondence for incomplete-information static games with finite action sets: for any rationalizable action s of any finite type, we can perturb (in the product topology of the universal type space) the beliefs of the type in such a way that s is uniquely rationalizable for the new type. Since the set of finite types is dense in the universal space (Mertens and Zamir (1985)) and the rationalizable correspondence is upper hemicontinuous (Dekel et al. (2006)), it follows that types with a unique rationalizable outcome are generic (i.e., the set of such types is open and dense) in the universal type space and no refinements are robust under this notion of perturbation.

The WY result relies on a "richness" assumption about the game, namely that every action for every player is strictly dominant for some payoff parameter. As WY observe, this richness assumption holds — in simultaneous-move games — if we assume away any knowledge of payoffs. However, this richness assumption essentially rules out all truly dynamic games. To see this, observe that, if an extensive game form is not equivalent to a simultaneous-move game, there must be some player i with an information set I that is not crossed by every path; hence, two strategies of i that differ only in the choice prescribed at I must yield the same outcome whenever the rest of the strategy profile implies a path that does not cross I. Since an important implication of the structure theorem of WY is that no refinement can produce robust predictions beyond rationalizability, and since refinements play an important role in dynamic games, it is natural to ask whether there is some weaker condition which ensures WY's result and is suitable for dynamic games.[1]

In this paper, we propose a primitive notion of richness in extensive-form games: *Extensive-form richness* (EF-richness). EF-richness means that for any pure strategy s, there is some payoff parameter under which throughout the course of play, following s is

[1]WY mention that in a dynamic game, one can introduce trembles and use a reduced form to satisfy their richness assumption. However, introducing trembles changes the game, and hence makes it unclear whether some refinements may still produce robust predictions beyond rationalizability in the game without trembles.

always strictly better than switching to anything else. We demonstrate that EF-richness can be satisfied in a class of extensive-form games with perfect recall by suitably choosing payoffs at terminal histories (Lemma 4). We then establish WY's result in the reduced normal form of these extensive-form games under EF-richness (Theorem 2).

To obtain these results, we first observe that it is both necessary and sufficient for WY's result that every rationalizable action be the unique rationalizable action for *some* type (Theorem 1). We call this nonprimitive condition, which is obviously weaker than WY's richness condition, *Richness in Uniquely Rationalizable Actions* (RURA).[2] The central step in proving our results is to show that if an extensive-form game satisfies EF-richness, then RURA holds in its reduced normal form (Lemma 6).

To sum up, in this paper we propose the notion of EF-richness which we show enables us to extend WY's structure theorem to any finite extensive-form game with perfect recall. Specifically, we show that for games that satisfy EF-richness, we obtain unique rationalizable outcomes for "almost all" types in the universal type space. This implies that such a uniqueness property holds for any refinement. The main implication of our result is that in these extensive-form games with incomplete information, for a prediction to be robust to alternative specifications of beliefs — in the sense proposed by WY and adopted here — it must hold for all rationalizable outcomes.

In the course of this work, we also strengthen one of WY's results (and hence our extensive-form version). Specifically, WY prove the selection result for finite types: any rationalizable action of any *finite* type t is the unique rationalizable action of a sequence of finite types converging to t. Ely and Pęski (2011) show that finite types are contained in a nongeneric set, so this leaves open the question of whether such a result holds in general. We show that the answer is yes. In other words, the selection result holds whether t is finite or infinite.

As WY mention, their approach to the robustness of refinements is closely related to that of Fudenberg et al. (1988) (hereafter, FKL) and Dekel and Fudenberg (1990) (hereafter, DF), although WY consider simultaneous-move games, while these two studies, like the current paper, prove their main results in the reduced normal form of finite extensive-form games with perfect recall. WY have discussed the differences between their results and

[2] We thank Jonathan Weinstein for suggesting this name to us.

those of FKL and DF (see Weinstein and Yildiz (2007, pp. 390-391)). Here we highlight two critical features of our analysis for dynamic games.

First, instead of pursuing an extension of the structure theorem to some version of rationalizability for extensive-form games (see below for more discussion), we prove exactly the same thesis as WY, i.e., a statement about ICR, a normal-form solution concept. Our paper thus belongs to the literature which adopts the normal form to analyze dynamic games.[3] As mentioned above, the normal-form approach is also adopted in FKL's and DF's robustness analyses of dynamic games.

Second, like WY, we assume that the players have no knowledge of payoffs. This enables us to consider a large set of perturbations in which players can entertain any doubt about payoffs. This is analogous to the idea of the elaboration with general types in FKL in which players might attach a small probability to the possibility that even their *own* payoffs are different from the original game. This rich set of perturbations is what drives our conclusion that the tightest robust solution in the normal form is also the tightest that is robust for the extensive form. We therefore formalize the (perhaps by now intuitive) idea that when we rule out any relevant knowledge of the payoff structure, the dynamic structure of extensive-form games offers no force for robust refinements of rationalizability.

An alternative would be to impose some knowledge of payoffs. That is, one may want to maintain some knowledge assumption even when the beliefs of a given type are perturbed. Intuitively, by allowing a smaller set of perturbations, we may get a tighter solution concept to be robust. An important special case is the elaboration with personal types in FKL and DF in which players know their *own* payoffs and no play of the opponents can raise doubts in their minds regarding their own payoffs, even though it can raise doubts about anything else. This approach is pursued in the independent work of Penta (2011).[4]

[3] Many papers in the literature follow von Neumann and Morgenstern in claiming that the normal form is sufficient to analyze every game. For example, Kohlberg and Mertens (1986) have made forceful arguments that only the reduced normal form of a game should matter or, more precisely, that all extensive-form games that have the same reduced normal form should be viewed as strategically equivalent by rational players.

[4] We note that the distinction between assuming away some versus assuming away all knowledge of payoffs plays no role in a static game where no player moves after observing a move-related event. One would therefore anticipate that WY's result for simultaneous-move games holds even when we keep some knowledge of payoffs. Indeed, Penta (2011) proves such a variation. We thank an associate editor for pointing this out to us.

Under EF-richness, Penta (2011) considers multi-stage games with observable actions and obtains generic uniqueness for a refinement of ICR called interim sequential rationalizability (ISR). Unlike ICR, ISR is an extensive-form solution concept based on initial common belief of sequential rationality.[5] Penta shows that ISR, like ICR, is upper hemicontinuous. This may seem to conflict with our claim that no refinement of ICR is robust. Indeed, together with our results, this implies that when all knowledge assumptions are assumed away, ISR coincides with ICR *everywhere* on the universal type space (Theorem 3).[6] The intuition behind this equivalence is that without any knowledge, ISR allows players to revise beliefs even about their *own* payoffs upon being surprised, while under EF-richness this makes every subsequent move "credible" and hence sequential rationality has no bite.[7]

In contrast, if we maintain some knowledge assumption (e.g., it is common knowledge that players know their own payoffs), ISR is no longer equivalent to ICR. For instance, when we impose the knowledge assumption of one's own payoffs in the entry deterrence game, the ICR action "fight" is not ISR and can no longer be made uniquely ICR in a perturbed game because the incumbent knows that her own payoffs are those in the original game. This shows that ISR is a solution that depends on whether an aspect of a game is modeled as knowledge (and cannot be unlearned) or belief (which is therefore subject to updating).[8]

Either approach seems sensible in the abstract, and which approach one might want to apply would depend on the economic environments under consideration. This is the case even in simultaneous-move games.[9] For instance, in some applications, such as auctions with interdependent values or an oligopoly with demand uncertainty, it is conceivable that

[5] See, for example, Ben-Porath (1997) and Battigalli and Siniscalchi (1999, 2007).

[6] We thank Eddie Dekel for making this observation.

[7] This intuition is analogous to that in FKL's results for elaboration with general types.

[8] To obtain the structure theorem for ISR, Penta (2011) also follows FKL and DF in allowing for an additional perturbation of payoffs: players know their own payoffs but the payoffs may be slightly different from the original game. To see that this is necessary, consider a complete-information simultaneous-move game in which a player has a weakly dominated rationalizable (and thus ICR and ISR) action. There is no way to perturb the player's beliefs such that the weakly dominated ISR action becomes the only (sequential) rational action if in the perturbed game the player knows her payoffs are exactly the same as those in the unperturbed game.

[9] More generally, some perturbations may seem more natural than others, so in a specific context one might want to define robustness differently. For example, the narrower class of perturbations of global games may seem more natural in some environments than the more general class allowed for by WY. See Morris and Shin (2007) for more discussion on this point.

a player's opponents have better information than the player himself regarding the player's payoffs. For robustness, we may thus consider perturbations in which, with small probability, players update beliefs about their own payoffs upon seeing an unexpected move of their opponents. On the other hand, models with private values are also widespread; in such models a player *knows* his payoff function (valuation), which rules out the perturbations considered in this paper, as knowledge cannot be unlearned. Hence, whether a type of model is common is not enough to rule out either approach. Here we view the case of assuming away any knowledge of payoffs as a natural benchmark which embodies a stringent robustness check.

2 The structure theorem under RURA

We first introduce some essential notation and definitions that we adopt from WY. Fix a finite set of players $N = \{1, 2, ..., n\}$. There is a (possibly unknown) payoff-relevant parameter $\theta \in \Theta$, where Θ is a finite set.[10] In this section, we consider only normal-form games. Each player i has a finite action space S_i and utility function $u_i : \Theta \times S \to \mathbb{R}$ (where $S \equiv \times_{j \in N} S_j$). Let $S_{-i} \equiv \times_{j \neq i} S_j$ be the set of all action profiles of player i's opponents. By a model, we mean a pair $(\Theta \times T, \kappa)$, where $T = T_1 \times T_2 \times \cdots \times T_n$ is a compact metric space of type profiles associated with beliefs $\kappa_{t_i} \in \Delta(\Theta \times T_{-i})$ for each type $t_i \in T_i$. Assume that $\Delta(\Theta \times T_{-i})$ is endowed with the weak* topology and that $t_i \mapsto \kappa_{t_i}$ is a continuous mapping. A finite model is a model such that $|T| < \infty$.

Given any type t_i in a model $(\Theta \times T, \kappa)$, we can compute the belief of t_i on Θ by setting $t_i^1 = \text{marg}_\Theta \kappa_{t_i}$, which is called the first-order belief of t_i. We can also compute the second-order belief of t_i (i.e., his belief about (θ, t_{-i}^1)) by setting

$$t_i^2(F) = \kappa_{t_i}\left(\left\{(\theta, t_{-i}) : \left(\theta, t_{-i}^1\right) \in F\right\}\right)$$

for each measurable $F \subseteq \Theta \times [\Delta(\Theta)]^{n-1}$. We can compute the entire hierarchy of beliefs $\left(t_i^1, t_i^2, ..., t_i^k, ...\right)$ by proceeding in this way and we can write $h_i(t_i) = \left(t_i^1, t_i^2, ..., t_i^k, ...\right)$ for the resulting hierarchy. Mertens and Zamir (1985) construct a universal type space T_i^* which contains all such belief hierarchies (i.e., coherent hierarchies that satisfy common belief of

[10] WY assume only that Θ is a compact metric space. Our structure theorem also holds in this case (see Appendix A.1).

coherency (see also Brandenburger and Dekel (1993))). Let $T^* = T_1^* \times \cdots \times T_n^*$ be the set of type profiles and $T_{-i}^* = \times_{j \neq i} T_j^*$ denote the set of profiles for players other than i.

Each T_i^* is endowed with the product topology, so that a sequence of types $(t_i(m))_{m=1}^{\infty}$ converges to a type t_i (denoted as $\lim_{m\to\infty} t_i(m) = t_i$) if and only if for every $k \geq 1$, $\left(t_i^k(m)\right)_{m=1}^{\infty}$ converges to t_i^k under the weak* topology. Likewise, T^* is also endowed with the product topology, so that a sequence of type profiles $(t(m))_{m=1}^{\infty}$ with $t(m) = (t_1(m), ..., t_n(m))$ converges to a type profile $t = (t_1, ..., t_n)$ ($\lim_{m\to\infty} t(m) = t$) if and only if $(t_i(m))_{m=1}^{\infty}$ converges to t_i for each player i. Mertens and Zamir (1985) show that T_i^* is a compact metrizable space and, moreover, that there is a homeomorphism κ_i^* between T_i^* and $\Delta\left(\Theta \times T_{-i}^*\right)$. Hence, $(\Theta \times T^*, \kappa^*)$ is a model according to the definition given above, where $\kappa_{t_i}^* \equiv \kappa_i^*(t_i)$ for every $t_i \in T_i^*$. A type t_i in T_i^* is called a finite type if $t_i = h_i(t_i')$ for some t_i' in a finite model.

For each $i \in N$ and for each belief $\pi \in \Delta(\Theta \times S_{-i})$, we write $BR_i(\pi)$ for the set of actions $s_i \in S_i$ that maximize the expected utility $u_i(\theta, s_i, s_{-i})$ under the probability distribution π. Following Dekel et al. (2007), we define the solution concept of interim correlated rationalizability (ICR) as follows.

Let $(\Theta \times T, \kappa)$ be a model. For $i \in N$ and type $t_i \in T_i$, set $S_i^0[t_i] = S_i$; define sets $S_i^k[t_i]$ for $k > 0$ iteratively by letting $s_i \in S_i^k[t_i]$ if and only if $s_i \in BR_i\left(\mathrm{marg}_{\Theta\times S_{-i}}\pi\right)$ for some $\pi \in \Delta(\Theta \times T_{-i} \times S_{-i})$ such that $\mathrm{marg}_{\Theta\times T_{-i}}\pi = \kappa_{t_i}$ and $\pi\left(\left\{(\theta, t_{-i}, s_{-i}) : s_{-i} \in S_{-i}^{k-1}[t_{-i}]\right\}\right) = 1$ where $S_{-i}^{k-1}[t_{-i}] = \Pi_{j\neq i} S_j^{k-1}[t_j]$. Then, define

$$S_i^{\infty}[t_i] = \bigcap_{k=0}^{\infty} S_i^k[t_i].$$

We write $S^{\infty}[t] = \Pi_{j\in N} S_j^{\infty}[t_j]$ and $S_{-i}^{\infty}[t_{-i}] = \Pi_{j\neq i} S_j^{\infty}[t_j]$. Say that $\pi \in \Delta(\Theta \times T_{-i} \times S_{-i})$ is *valid* for a type $t_i \in T_i$ if $\mathrm{marg}_{\Theta\times T_{-i}}\pi = \kappa_{t_i}$ and $\pi\left(\left\{(\theta, t_{-i}, s_{-i}) : s_{-i} \in S_{-i}^{\infty}[t_{-i}]\right\}\right) = 1$. Dekel et al. (2007) show that $s_i \in S_i^{\infty}[t_i]$ if and only if $s_i \in BR_i\left(\mathrm{marg}_{\Theta\times S_{-i}}\pi\right)$ for some π which is valid for t_i. Moreover, for any t_i in any model, $S_i^{\infty}[t_i] = S_i^{\infty}[h_i(t_i)]$, in other words, $S_i^{\infty}[\cdot]$ only depends on the belief hierarchy of a type.

In proving our main result, we will adopt the following notion defined in WY. In a model $(\Theta \times T, \kappa)$, we say that a profile of correspondences $(V_i)_{i\in N}$ with $V_i : T_i \rightrightarrows S_i$ satisfies the *strict best reply property* if each $s_i \in V_i[t_i]$ is a strict best reply to a belief $\pi \in \Delta(\Theta \times T_{-i} \times S_{-i})$ with $\mathrm{marg}_{\Theta\times T_{-i}}\pi = \kappa_{t_i}$ and $\pi(\{(\theta, t_{-i}, s_{-i}) : s_{-i} \in V_{-i}[t_{-i}]\}) = 1$.

We now state the richness assumption in WY.

Definition 1 *The game satisfies Richness if for each player $i \in N$ and each action $s_i \in S_i$, there exists a payoff parameter $\theta_{s_i} \in \Theta$ such that*

$$u_i(\theta_{s_i}, s_i, s_{-i}) > u_i(\theta_{s_i}, s_i', s_{-i}), \ \forall s_i' \neq s_i, \ \forall s_{-i} \in S_{-i}.$$

The following notion plays a crucial role in our analysis.

Definition 2 *The game satisfies Richness in Uniquely Rationalizable Actions (RURA) if for each player i and each action $s_i \in \cup_{t_i' \in T_i^*} S_i^\infty[t_i']$, there exists a type $t_i \in T_i^*$ such that $S_i^\infty[t_i] = \{s_i\}$.*

Richness implies RURA, as can be seen by choosing any type that assigns probability 1 to θ_{s_i}. However, RURA can be satisfied even when Richness is not. For instance, in the entry deterrence game, payoffs at terminal histories can be assigned so that RURA is satisfied for the actions of the incumbent, but there is no way to satisfy Richness. In Section 3, we will generalize this observation by demonstrating that RURA is satisfied in the reduced normal form of any finite extensive-form game with perfect recall and suitably rich payoffs.

A precursor of RURA occurs, in a different and more specialized context, in Frankel et al. (2003). There they note that — in the context of supermodular games — one could replace types that have dominant actions with types that have unique Nash equilibrium actions (i.e., types with unique rationalizable actions within the class of supermodular games) to start their "global-game infection arguments." Theorem 1 below shows that types with unique rationalizable actions suffice to start the infection arguments in the less structured context of WY, as they do in the more structured context of global games.[11] As in WY, Theorem 1 immediately implies the generic uniqueness result stated in Corollary 1.

Theorem 1 *RURA holds if and only if for any type $t_i \in T_i^*$ and any action $s_i \in S_i^\infty[t_i]$, there is a sequence of types $(t_i(m))_{m=1}^\infty$ such that $\lim_{m\to\infty} t_i(m) = t_i$ and $S_i^\infty[t_i(m)] = \{s_i\}$ for all m.*

[11] We thank Stephen Morris for pointing this out to us.

Corollary 1 *Under RURA,* $\{t \in T^* : |S^\infty[t]| = 1\}$ *is open and dense in the product topology.*

Observe that the "if" part of Theorem 1 is immediate: if an action is never the unique rationalizable action of any type, the sequence in Theorem 1 cannot exist. The "only if" part of Theorem 1 strengthens WY's Proposition 1 in two ways. First, it replaces the Richness condition with the RURA condition which is weaker and in fact equivalent to the property stated in the thesis of the selection result. This will be useful for our analysis of dynamic games in Section 3. Second, in WY's Proposition 1, $t_i \in T_i^*$ is a finite type, whereas in our Theorem 1, t_i can be either finite or infinite. Infinite types are necessary to model the information structure in global games as well as in other applications; recall that WY use their Proposition 1 to conclude that any refinement of rationalizability is not robust. In Section 4, we will use the "only if" part (and in particular the fact that it works for any type) to prove that ICR is equal to ISR (as defined in Penta (2011)) everywhere on the universal type space.

The proof of the "only if" part requires three lemmas. First, note that Lemmas 1 and 2 are the counterparts under RURA of Lemmas 6 and 7 of WY. The proofs follow closely the ideas of Lemmas 6 and 7 of WY and are presented in an Online Appendix linked to the author's website.

Lemma 1 *Under RURA, for any finite type* $t_i \in T_i^*$ *and any action* $s_i \in S_i^\infty[t_i]$*, there exists a sequence of finite models* $((\Theta \times T^m, \kappa^m))_{m=1}^\infty$ *and a sequence of finite types* $(t_i(m))_{m=1}^\infty$ *such that* $t_i(m) \in T_i^m$ *and* $s_i \in V_i^m[t_i(m)]$ *for some profile of correspondences* $\left(V_j^m\right)_{j\in N}$ *with* $V_j^m : T_j^m \rightrightarrows S_j$ *which satisfies the strict best reply property, for all* m*, and* $\lim_{m\to\infty} t_i(m) = t_i$.

Lemma 2 *Let* $(\Theta \times T, \kappa)$ *be a finite model. Under RURA, for any type* $t_i \in T_i$*, any action* $s_i \in V_i[t_i]$ *for some profile of correspondences* $(V_j)_{j\in N}$ *with* $V_j : T_j \rightrightarrows S_j$ *which satisfies the strict best reply property, and integer* $k \geq 1$*, there exists a finite type* $\widetilde{t}_i$ *such that* $\widetilde{t}_i^{k'} = t_i^{k'}$ *for all* $k' \leq k$ *and* $S_i^\infty\left[\widetilde{t}_i\right] = \{s_i\}$.

Lemma 3 is an additional step to deal with the case in which t_i is an infinite type.

Lemma 3 *For any type $t_i \in T_i^*$, there is a sequence of finite types $(t_i(m))_{m=1}^{\infty}$ such that $S_i^{\infty}[t_i(m)] = S_i^{\infty}[t_i]$ and $\lim_{m\to\infty} t_i(m) = t_i$.*

Lemma 3 is trivial when t_i is finite. When t_i is infinite, however, it shows that there is a sequence of finite types converging to t_i in the product topology, and moreover, every type in the sequence has the same set of rationalizable actions as t_i. The interesting aspect of type t_i and the approaching sequence is that along the sequence not even lower hemicontinuity fails at t_i. With Lemma 3, the result with an infinite t_i in Theorem 1 is implied by the result with finite types (by applying Theorem 1 to every finite type $t_i(m)$ in Lemma 3 and taking an appropriate diagonal sequence). Lemma 3 is closely related to Theorem 3.1 in Mertens and Zamir (1985), which states that finite types are dense in the product topology, but Lemma 3 is not directly implied by the latter result due to possible failure of lower hemicontinuity. Indeed, the proof of Lemma 3, which can be found in Appendix A.2, is similar to the proof of Theorem 3.1 in Mertens and Zamir (1985) but addresses the issue of lower hemicontinuity by adopting the idea in the proof of Theorem 3 in Dekel et al. (2006).[12]

3 The structure theorem under EF-richness

In this section, we demonstrate that RURA can be satisfied in the reduced normal form of any finite extensive-form game with perfect recall and a sufficiently rich set of payoff states. To motivate our setup and results, consider Example 1 in Figure 1. In this game, the two pure strategies $(D1, C3)$ and $(D1, D3)$ of player 1 are realization-equivalent: the history $(C1, C2)$ is precluded by either $(D1, C3)$ or $(D1, D3)$ and hence $(D1, C3)$ and $(D1, D3)$ induce the same outcome no matter what player 2 chooses. If we treat $(D1, C3)$ and $(D1, D3)$ as different strategies, then for any possible payoffs and belief, player 1 cannot have either $(D1, C3)$ or $(D1, D3)$ as the unique best reply. Thus, RURA is not applicable.

[12] Theorem 3 in Dekel et al. (2006) shows that finite types are dense in the strategic topology defined in their paper. Strategic topology on types is the coarsest topology in which rationalizable behaviors are continuous. More precisely, Theorem 3 in Dekel et al. (2006) shows that for any type t_i, any finite game G, and any $\varepsilon > 0$, there is a finite type t_i' such that every rationalizable action of t_i in G is $\varepsilon-$rationalizable for t_i'. Here we can replace $\varepsilon-$rationalizable with $(0-)$rationalizable because we consider a fixed finite game (see Lemma 12 in Dekel et al. (2006)).

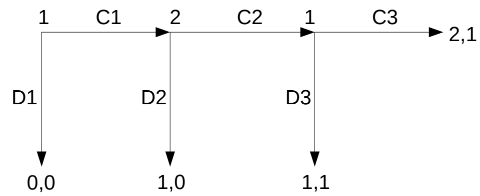

Figure 1: Example 1

One solution to this problem is to define the reduced form by identifying equivalent extensive-form pure strategies. That is, we take the quotient space of the pure strategy space with respect to the realization-equivalent relation. This idea dates back to early work by Kuhn (1950, 1953) and is also suggested by WY. In Example 1 it means that we should identify $(D1, C3)$ and $(D1, D3)$ in defining the reduced form. Essentially, this means that we will just "ignore" the part of a strategy that a player's own strategy precludes. Specifically, Extensive-form Richness (as in Definition 3) says that for any such pure strategy s_i in the reduced form, there is some payoff parameter θ_{s_i} such that throughout the course of play, following s_i is always strictly better than switching to anything else.

To formalize the idea, we introduce some notation and definitions from Osborne and Rubinstein (1994, Chapters 6 and 11). Let $\Gamma = \left(N, H, (\mathcal{I}_i)_{i\in N}\right)$ be a finite extensive game form with perfect recall: H is a finite set of histories, and $\mathcal{I}_i$ is player i's information partition on H_i, where H_i is the set of histories after which player i moves. For any history $h \in H_i$, let $I_i(h) \in \mathcal{I}_i$ be the information set which contains h, and let $A_i(h) = \{a_i : (h, a_i) \in H\}$ be the set of actions available to player i at h. It is required that $A_i(\cdot)$ be $\mathcal{I}_i$-measurable, i.e., $A_i\left(\overline{h}\right) = A_i(h)$ for any $\overline{h} \in I_i(h)$. Note that each history is a finite sequence of actions and we do not allow more than one player to move after any history.[13] Let Z be the set of terminal histories and $U_i : \Theta \times Z \to \mathbb{R}$ be the payoff function of player i. Then, $\left(\Gamma, (U_i)_{i\in N}\right)$ is a finite extensive-form game.

A pure strategy of player i is an $\mathcal{I}_i$-measurable mapping $\widetilde{s}_i$ from each $h \in H_i$ to some action $\widetilde{s}_i(h) \in A_i(h)$. For any pure strategy $\widetilde{s}_i$, let $H(\widetilde{s}_i)$ be the set of all the histories in which all the actions of player i are those dictated by $\widetilde{s}_i$, i.e., $h \in H(\widetilde{s}_i)$ if and only if for any proper subhistory $h' \in H_i$ of h, $(h', \widetilde{s}_i(h'))$ is also a subhistory of h. For each pure strategy

[13]However, as explained in Osborne and Rubinstein (1994, p. 202), we can still use this framework to formulate simultaneous-move games or multi-stage games with observation actions. See Section 4 for details.

profile $\widetilde{s}$, denote by $z(\widetilde{s})$ the unique terminal history induced by $\widetilde{s}$, that is, $z(\widetilde{s}) \in H(\widetilde{s}_i)$ for all i.

Two pure strategies $\widetilde{s}_i$ and $\widetilde{s}'_i$ of player i are said to be (realization-)*equivalent* if $z(\widetilde{s}_i, \widetilde{s}_{-i}) = z(\widetilde{s}'_i, \widetilde{s}_{-i})$ for any player i's opponents' pure strategy profile $\widetilde{s}_{-i}$. Let S_i be the quotient space of the pure strategies with respect to the equivalence relation, and each $s_i \in S_i$ is called a *reduced pure strategy*. By Theorem 1 of Kuhn (1953), pure strategy $\widetilde{s}_i$ is equivalent to pure strategy $\widetilde{s}'_i$ if and only if $H(\widetilde{s}_i) = H(\widetilde{s}'_i)$ and $\widetilde{s}_i(h) = \widetilde{s}'_i(h)$ for all $h \in H(\widetilde{s}_i) \cap H_i$. Hence, for each reduced pure strategy s_i, $H(s_i) \equiv H(\widetilde{s}_i)$ and $s_i(h) \equiv \widetilde{s}_i(h)$ for any $h \in H(s_i)$ are defined independent of the pure strategy $\widetilde{s}_i$ chosen from s_i. Likewise, for any reduced pure strategy profile s, the unique terminal history $z(s) \equiv z(\widetilde{s})$ induced from s is independent of $\widetilde{s}_i$ chosen from s_i. Hereafter, we will consider only reduced pure strategies and omit the term *reduced*.

Let $S_i(h)$ be the set of player i's pure strategies consistent with h, i.e., $s_i \in S_i(h)$ if and only if $h \in H(s_i)$. Let $S_{-i}(h) = \times_{j \neq i} S_j(h)$. We say that a pure strategy profile s reaches an information set $I_i(h)$ if $I_i(h)$ contains a subhistory of $z(s)$. Let $G^{\Gamma,U} \equiv \left(N, (S_i)_{i \in N}, (u_i)_{i \in N}\right)$ be the reduced normal-form representation of $\left(\Gamma, (U_i)_{i \in N}\right)$ where $u_i(\theta, s) = U_i(\theta, z(s))$ for all θ and s. We will now apply the notions and results in the previous section to study $G^{\Gamma,U}$.

Definition 3 $\left(\Gamma, (U_i)_{i \in N}\right)$ *satisfies Extensive-form Richness (EF-richness) if for any $i \in N$ and any $s_i \in S_i$, there is some $\theta_{s_i} \in \Theta$ such that for any $s'_i \in S_i$ and any $s_{-i} \in S_{-i}$, if $s'_i(h') \neq s_i(h')$ for some subhistory $h' \in H_i$ of $z(s_i, s_{-i})$, then $U_i(\theta_{s_i}, z(s_i, s_{-i})) > U_i(\theta_{s_i}, z(s'_i, s_{-i}))$.*

One can show that EF-richness requires that each $s_i \in S_i$ be conditionally dominant (in the sense of Shimoji and Watson (1998)) at some payoff state θ. For instance, to satisfy EF-richness for the pure strategy $(C1, C3)$ of player 1 and for the pure strategy $C2$ of player 2, the payoffs under $\theta_{(C1,C3)}$ and θ_{C2} can be assigned as in Figure 1.

Lemma 4 below states that EF-richness can be satisfied in a class of extensive-form games with perfect recall by suitably choosing payoffs at terminal histories. The proof of Lemma 4 is similar to the way that payoffs of "crazy types" are specified in Fudenberg et al. (1988, p. 366) and can be found in Appendix A.4. Lemma 5 follows from Theorem 1 of Kuhn (1953) and will be used later.

Lemma 4 *There are some finite set of payoff parameters* Θ *and utility functions* $(U_i)_{i\in N}$ *with* $U_i : \Theta \times Z \to \mathbb{R}$ *such that* $\left(\Gamma, (U_i)_{i\in N}\right)$ *satisfies EF-richness.*

Lemma 5 *For any* $s_i, s_i' \in S_i$*, if for any history* $h \in H(s_i) \cap H_i$*,* $h \in H(s_i')$ *and* $s_i'(h) = s_i(h)$*, then* $s_i = s_i'$*.*

We now present our main result.

Theorem 2 *Suppose that the extensive-form game* $\left(\Gamma, (U_i)_{i\in N}\right)$ *satisfies EF-richness. Then, for any type* t_i *and any* $s_i \in S_i^\infty[t_i]$ *in* $G^{\Gamma,U}$*, there is a sequence of types* $(t_i(m))_{m=1}^\infty$ *such that* $\lim_{m\to\infty} t_i(m) = t_i$ *and* $S_i^\infty[t_i(m)] = \{s_i\}$ *for any* m*. Moreover,* $\{t \in T^* : |S^\infty[t]| = 1\}$ *is open and dense in the product topology.*

Theorem 2 immediately follows from Theorem 1 and Lemma 6 below. Lemma 6 shows that RURA holds in the reduced normal-form representation $G^{\Gamma,U}$ whenever the extensive-form game $\left(\Gamma, (U_i)_{i\in N}\right)$ satisfies EF-richness.

Lemma 6 *Suppose that the extensive-form game* $\left(\Gamma, (U_i)_{i\in N}\right)$ *satisfies EF-richness. Then, for any* $s_i \in S_i$ *in* $G^{\Gamma,U}$*, there is a finite type* $t_i \in T_i^*$ *such that* $S_i^\infty[t_i] = \{s_i\}$*. Hence, RURA holds in* $G^{\Gamma,U}$*.*

To prove this lemma, we define a finite model $(\Theta \times T, \kappa)$ as follows. For each i and s_i, fix some θ_{s_i} as per Definition 3. Then, let

$$\begin{aligned} T_i &= \{t_{i,s_i} : s_i \in S_i\}, \forall i \in N; \\ \kappa_{t_{i,s_i}}(\{(\theta, t_{-i})\}) &= \begin{cases} \frac{1}{|T_{-i}|}, & \text{if } \theta = \theta_{s_i}; \\ 0, & \text{otherwise}, \end{cases} \quad \forall (\theta, t_{-i}) \in \Theta \times T_{-i}, \forall i \in N. \end{aligned} \tag{$\bigstar$}$$

Note that in this model, for every $s_i \in S_i$, player i has a type t_{i,s_i} which assigns probability one to θ_{s_i} and positive probability to every $t_{-i,s_{-i}}$ (where $t_{-i,s_{-i}} \equiv \left(t_{j,s_j}\right)_{j\neq i}$ for any $s_{-i} = (s_j)_{j\neq i}$); moreover, these are common belief among all players. Claim 1 below, together with Lemma 5, then implies that $S_i^\infty[t_{i,s_i}] = \{s_i\}$. Since $S_i^\infty[\cdot]$ depends only on the belief hierarchy of a type, there is a finite type $t_i \in T_i^*$ such that $S_i^\infty[t_i] = \{s_i\}$.

The key idea to prove the claim is to make all information sets be reached with positive probability. The proof proceeds by induction on the lengths of histories: the initial step follows because the initial history is trivially reached and t_{i,s_i} assigns probability one to θ_{s_i}. For the induction step, let s_i' be a pure strategy and h be some history in $H(s_i) \cap H_i$ with length $m+1$. First, if $h \notin H(s_i')$, the induction hypothesis immediately implies that $s_i' \notin S_i^\infty[t_{i,s_i}]$. Second, suppose that $h \in H(s_i')$ but $s_i'(h) \neq s_i(h)$. Since t_{i,s_i} assigns positive probability to every $t_{-i,s_{-i}}$, perfect recall and the induction hypothesis then ensure that $I_i(h)$ is reached with positive probability under any valid belief π of t_{i,s_i}. Moreover, since t_{i,s_i} assigns probability one to θ_{s_i}, s_i generates a strictly higher expected utility than s_i' under π, and thus $s_i' \notin S_i^\infty[t_{i,s_i}]$.

Claim 1 *Let $s_i \in S_i$. Then, $s_i' \in S_i^\infty[t_{i,s_i}]$ implies that for any $h \in H(s_i) \cap H_i$, $h \in H(s_i')$ and $s_i'(h) = s_i(h)$.*

Proof. We prove this claim by induction. First, consider player i who moves at the initial history h. Since the initial history is a subhistory of any history and t_{i,s_i} assigns probability one to θ_{s_i}, then for any pure strategy s_i' with $s_i'(h) \neq s_i(h)$, $U_i(\theta_{s_i}, z(s_i, s_{-i})) > U_i(\theta_{s_i}, z(s_i', s_{-i}))$ for any $s_{-i} \in S_{-i}$. Hence, $s_i' \in S_i^\infty[t_{i,s_i}]$ only if $s_i'(h) = s_i(h)$.

Now suppose that our claim holds for every player j and every history with length less than or equal to m where $m \geq 0$; we then have the following:

$$\begin{aligned} s_j' &\in S_j^\infty\left[t_{j,s_j}\right] \Rightarrow \\ \text{for any } h' &\in H(s_i) \cap H_i \text{ with length} \leq m,\ h' \in H(s_i') \text{ and } s_j'(h') = s_j(h'). \end{aligned} \tag{IH}$$

Fix a history $h \in H(s_i) \cap H_i$ with length $m+1$. We now show that under (IH) any $s_i' \in S_i$ cannot be rationalizable for t_{i,s_i}, either in the case where $h \notin H(s_i')$ or in the case where $h \in H(s_i')$ and $s_i'(h) \neq s_i(h)$. First, if $h \notin H(s_i')$, then $s_i'(h') \neq s_i(h')$ for a proper subhistory $h' \in H(s_i) \cap H_i$ of h. Since the length of h' is smaller than m, it follows from the induction hypothesis that $s_i' \notin S_i^\infty[t_{i,s_i}]$. Second, if $h \in H(s_i')$ and $s_i'(h) \neq s_i(h)$, we show that $s_i' \notin S_i^\infty[t_{i,s_i}]$ by proving that for any $\pi \in \Delta(\Theta \times T_{-i} \times S_{-i})$ which is valid for t_{i,s_i},

$$\sum_{(\theta,t_{-i},s_{-i}) \in \Theta \times T_{-i} \times S_{-i}} \left[u_i(\theta, s_i, s_{-i}) - u_i(\theta, s_i', s_{-i})\right] \pi(\{(\theta, t_{-i}, s_{-i})\}) > 0. \tag{$\blacklozenge$}$$

We divide the proof of ($\blacklozenge$) into two steps:

Step 1 *Under (IH), for any $\pi \in \Delta(\Theta \times T_{-i} \times S_{-i})$ which is valid for t_{i,s_i}, $\pi(\{s_{-i}\}) > 0$ for some action $s_{-i} \in S_{-i}$ such that h is a subhistory of $z(s_i, s_{-i})$.*

Recall that for any history h', $s_{-i} = (s_j)_{j \neq i} \in S_{-i}(h')$ if and only if $h' \in H(s_j)$ for all $j \neq i$. First, $S_{-i}(h) \neq \varnothing$ because perfect recall ensures that not more than one subhistory of h can be contained in the same information set of a player. Let $s'_{-i} = (s_j)_{j \neq i} \in S_{-i}(h)$. Second, by (IH), $s''_j \in S_j^{\infty}\left[t_{j,s'_j}\right]$ implies that $h \in H(s''_i)$ and $s''_j(h') = s'_j(h')$, for any $h' \in H_j(s'_j) \cap H_j$ which is a proper subhistory of h (and hence is of length smaller than m). Then, since $\kappa_{t_{i,s_i}}\left(\left\{t_{-i,s'_{-i}}\right\}\right) > 0$ and π is valid for t_{i,s_i}, it follows that $\pi(\{s_{-i}\}) > 0$ for some $s_{-i} \in S_{-i}(h)$. Finally, since $s_{-i} \in S_{-i}(h)$ and $h \in H(s_i)$, it follows that $h \in H(s_j)$ for all $j \in N$. Therefore, h is a subhistory of $z(s_i, s_{-i})$.

Step 2 *Under (IH), any $s'_i \in S_i$ with $s'_i(h) \neq s_i(h)$ is not rationalizable for t_{i,s_i}.*

Suppose that $\pi \in \Delta(\Theta \times T_{-i} \times S_{-i})$ is valid for t_{i,s_i}. Since t_{i,s_i} assigns probability one to θ_{s_i}, we can replace θ by θ_{s_i} in the summand of (♦). Let $s_{-i} \in S_{-i}$. There are then two cases: (i) If $s'_i(h') \neq s_i(h')$ for some subhistory $h' \in H_i$ of $z(s_i, s_{-i})$, then $u_i(\theta_{s_i}, s_i, s_{-i}) > u_i(\theta_{s_i}, s'_i, s_{-i})$ by the definition of θ_{s_i}. (ii) If $s'_i(h') = s_i(h')$ for any subhistory $h' \in H_i$ of $z(s_i, s_{-i})$, then $z(s_i, s_{-i}) = z(s'_i, s_{-i})$ and hence $u_i(\theta_{s_i}, s_i, s_{-i}) = u_i(\theta_{s_i}, s'_i, s_{-i})$. By step 1, case (i) happens with positive probability under π. Therefore, (♦) holds and s'_i is not rationalizable for t_{i,s_i}.■

We conclude the section by elaborating the differences between the arguments in FKL and DF (see FKL's Theorem 3 and DF's Proposition 4.2) and the proof of Lemma 6. Both FKL and DF start with a complete-information game, that is, a game in which payoffs are common knowledge. A player with the information and payoffs in this complete-information game is called a "sane type." A perturbed game is an incomplete-information game, where with small probability, a player becomes a "crazy type" whose payoffs and information differ drastically from her sane type. FKL show, for instance, that every Nash equilibrium profile in the complete-information game can be approximated by some strict Nash equilibrium profile in the perturbed game.

To facilitate comparison, types with multiple rationalizable actions can be viewed here as sane types, whereas types with unique rationalizable actions can be viewed as crazy types.

Our proof of Lemma 6 shares the feature of FKL's and DF's proofs that an incomplete-information game with crazy types is constructed such that every information set is reached with positive probability. However, there is no sane type in the model constructed in ($\bigstar$). This is because Lemma 6 is only one step toward the structure theorem. Sane types appear when we combine Lemma 6 and Theorem 1 to prove that crazy types are dense.

For instance, consider for simplicity's sake a rationalizable strategy s_i of a finite type t_i. In proving Theorem 1, we first use Lemma 1 to make s_i belong to $V_i\left[t_{i,0}^{s_i}\right]$ for some finite type $t_{i,0}^{s_i}$ which is close to t_i and some profile of correspondences $(V_j)_{j\in N}$ which satisfies the strict best reply property. The crazy types constructed in Lemma 6 are then used to start the following "infection." First, the belief π under which s_i is a strict best reply for $t_{i,0}^{s_i}$, taken together with the mapping $(\theta, t_{-i}, s_{-i}) \mapsto \left(\theta, t_{-i,s_{-i}}\right)$, identifies a crazy type $t_{i,1}^{s_i}$ who has the same first-order belief as $t_{i,0}^{s_i}$ and who plays s_i. Second, the belief π and the mapping $(\theta, t_{-i}, s_{-i}) \mapsto \left(\theta, t_{-i,1}^{s_{-i}}\right)$ identify another crazy type $t_{i,2}^{s_i}$ who has the same second-order belief as $t_{i,0}^{s_i}$ and plays s_i, and so on. In this way we produce more and more crazy types, eventually approaching every sane type. The perturbed information structure is defined recursively. Thus, this construction is close to WY but differs from FKL and DF.

Another difference is that in proving their results, FKL (and, similarly, DF) aim to show that some profile constitutes a strict Nash equilibrium, whereas in Lemma 6 we show that the strategy s_i is uniquely rationalizable for the crazy type t_{i,s_i}. In particular, the Bayesian game strategy profile $(\sigma_i)_{i\in N}$ with $\sigma_i\left[t_{i,s_i}\right] = s_i$ clearly constitutes a strict Nash equilibrium in the model defined in ($\bigstar$): given σ_{-i}, every h in $H(s_i)$ occurs with positive probability and hence s_i does strictly better than any other s_i' for t_{i,s_i}. Here, without fixing the strategy of t_{j,s_j} being s_j, we can know that every h in $H(s_i)$ occurs with positive probability only by proving inductively that a rationalizable strategy for t_{i,s_i} must coincide with s_i along histories in $H(s_i) \cap H_i$.

4 Equivalence between ICR and ISR

Penta (2011) considers a finite multi-stage game with observable actions. Specifically, in a multi-stage game with observable actions, a typical history (up to player i's move at stage

L) is of the form

$$h_{L,i} = (a_{1,1}, ..., a_{1,n}, ..., a_{L-1,1}, ..., a_{L-1,n}, a_{L,1}, ..., a_{L,i}),$$

where $(a_{l,1}, ..., a_{l,n})$ is the action profile taken at stage l for $1 \leq l \leq L-1$ and $(a_{L,1}, ..., a_{L,i})$ are the actions taken at stage L up to player i; note that $h_{1,i} = (a_{1,1}, ..., a_{1,i})$. Since the game is finite, there is a natural number L^* such that $h_{L^*,n}$ represents a typical terminal history. Within each stage, players move in order, observing all the actions taken in previous stages but none of the actions taken in the current stage.[14] Hence, moves within a stage can be viewed as simultaneous. Let $\left(\Gamma, (U_i)_{i\in N}\right)$ be such a multi-stage game with observable actions and $G^{\Gamma,U}$ be the reduced normal-form representation of $\left(\Gamma, (U_i)_{i\in N}\right)$.

Let $(\Theta \times T, \kappa)$ be a model. We call a history $h = h_{L,n}$ for some L a *stage history*. A conditional probability system π on $\Theta \times T_{-i} \times S_{-i}$ specifies, for any stage history h, some $\pi(\cdot|h) \in \Delta(\Theta \times T_{-i} \times S_{-i}(h))$ such that for any measurable set $E \subseteq \Theta \times T_{-i} \times S_{-i}(h)$ and stage histories h and h' with $S_{-i}(h) \subseteq S_{-i}(h')$, the following equation holds:

$$\pi(E|h') = \pi(E|h)\,\pi(\Theta \times T_{-i} \times S_{-i}(h)\,|h').$$

This latter requirement amounts to saying that π is consistent with Bayes' rule whenever possible. Let s_i belonging to S_i be a *sequential best reply* to a conditional probability system π on $\Theta \times T_{-i} \times S_{-i}$ if and only if for any stage history $h \in H(s_i)$, the following condition holds:

$$s_i \in \arg\max_{s_i' \in S_i(h)} \int_{\Theta\times T_{-i}\times S_{-i}(h)} u_i(s_i', s_{-i}, \theta)\, d\pi((\theta, t_{-i}, s_{-i})\,|h).$$

Penta (2011) defines the solution concept of Interim Sequential Rationalizability (ISR) as follows. For each player i and type $t_i \in T_i$, set $ISR_i^0[t_i] = S_i$; define sets $ISR_i^k[t_i]$ for $k > 0$ iteratively by letting action $s_i \in ISR_i^k[t_i]$ if and only if s_i is a sequential best reply to a conditional probability system π on $\Theta \times T_{-i} \times S_{-i}$ such that $\mathrm{marg}_{\Theta\times T_{-i}}\pi(\cdot|\phi) = \kappa_{t_i}$ and $\pi\left(\left\{(\theta, t_{-i}, s_{-i}) : s_{-i} \in ISR_{-i}^{k-1}[t_{-i}]\right\}|\phi\right) = 1$ where ϕ is the initial history and $ISR_{-i}^{k-1}[t_{-i}] = \Pi_{j\neq i} ISR_j^{k-1}[t_j]$. Then, define $ISR_i^\infty[t_i] = \bigcap_{k=0}^{\infty} ISR_i^k[t_i]$.

As we mentioned in the introduction, in the models studied here which assume away any knowledge of payoffs, ISR allows players to revise beliefs about their own payoffs upon

[14] That is, $\varnothing \in H_1$, and moreover, if $i < n$ and $L \leq L^*$, then $h_{L,i} \in H_{i+1}$ and $I_{i+1}(h_{L,i})$ contains all $h'_{L,i} \in H_{i+1}$ with the same actions taken from stage 1 to stage $L-1$ as $h_{L,i}$; if $i = n$ and $L < L^*$, then $h_{L,i} \in H_1$ and $I_1(h_{L,i})$ contains all $h'_{L,i} \in H_1$ with the same actions taken from stage 1 to stage L as $h_{L,i}$.

being surprised. More precisely, the only restriction that ISR puts on the beliefs held at zero-probability histories comes from the definition of a model. This means that once surprised, type t_i may assign positive probability to pairs (θ, t_{-i}) that were initially given zero probability by the belief κ_{t_i}. Under EF-richness, this makes every subsequent move "credible" and hence sequential rationality has no bite. The following result formalizes the idea.

Theorem 3 *If the extensive-form game $\left(\Gamma, (U_i)_{i \in N}\right)$ satisfies EF-richness, then $ISR_i^\infty [t_i] = S_i^\infty [t_i]$ for all $t_i \in T_i^*$.*

Theorem 3 is a direct consequence of Theorem 1 and the following proposition due to Penta (2011). See Appendix A.5 for a proof.

Proposition 1 (Penta (2011)) *$ISR_i^\infty [\cdot]$ is nonempty and upper hemicontinuous.*

5 Concluding remarks

We conclude with some remarks on other related works. First, Weinstein and Yildiz (2010) have further extended our result to prove a structure theorem for ICR in the normal form of infinite-horizon games. Weinstein and Yildiz apply this structure theorem for infinite-horizon games to study important applications, such as the Rubinstein bargaining model and repeated games. Second, Oury and Tercieux (2011) assume a weaker richness condition and use a version of WY's argument in their study of robust implementation.

Two other recent papers are closely related to Weinstein and Yildiz (2007). Neither assumes any richness condition and neither deals with dynamic games. First, Weinstein and Yildiz (2011) characterize, for any fixed finite order, the set of actions that can be played in a Bayesian Nash equilibrium by some type whose lower-order beliefs are all as in the original type in nice games (where action spaces are compact intervals, and utilities are continuous and strictly concave in own action). Second, in a fixed game, the rationalizable correspondence is continuous in the product topology around types with unique rationalizable actions. Ely and Pęski (2011) generalize this observation to define a type to be regular if it exhibits a similar notion of continuity in *every* finite game. They show that regular types are generic in the universal type space.

A Appendix

A.1 Compact metric Θ

Here we sketch how to extend our result when Θ is a compact metric space, as assumed in WY. This includes the case where $\Theta = \Theta_1 \times \cdots \times \Theta_n$ and $\Theta_i = [0,1]^Z$. Our arguments for the "only if" part of Theorem 1 (which is the direction we use in proving Theorem 2) still go through if we strengthen RURA as follows (see the Online Appendix for more details).

RURA$'$ *There is a finite set of payoff parameters $\overline{\Theta} \subset \Theta$ such that for each player i and each action $s_i \in \cup_{t'_i \in T^*_i} S^\infty_i [t'_i]$, there exists a finite type $t_i \in T^*_i$ such that $S^\infty_i [t_i] = \{s_i\}$ and $\kappa_{t_i} \left[\overline{\Theta}\right] = 1$.*

We then follow WY in assuming that every utility function u_i is continuous; note that a finite model is now a model $(\Theta' \times T, \kappa)$ such that $\Theta' \subset \Theta$ and $|\Theta' \times T| < \infty$. To see that EF-richness implies RURA$'$, let $\overline{\Theta} = \{\theta_{s_i} : s_i \in S_i,\ i \in N\}$ and apply the proof of Lemma 6.

A.2 Proof of Lemma 3

In this proof, we require only that Θ be a compact metric space equipped with metric d^0. Let $j \in \{1, 2, ..., n\}$ denote a generic player. Recall that the universal type space T^*_j endowed with the product topology is a compact metrizable space. The compatible metric d_j on T^*_j used in the proof is the one obtained from the Prohorov distance between beliefs of the same order.[15] Specifically, for any $t_j, t'_j \in T^*_j$, let $d^1_j \left(t^1_j, t'^1_j\right)$ be the Prohorov distance between t^1_j and t'^1_j (recall that $t^1_j, t'^1_j \in \Delta(\Theta)$). Recursively, for any integer $k \geq 2$, and $t_j, t'_j \in T^*_j$, let $d^k_j \left(t^k_j, t'^k_j\right)$ be the Prohorov distance between t^k_j and t'^k_j, where $t^k_j, t'^k_j \in \Delta\left(\Theta \times T^{k-1}_{-j}\right)$; here T^{k-1}_{-j} is the space

[15] Let Y be an arbitrary compact metric space endowed with metric ρ and the Borel σ-algebra. For any two $\mu, \mu' \in \Delta(Y)$, the Prohorov distance between μ and μ' is defined as

$$d(\mu, \mu') = \inf\{\varepsilon > 0 : \mu(E) \leq \mu'(E^\varepsilon) + \varepsilon \text{ for all Borel set } E \subseteq Y\},$$

where $A^\varepsilon \equiv \{y \in Y : \inf_{y' \in E} \rho(y, y') < \varepsilon\}$. It is known that the Prohorov metric metrizes the weak*-topology on $\Delta(Y)$ (see 11.3.3. Theorem in Dudley (2002)).

of all $(k-1)^{th}$-order beliefs of player j's opponents and $\Theta \times T_{-j}^{k-1}$ is equipped with the metric ρ_{-j}^{k-1} defined as $\rho_{-j}^{k-1}\left(\left(\theta, t_{-j}^{k-1}\right), \left(\theta', t_{-j}'^{k-1}\right)\right) \equiv \max\left(d^0\left(\theta, \theta'\right), \max_{j' \neq j} d_{j'}^{k-1}\left(t_{j'}^{k-1}, t_{j'}'^{k-1}\right)\right)$. Let $d_j\left(t_j, t_j'\right) \equiv \sum_{k=1}^{\infty} 2^{-k} d_j^k\left(t_j^k, t_j'^k\right)$, in other words, d_j is the product metric which metrizes the product topology on T_j^*.

Lemma 3 *For any type $\bar{t}_i \in T_i^*$, there is a sequence of finite types $\left(t_i\left(m\right)\right)_{m=1}^{\infty}$ such that $S_i^{\infty}\left[t_i\left(m\right)\right] = S_i^{\infty}\left[\bar{t}_i\right]$ for all m and $\lim_{m \to \infty} t_i\left(m\right) = \bar{t}_i$.*

Proof. We construct the sequence of finite types as follows. Since T_j^* is a compact metric space, then for each natural number m, T_j^* can be covered by finitely many open balls with radius $1/2m$. Let $\mathcal{T}_{j,m}$ be the finite measurable partition of T_j^* induced from these open balls; thus, for any $T_j \in \mathcal{T}_{j,m}$, and t_j and t_j' in T_j, it follows that $d_j\left(t_j, t_j'\right) < 1/m$. Second, let $\mathcal{T}_{j,0}$ be the finite measurable partition induced by rationalizable sets, i.e., for any $T_j \in \mathcal{T}_{j,0}$, $t_j, t_j' \in T_j$ if and only if $S_j^{\infty}\left[t_j\right] = S_j^{\infty}\left[t_j'\right]$.[16]

We refine the partition $\mathcal{T}_{j,m}$ by "joining" $\mathcal{T}_{j,m}$ with $\mathcal{T}_{j,0}$. Let $\widetilde{\mathcal{T}}_{j,m}$ denote the join (the coarsest common refinement) of $\mathcal{T}_{j,0}$ and $\mathcal{T}_{j,m}$. Let $f_{j,m} : T_j^* \to \widetilde{\mathcal{T}}_{j,m}$ be the mapping such that $f_{j,m}\left(t_j\right) = \widetilde{t}_{j,m}$ if and only if $t_j \in \widetilde{t}_{j,m}$. Moreover, for each $\widetilde{t}_{j,m} \in \widetilde{\mathcal{T}}_{j,m}$, select arbitrarily a type $t_{j,m} \in \widetilde{t}_{j,m}$. Define a sequence of finite models $\left(\left(\Theta \times \widetilde{T}^m, \widetilde{\kappa}^m\right)\right)_{m=1}^{\infty}$ by letting $\widetilde{T}_j^m \equiv \widetilde{\mathcal{T}}_{j,m}$, and for each $\widetilde{t}_{j,m} \in \widetilde{T}_j^m$,

$$\widetilde{\kappa}_{\widetilde{t}_{j,m}}^m\left[\left(\theta, \widetilde{t}_{-j,m}\right)\right] \equiv \kappa_{t_{j,m}}^*\left[\left\{\left(\theta, t_{-j}\right) : f_{-j,m}\left(t_{-j}\right) = \widetilde{t}_{-j,m}\right\}\right], \forall \left(\theta, \widetilde{t}_{-j,m}\right) \in \Theta \times \widetilde{T}_{-j}^m.$$

Note that $\widetilde{t}_{j,m}$ denotes both a subset of T_j^* and a type in the model $\left(\Theta \times \widetilde{T}^m, \widetilde{\kappa}^m\right)$.

Finally, let $\bar{t}_i\left(m\right) \equiv f_{i,m}\left(\bar{t}_i\right)$ for every m. First, $\lim_{m \to \infty} \sup_{t_j \in T_j^*} d_j\left(f_{j,m}\left(t_j\right), t_j\right) = 0$ (the idea is similar to Mertens and Zamir (1985, pp. 15-19). In particular, $\lim_{m \to \infty} \bar{t}_i\left(m\right) = \bar{t}_i$. Second, we can also show that $S_j^{\infty}\left[f_{j,m}\left(t_j\right)\right] \supseteq S_j^{\infty}\left[t_j\right]$ for all $t_j \in T_j^*$ and for all m. In particular, $S_i^{\infty}\left[\bar{t}_i\left(m\right)\right] \supseteq S_i^{\infty}\left[\bar{t}_i\right]$ for all m.[17] Since $S_i^{\infty}\left[\cdot\right]$ is upper hemicontinuous and $\lim_{m \to \infty} \bar{t}_i\left(m\right) = \bar{t}_i$, it follows that $S_i^{\infty}\left[\bar{t}_i\left(m\right)\right] = S_i^{\infty}\left[\bar{t}_i\right]$ for sufficiently large m, say, $m \geq \overline{m}$.

[16] Measurability follows from the upper hemicontinuity of $S_j^{\infty}\left[\cdot\right]$: If $S_j' \subseteq S_j$ is $1-$minimal in the sense that there is no type t_j with $S_j^{\infty}\left[t_j\right] \subsetneq S_j'$, then upper hemicontinuity implies that $\left\{t_j : S_j^{\infty}\left[t_j\right] = S_j'\right\} = \left\{t_j : S_j^{\infty}\left[t_j\right] \subseteq S_j'\right\}$ is open and hence measurable. If $S_j' \subseteq S_j$ is $2-$minimal in the sense that $S_j^{\infty}\left[t_j\right] \subsetneq S_i'$ if and only if $S_j^{\infty}\left[t_j\right]$ is $1-$minimal, then $\left\{t_j : S_j^{\infty}\left[t_j\right] = S_j'\right\} = \left\{t_j : S_j^{\infty}\left[t_j\right] \subseteq S_j'\right\} \setminus \left\{t_j : S_j^{\infty}\left[t_j\right] \text{ is } 1-\text{minimal}\right\}$ is measurable, and so on. Since S_j is a finite set, every $S_j^{\infty}\left[t_j\right]$ is $k-$minimal for some k and thus $\left\{t_j : S_j^{\infty}\left[t_j\right] = S_j'\right\}$ is measurable for all $S_j' \subseteq S_j$.

[17] We present a detailed proof of these two steps in the Online Appendix linked to the author's website.

We then define $t_i(m) = \bar{t}_i(\overline{m} + m)$ for all m and $(t_i(m))_{m=1}^{\infty}$ is the desired sequence.■

A.3 Proof of Theorem 1

Theorem 1 *RURA holds if and only if for any type $t_i \in T_i^*$ and for any action $s_i \in S_i^{\infty}[t_i]$, there is a sequence of types $(t_i(m))_{m=1}^{\infty}$ such that $\lim_{m\to\infty} t_i(m) = t_i$ and $S_i^{\infty}[t_i(m)] = \{s_i\}$ for all m.*

Proof. The "if" part is immediate. We now prove the "only if" part. Suppose RURA holds. We first prove the result for the case where t_i is a finite type. Let t_i be a finite type and $s_i \in S_i^{\infty}[t_i]$. By Lemma 1, there is a a sequence of finite models $((\Theta \times T^m, \kappa^m))_{m=1}^{\infty}$ and a sequence of finite types $(t_i(m))_{m=1}^{\infty}$ such that $t_i(m) \in T_i^m$ and $s_i \in V_i^m[t_i(m)]$ for some profile of correspondences $\left(V_j^m\right)_{j\in N}$ with $V_j^m : T_j^m \rightrightarrows S_j$ which satisfies the strict best reply property, for all m, and $\lim_{m\to\infty} t_i(m) = t_i$. Moreover, by Lemma 2, for each m, there is a sequence of types $(t_i(m,k))_{k=1}^{\infty}$ such that $\lim_{k\to\infty} t_i(m,k) = t_i(m)$ and $\{s_i\} = S_i^{\infty}[t_i(m,k)]$ for every k. Since T_i^* is a metric space, there is some sequence of types $(t_i(m,k_m))_{m=1}^{\infty}$ such that $\lim_{m\to\infty} t_i(m,k_m) = t_i$ and $S_i^{\infty}[t_i(m,k_m)] = \{s_i\}$.

Now consider the case where t_i is an infinite type. Let $s_i \in S_i^{\infty}[t_i]$. By Lemma 3, there is a sequence of finite types $(t_i(m))_{m=1}^{\infty}$ such that $s_i \in S_i^{\infty}[t_i(m)]$ for every m and $\lim_{m\to\infty} t_i(m) = t_i$. For each m, since $t_i(m)$ is a finite type, then by our previous step there is a sequence of types $(t_i(m,k))_{k=1}^{\infty}$ such that $S_i^{\infty}[t_i(m,k)] = \{s_i\}$ and $\lim_{k\to\infty} t_i(m,k) = t_i(m)$. Since T_i^* is a metric space, it follows that there is some sequence of types $(t_i(m,k_m))_{m=1}^{\infty}$ such that $S_i^{\infty}[t_i(m,k_m)] = \{s_i\}$ and $\lim_{m\to\infty} t_i(m,k_m) = t_i$.■

A.4 Proof of Lemma 4

Lemma 4 *There are some set of payoff parameters Θ and utility functions $(U_i)_{i\in N}$ with $U_i : \Theta \times Z \to \mathbb{R}$ such that $\left(\Gamma, (U_i)_{i\in N}\right)$ satisfies EF-richness.*

Proof. Let $\Theta = \{\theta_{s_i} : s_i \in S_i, i \in N\}$. For any histories h' and h, we write $h' \preceq h$ to mean that h' is a subhistory of h. Then, define a function $\alpha : H \times S_i \to \mathbb{R}$ as follows:

$$\alpha(h, s_i) = |\{h' \in H(s_i) : h' \preceq h\}|, \forall h \in H, \forall s_i \in S_i.$$

where $|\cdot|$ denotes the cardinality of "$\cdot$." Define the utility function U_i as follows:

$$U_i\left(\theta_{s_i}, z\right) \equiv \alpha\left(z, s_i\right), \forall z \in Z,$$

and for any $j \neq i$ and $s_i \in S_i$, define $U_j\left(\theta_{s_i}, \cdot\right)$ arbitrarily. We now show that $\left(\Gamma, \left(U_i\right)_{i \in N}\right)$ satisfies EF-richness.

Let $s_i, s_i' \in S_i$ and $s_{-i} \in S_{-i}$. Suppose that $s_i'(h) \neq s_i(h)$ for some subhistory $h \in H_i$ of $z\left(s_i, s_{-i}\right)$. We show that $\alpha\left(z\left(s_i, s_{-i}\right), s_i\right) \geq \alpha\left(z\left(s_i', s_{-i}\right), s_i\right) + 1$. First, since $s_i'(h) \neq s_i(h)$, for any history h' with $\left(h, s_i'(h)\right) \preceq h' \preceq z\left(s_i', s_{-i}\right)$, it follows that $h' \notin H\left(s_i\right)$ and hence:

$$\alpha\left(z\left(s_i', s_{-i}\right), s_i\right) = \alpha\left(h, s_i\right). \tag{1}$$

Second, observe that $\left(h, s_i(h)\right) \in H\left(s_i\right)$. Then, since $\left(h, s_i(h)\right)$ is a subhistory of $z\left(s_i, s_{-i}\right)$ and is not a subhistory of h, it follows that

$$\alpha\left(z\left(s_i, s_{-i}\right), s_i\right) \geq \alpha\left(h, s_i\right) + 1. \tag{2}$$

By (1) and (2), we conclude that $\alpha\left(z\left(s_i, s_{-i}\right), s_i\right) \geq \alpha\left(z\left(s_i', s_{-i}\right), s_i\right) + 1$.■

A.5 Proof of Theorem 3

Theorem 3 *If the extensive-form game $\left(\Gamma, \left(U_i\right)_{i \in N}\right)$ satisfies EF-richness, then $ISR_i^{\infty}\left[t_i\right] = S_i^{\infty}\left[t_i\right]$ for all $t_i \in T_i^*$.*

Proof. Clearly, if s_i is a sequential best reply to a conditional probability system π on $\Theta \times T_{-i}^* \times S_{-i}$, then $\mathrm{marg}_{\Theta \times T_{-i}^*}\pi\left(\cdot|\phi\right) = \kappa_{t_i}$ $s_i \in BR_i\left(\mathrm{marg}_{\Theta \times S_{-i}}\pi\left(\cdot|\phi\right)\right)$. Hence, $ISR_i^k\left[t_i\right] \subseteq S_i^k\left[t_i\right]$ for every $k \geq 0$, and therefore $ISR_i^{\infty}\left[t_i\right] \subseteq S_i^{\infty}\left[t_i\right]$ for every $t_i \in T_i^*$. Conversely, suppose that $t_i \in T_i^*$ and $s_i \in S_i^{\infty}\left[t_i\right]$. We show that $s_i \in ISR_i^{\infty}\left[t_i\right]$. By Theorem 2, there is a sequence of types $\left(t_i(m)\right)_{m=1}^{\infty}$ such that $\lim_{m \to \infty} t_i(m) = t_i$ and $\left\{s_i\right\} = S_i^{\infty}\left[t_i(m)\right]$ for every m. Since $\varnothing \neq ISR_i^{\infty}\left[t_i\right] \subseteq S_i^{\infty}\left[t_i\right]$ for every $t_i \in T_i^*$, it follows that $\left\{s_i\right\} = ISR_i^{\infty}\left[t_i(m)\right]$ for every m. Since $ISR_i^{\infty}\left[\cdot\right]$ is upper hemicontinuous on T_i^*, we have $s_i \in ISR_i^{\infty}\left[t_i\right]$. Thus, $ISR_i^{\infty}\left[t_i\right] = S_i^{\infty}\left[t_i\right]$ for all $t_i \in T_i^*$.■

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