Robust Refinement of Rationalizability*

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Abstract

Following Fudenberg, Kreps, and Levine (1988) and Dekel and Fudenberg (1990), we say that a refinement of a solution concept is robust if its outcome correspondence satisfies the closed-graph property. Specifically, we characterize these robust refinements of rationalizability with respect to perturbation of higher-order beliefs of the players. We demonstrate how the characterization pins down a novel robust refinement of rationalizability in arbitrary finite games as well as in specific examples such as first-price auctions and the Cournot competition. We also apply our characterization to study the critique raised by Weinstein and Yildiz (2007b) to the widely-adopted global-game equilibrium refinement approach. Without imposing any payoff richness condition, we provide a necessary and sufficient condition under which the Weinstein-Yildiz critique remains valid.

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1 Introduction

Game-theoretic models in economics typically have multiple equilibria and a large set of rationalizable actions. An important research agenda is thus to refine the large set of outcomes in order to make sharp predictions. Starting from the seminal paper by Carlsson and Van Damme (1993), the global game literature in particular focuses on identifying such refinements via perturbing higher-order beliefs. As such a refinement is typically obtained from one particular perturbation, it leaves open the question as to how we can make refinements which are robust with respect to *all* perturbations.

Weinstein and Yildiz (2007b) (hereafter, WY) prove a striking result which casts doubt on the global-game refinement methodology. In particular, WY prove what they call a *structure theorem* (for rationalizability): any rationalizable action of any type can be selected as the unique outcome via perturbation of higher-order beliefs. The structure theorem implies that for a given type, the only robust refinement is the trivial one. Combined with the upper hemicontinuity of the rationalizability correspondence (Dekel, Fudenberg, and Morris (2006)), the structure theorem also implies *generic uniqueness*, namely that the set of types with uniquely rationalizable actions is generic (i.e., open and dense) in the universal type space. In other words, it is generically unnecessary to refine rationalizable outcomes.

It is crucial to note that the results of WY rely on a "richness" assumption, namely that every action is strictly dominant for some payoff parameter. As WY observe, this assumption holds—in simultaneous-move games— if there is no common knowledge restriction on payoffs. Nevertheless, this condition entails an unnecessarily demanding robustness test once some common knowledge payoff restrictions need to be maintained.[1] Such common knowledge payoff restriction may arise from a game tree (Chen, 2012; Penta, 2012), an auction model in which no bidder has a strictly dominant bid (Penta (2013)), or a particular cost structure in oligopolistic competition (Weinstein and Yildiz, 2011).[2] To the extent that these structural restrictions are kept in the robustness analysis, the richness assumption inevitably restricts

[1]This distinction is reminiscent of the well known Wilson's doctrine, "Game theory has a great advantage in explicitly analyzing the consequences of trading rules that presumably are really common knowledge; it is deficient to the extent it assumes other features to be common knowledge, such as one player's probability assessment about another's preferences or information."

[2]We will scrutinize these examples in Section 5.

the scope of the WY critique.[3] Relaxing the richness assumption invites the prominent question of how we can robustly refine rationalizable outcomes.

This paper proposes a new approach to characterize robust refinements of rationalizability without presupposing or precluding any structure on the payoff uncertainty. Formally, a refinement identifies for each type of a player a subset of rationalizable actions (see Definition 3 in Section 3). Following the idea of Fudenberg, Kreps, and Levine (1988) and Dekel and Fudenberg (1990), we say that a refinement is robust for type t_i if any action that is ruled out by the refinement for t_i is also ruled out by the refinement at any type sufficiently close to t_i. In other words, the refinement, viewed as a correspondence defined on the set of types of player i, is upper hemicontinuous at t_i. In this vein, WY's structure theorem implies that no (nontrivial) refinement of rationalizability is robust under their richness condition.

We aim to fully characterize robust refinement. We first observe that a set of actions can be prescribed by a robust refinement if and only if it overlaps with the rationalizable set of any type sufficiently close to t_i (Proposition 1). It follows that identifying robust refinements also amounts to identifying for each type a subset of actions which intersects the rationalizable actions of any nearby type.[4] By means of this equivalence result, we propose a novel two-step procedure to identify such subsets of actions:

1. We show that every finite game is endowed with an global-upper ICR collection denoted by $\mathcal{R}_i^{\uparrow}$ (resp. a global-lower ICR collection denoted by $\mathcal{R}_i^{\downarrow}$), which is the collection of all action sets that contain (resp. are contained in) the rationalizable action set for

[3] A similar observation has been made in the global game literature. In particular, in a global game with a one-sided dominance region, we may not be able to select some action as a unique prediction (Morris and Shin, 2000; Goldstein and Pauzner, 2005; Bueno de Mesquita, 2011; Shadmehr and Bernhardt, 2012). There are also papers which relax the richness assumption in addressing similar but different issues. For instance, Oury and Tercieux (2012) assume a weaker richness condition and use a version of WY's argument in their study of continuous implementation. Ely and Peski (2011) generalize generic uniqueness to the genericity of regular types (i.e., types which display strategic continuity) without imposing the richness assumption.

[4] Extending the notion of robust equilibrium in Kajii and Morris (1997) to a set-valued notion, Morris and Ui (2005) define a robust equilibrium set as a set of Nash equilibria of a complete-information game such that every nearby incomplete-information game has some Bayesian Nash equilibrium that approximates some action profile of the set. The notion of robust equilibrium set is analogous to the set of outcomes prescribed by a robust refinement, but the former focuses on equilibrium, while the latter on rationalizability.

some type;

2. Based on $\mathcal{R}_i^{\uparrow}$, we show that each finite type t_i is endowed with a local-upper ICR collection $\mathcal{S}_i(t_i)$, which is the collection of all action sets that contain the rationalizable actions for some sequence of types converging to t_i.[5]

We use the local-upper ICR collection to fully characterizes the robust refinement. Indeed, a refinement is robust for any finite type t_i if and only if it prescribes a set of actions which intersects every set in $\mathcal{S}_i(t_i)$ (Theorem 3). By invoking this characterization, we identify a solution concept ICR_W which is a robust refinement of rationalizability. In particular, ICR_W prescribes the set of action profiles which survive iterated deletion of weakly dominated actions followed by interim strictly dominated actions.[6] We demonstrate via an auction example that ICR_W can produce a unique robust refinement which refines those prescribed under rationalizability.

We also use the characterization to delineate the boundary of the WY critique on the global-game literature. In particular, we show that the structure theorem holds if and only if every rationalizable action is uniquely rationalizable for some type (Corollary 2). Moreover, we show that the generic uniqueness result holds if and only if any minimal set of rationalizable actions is a singleton (Corollary 3). Finally, we demonstrate how our conditions can be applied to (in)validate the structure theorem or generic uniqueness in an example of Cournot competition (see Section 5.3).[7]

Our two-step procedure also generalizes the idea in Penta (2013) which proposes a sufficient condition for an action to be played as the unique rationalizable action for some

[5] An extension to infinite types will be discussed in Section 6.

[6] ICR_W resembles the well known Dekel-Fudenberg procedure under complete information with one important difference: for ICR_W the iterative deletion of weakly dominated actions applies directly to the payoff function with no reference to the type space (such as the one with complete information). The interim/type space information matters only during the iterative deletion of interm strictly dominated actions.

[7] Instead of relaxing the richness assumption, Germano, Weinstein, and Zuazo-Garin (2020) examine to what extent the WY critique remains valid with departures from common belief in rationality. Specifically, they show that if players' confidence in mutual rationality persists at high orders, (under the richness assumption) the WY critique persists and no refinement of rationalizability is robust. In contrast, if their confidence vanishes at high orders, any subset of rationalizable actions of any type t_i is prescribed by some robust refinement for t_i.

perturbation of high-order beliefs. Specifically, Penta (2013) assumes that every player has some dominant actions, which generate uniquely rationalizable actions in the universal type space (corresponding to our Step 1). Based on such uniquely rationalizable actions, he proposes a condition for a rationalizable action of a type to be selected as the unique rationalizable action via a sequence of types (corresponding to our Step 2). Instead of assuming the existence of dominant actions, our Step 1 exploits the richness of possible higher-order beliefs to identify from primitives all actions that are uniquely rationalizable for some type. Our Step 2 also highlights the necessity of considering non-singleton rationalizable action sets in characterizing the selection via perturbation of higher-order beliefs (see Section 5.1). Moreover, since Penta (2013) only studies selection of an action as opposed to an action set, his result yields necessary but not sufficient condition for the existence of non-trivial robust refinements

The remainder of the paper proceeds as follows: Section 2 describes the preliminaries. Section 3 introduces the notions of robust refinement. Section 4 presents the main results. Section 4.4 defines the solution concept ICR_W and prove its robustness. Section 5 illustrates the results by means of economic examples. Section 6 extends our characterization to infinite types. Section 7 concludes.

2 Preliminaries

Fix a game $G = (A_i, u_i)_{i \in I}$, where each player $i \in I$ is endowed with a set of actions A_i and a payoff function u_i that depends on the action profile $a \in A := \prod_{i \in I} A_i$ and a payoff-relevant parameter $\theta \in \Theta$. Assume that I, A, and Θ are nonempty finite sets. While we will not impose any condition on G, we state here WY's richness condition for the ease of reference:

Definition 1 $G = (A_i, u_i)_{i \in I}$ *satisfies the richness condition if for every* $i \in N$ *and every* $a_i \in A_i$*, there exists* $\theta^{a_i} \in \Theta$ *such that* $u_i(\theta^{a_i}, a_i, a_{-i}) > u_i(\theta^{a_i}, a_i', a_{-i})$ *for every* $a_i' \in A_i \setminus \{a_i\}$ *and every* $a_{-i} \in A_{-i}$.

For any $\pi_i \in \Delta(\Theta \times A_{-i})$, we use $BR_i(\pi_i)$ to denote the set of best replies to π_i. That is,

$$BR_i(\pi_i) = \arg\max_{a_i \in A_i} \sum_{\theta, a_{-i}} u_i(\theta, a_i, a_{-i}) \pi_i[\theta, a_{-i}].$$

A *model* is a tuple (T, κ) where $T = \prod_{i \in I} T_i$ is a (metrizable) type space which associates a belief $\kappa_{t_i} \in \Delta(\Theta \times T_{-i})$ for each type $t_i \in T_i$.[8] Assume that $t_i \mapsto \kappa_{t_i}$ is a continuous mapping. Given a type $t_i \in T_i$, we can compute the first-order belief of t_i (i.e., his belief about Θ) by setting t_i^1 equal to the marginal distribution of κ_{t_i} on Θ. We can also compute the second-order belief of t_i (i.e., his belief about (θ, t_{-i}^1)) by setting

$$t_i^2[E] = \kappa_{t_i}\left[\left\{(\theta, t_{-i}) : \left(\theta, t_{-i}^1\right) \in E\right\}\right]$$

for every measurable set $E \subset \Theta \times (\Delta(\Theta))^{|I|-1}$. We can compute the entire hierarchy of beliefs $(t_i^1, t_i^2, \ldots, t_i^n, \ldots)$ by proceeding in this way. A model is said to be finite if $|T| < \infty$.

We collect all such hierarchies and construct the universal type space T_i^*. Endowed with the product topology, T_i^* is a compact metrizable space and admits a homeomorphism $\kappa_i^* \colon T_i^* \to \Delta(\Theta \times T_{-i}^*)$ (Mertens and Zamir, 1985). Thus, we can regard (T^*, κ^*) as a model, where $\kappa_{t_i}^* := \kappa_i^*(t_i)$ for every $t_i \in T_i^*$. Moreover, the hierarchy of beliefs induced by type $t_i \in T_i^*$ in the model (T^*, κ^*) is also t_i. A type $t_i \in T_i^*$ is said to be a *finite type* if t_i has the same hierarchy of belief as a type in a finite model.

Let (T, κ) be a model. We define the solution concept of interim correlated rationalizability (ICR) (Dekel, Fudenberg, and Morris, 2006, 2007) as follows. For $i \in I$ and type $t_i \in T_i$, set $ICR_i^0(t_i) = A_i$; define sets $ICR_i^n(t_i)$ for $n > 0$ iteratively such that $a_i \in ICR_i^n(t_i)$ if and only if there is some *conjecture* $\nu_i \in \Delta(\Theta \times T_{-i} \times A_{-i})$ which satisfies:

(i) $\mathrm{marg}_{\Theta \times T_{-i}} \nu_i = \kappa_{t_i}$;

(ii) $\nu_i\left[\left\{(\theta, t_{-i}, a_{-i}) : a_{-i} \in ICR_{-i}^{n-1}(t_{-i})\right\}\right] = 1$ (where $ICR_{-i}^{n-1}(t_{-i}) = \prod_{j \neq i} ICR_j^{n-1}(t_j)$);

(iii) $a_i \in BR_i\left(\mathrm{marg}_{\Theta \times A_{-i}} \nu_i\right)$.

[8] Throughout the paper, for any metrizable space Y, we use $\Delta(Y)$ to denote the space of probability measures on the Borel σ-algebra of Y. We endow $\Delta(Y)$ with the weak* topology. Moreover, we endow a product space with the product topology, a subspace with the relative topology, and a finite set with the discrete topology. Let $|E|$ denote the cardinality of a set E.

Then, define

$$ICR_i(t_i) = \bigcap_{n=0}^{\infty} ICR_i^n(t_i).$$

We write $ICR_{-i}(t_{-i}) = \prod_{j \neq i} ICR_j(t_j)$. Say conjecture $\nu_i \in \Delta(\Theta \times T_{-i} \times A_{-i})$ is *valid for* t_i if $\text{marg}_{\Theta \times T_{-i}} \nu_i = \kappa_{t_i}$ and $\nu_i[a_{-i} \in ICR_{-i}(t_{-i})] = 1$. Dekel, Fudenberg, and Morris (2007, Proposition 4) show that

$$ICR_i(t_i) = \bigcup_{\nu_i \text{ is a valid conjecture for } t_i} BR_i\left(\text{marg}_{\Theta \times A_{-i}} \nu_i\right); \tag{1}$$

moreover, $ICR_i(\cdot)$ only depends on the belief hierarchy of a type.

We will hereafter identify a type with its belief hierarchy. With a further abuse of notations, we say that a sequence of types $\{t_{i,m}\}_{m=0}^{\infty}$ on T_i^* converges to a type t_i in some (not necessarily the universal) model, denoted as $t_{i,m} \to t_i$, if for every n, $t_{i,m}^n$ converges to t_i^n in the weak* topology as $m \to \infty$. We reproduce Chen (2012, Lemma 3) here for the sake of later use. The lemma says that for any type t_i, we can find a sequence of finite types that approximate t_i in beliefs and have the same rationalizable behaviors as t_i.

Lemma 1 *For any type $t_i \in T_i^*$, there is a sequence of finite types $\{t_{i,m}\}_{m=0}^{\infty} \subset T_i^*$ such that $t_{i,m} \to t_i$ and $ICR_i(t_{i,m}) = ICR_i(t_i)$ for every m.*

Following WY, we say that an action can be selected for t_i if there is a sequence of types $\{t_{i,m}\}$ converging to t_i along which a_i is uniquely rationalizable.[9] Namely, a modeler who knows the belief of a type t_i of interest only approximately cannot preclude the possibility that a_i is a uniquely rationalizable action for some "true type" $t_{i,m}$.

Definition 2 *Given a model (T, κ), an action $a_i \in A_i$ is said to be selected for type $t_i \in T_i$ if there is a sequence of types $\{t_{i,m}\}_{m=0}^{\infty} \subset T_i^*$ such that $t_{i,m} \to t_i$ and $ICR_i(t_{i,m}) = \{a_i\}$ for every m.*

[9] Chen, Takahashi, and Xiong (2014) use the term "WY-selection" to differentiate it from another selection notion "robust selection".

3 Robust Refinement

A *refinement* is mapping $\psi_i : T_i^* \to 2^{A_i} \setminus \{\varnothing\}$ such that $\psi_i(t_i) \subset ICR_i(t_i)$ for every $t_i \in T_i^*$. We now define the notion of robust refinement.

Definition 3 *A refinement ψ_i is robust for type t_i if ψ_i is upper hemicontinuous at t_i.*

That is, we say that a refinement is robust for type t_i if there is a neighborhood of t_i in which the refinement never prescribes an outcome which it does not prescribe for t_i. The idea of formulating robustness of refinement as an upper hemicontinuity/closed-graph property dates back to Fudenberg, Kreps, and Levine (1988) and Dekel and Fudenberg (1990). For instance, Fudenberg, Kreps, and Levine (1988) show that by perturbing the payoffs/information of the players, any Nash equilibrium in a complete-information game is "near strict" in the sense of being approximated by a sequence of strict equilibria in a sequence of nearby incomplete-information games. As a result, any refinement of Nash equilibrium which is not a refinement of strict equilibrium is not robust. The upper hemicontinuity property is also relevant and has been applied to study the (local) robustness of full implementation (see Chung and Ely (2003) and Aghion, Fudenberg, Holden, Kunimoto, and Tercieux (2012)).

Since $ICR_i(\cdot)$ is an upper hemicontinuous correspondence, $ICR_i(\cdot)$ is trivially a robust refinement. The following proposition characterizes the set of actions which can be prescribed by a robust refinement for a type. The proof is in Appendix A.1.

Proposition 1 *A set of actions P_i is equal to $\psi_i(t_i)$ for some robust refinement ψ_i for type t_i if and only if there exists an open neighborhood $E_i \subset T_i^*$ of t_i such that $P_i \cap ICR_i(s_i) \neq \varnothing$ for every $s_i \in E_i$.*

Proposition 1 says that a set of actions P_i can be prescribed by a robust refinement ψ_i for t_i if and only if for every type s_i sufficiently close to t_i, some action in P_i is rationalizable for s_i. Under the richness condition, WY show that every rationalizable action can be selected, and hence $ICR_i(\cdot)$ is the only robust refinement.

By Proposition 1, the notion of robust refinement for t_i is reminiscent of the approach adopted in Kajii and Morris (1997) and Morris and Ui (2005). Kajii and Morris (1997) define a robust equilibrium as a Nash equilibrium in a complete-information game which is played with sufficiently large probability in some Bayesian Nash equilibrium on any common priors which are sufficiently close to the complete-information scenario. In extending the notion of robust equilibrium in Kajii and Morris (1997) to a set-valued notion, Morris and Ui (2005) define a robust equilibrium *set* as a set of Nash equilibria of a complete-information game such that every nearby incomplete-information game has some Bayesian Nash equilibrium that approximates some action profile of the set. The notion of robust equilibrium set is analogous to the set $\psi_i(t_i)$ prescribed by a robust refinement for t_i, but the former is a set of equilibria, while $\psi_i(t_i)$ is a set of rationalizable outcomes.[10]

4 Main Results

We provide our main results in this section. Specifically, Subsection 4.1 defines the notions of the upper ICR collections and the lower ICR collections. Based on the notion of the upper ICR collections, Subsection 4.2 further defines the notion of the local upper ICR collections which is then used to provide a characterization for robust refinement as well as when an action can be selected for a (finite) type. Subsection 4.3 fully characterizes the structure theorem as well as generic uniqueness. Finally, Subsection 4.4 defines a novel solution concept ICR_W which we prove is a robust refinement of rationalizability.

4.1 The Upper and Lower ICR Collections

We denote by $\mathcal{A}_i$ the collection of all nonempty subsets of A_i. For each $(\mathcal{B}_j)_{j\neq i}$ with $\mathcal{B}_j \subset \mathcal{A}_j$, we denote by $\mathcal{B}_{-i}$ the collection of all product sets $B_{-i} = \prod_{j\neq i} B_j$ with $B_j \in \mathcal{B}_j$. Say that $\pi_i \in \Delta(\Theta \times A_{-i})$ is *consistent with* $\mu_i \in \Delta(\Theta \times \mathcal{A}_{-i})$ if there exists a function $\varphi_i \colon \Theta \times \mathcal{A}_{-i} \to$

[10] Oyama and Tercieux (2010) further extends the analysis of Kajii and Morris (1997) and Morris and Ui (2005) to allow for perturbations with non-common priors.

$\Delta(A_{-i})$ such that

$$\varphi_i(\theta, R_{-i})[a_{-i}] > 0 \Rightarrow a_{-i} \in R_{-i}; \tag{2}$$

$$\pi_i[\theta, a_{-i}] = \sum_{R_{-i} \in \mathcal{A}_{-i}} \mu_i[\theta, R_{-i}] \varphi_i(\theta, R_{-i})[a_{-i}]. \tag{3}$$

For a given $\mu_i \in \Delta(\Theta \times \mathcal{A}_{-i})$, we denote by $\Pi_i^{\mu_i}$ the set of π_i's that are consistent with μ_i. Namely, $\Pi_i^{\mu_i}$ is the set of player i's beliefs over $\Theta \times A_{-i}$ if state θ and a nonempty product set of actions R_{-i} realize according to μ_i, and the opponents choose actions (possibly stochastically) from R_{-i}.[11]

We define the *upper ICR collection* $\mathcal{R}_i^{\uparrow}$ and the *lower ICR collection* $\mathcal{R}_i^{\downarrow}$ as follows:

$$\mathcal{R}_i^{\uparrow} := \{R_i \in \mathcal{A}_i : \exists t_i \in T_i^* \text{ s.t. } R_i \supset ICR_i(t_i)\},$$
$$\mathcal{R}_i^{\downarrow} := \{R_i \in \mathcal{A}_i : \exists t_i \in T_i^* \text{ s.t. } R_i \subset ICR_i(t_i)\}.$$

Note that identifying $\mathcal{R}_i^{\uparrow}$ is equivalent to identifying all minimal ICR sets; identifying $\mathcal{R}_i^{\downarrow}$ is equivalent to identifying all maximal ICR sets.

Both the upper and lower ICR collections will play important roles in our characterization results. For example, we will show that the structure theorem holds if and only if every rationalizable action is uniquely rationalizable for some type. Using $\mathcal{R}_i^{\uparrow}$ and $\mathcal{R}_i^{\downarrow}$, we can rewrite the latter condition as "for any $i \in I$, $a_i \in A_i$, and $R_i \in \mathcal{R}_i^{\downarrow}$, if $a_i \in R_i$, then $\{a_i\} \in \mathcal{R}_i^{\uparrow}$" (see Corollary 2 in Subsection 4.3).

We now provide algorithms to compute $\mathcal{R}_i^{\uparrow}$ and $\mathcal{R}_i^{\downarrow}$ from the primitives, which terminates in finite steps.[12] For the algorithm to compute $\mathcal{R}_i^{\uparrow}$, let $\mathcal{R}_i^{\uparrow,0} := \{A_i\}$ for each $i \in I$.

[11] Each μ_i imposes finitely many linear inequality constraints on π_i since $\pi_i \in \Pi_i^{\mu_i}$ if and only if $\sum_{a_{-i} \in B_{-i}} \pi_i[\theta, a_{-i}] \geq \sum_{R_{-i} \subset B_{-i}} \mu_i[\theta, R_{-i}]$ for any $\theta \in \Theta$ and $B_{-i} \subseteq A_{-i}$ (Strassen, 1964).

[12] We are not aware of any finite-step finite-dimensional algorithm to compute all (not necessarily minimal or maximal) ICR sets. For example, if we let $\mathcal{R}_i^{=,0} := \{A_i\}$ and

$$\mathcal{R}_i^{=,n} := \left\{ R_i \in \mathcal{A}_i : \exists \mu_i \in \Delta\left(\Theta \times \mathcal{R}_{-i}^{=,n-1}\right) \text{ s.t. } R_i = \bigcup_{\pi_i \in \Pi_i^{\mu_i}} BR_i(\pi_i) \right\},$$

we do not know if the algorithm terminates in finite steps or not; the sequence $\mathcal{R}_i^{=,n}$ may enter a repeating cycle. Fortunately, in order to characterize all selections, robust refinements, the structure theorem, and generic uniqueness, it is enough to use $\mathcal{R}_i^{\uparrow}$ and $\mathcal{R}_i^{\downarrow}$.

For each $i \in I$ and $n \geq 1$, we define $\mathcal{R}_i^{\uparrow,n}$ inductively as follows:

$$\mathcal{R}_i^{\uparrow,n} := \left\{ R_i \in \mathcal{A}_i : \exists \mu_i \in \Delta\left(\Theta \times \mathcal{R}_{-i}^{\uparrow,n-1}\right) \text{ s.t. } R_i \supset \bigcup_{\pi_i \in \Pi_i^{\mu_i}} BR_i(\pi_i) \right\}.$$

First, note that each step is a finite-dimensional and linear problem. Second, it is without loss of generality that μ_i puts positive probabilities only on minimal sets in $\mathcal{R}_{-i}^{\uparrow,n-1}$.[13] Third, observe that $\mathcal{R}_i^{\uparrow,n}$ is increasing in the set-inclusion order, i.e., $\mathcal{R}_i^{\uparrow,0} \subset \mathcal{R}_i^{\uparrow,1} \subset \mathcal{R}_i^{\uparrow,2} \subset \cdots$. Moreover, $\mathcal{R}_i^{\uparrow,n'} = \mathcal{R}_i^{\uparrow,n}$ for all $i \in I$ and $n' \geq n$ whenever $\mathcal{R}_i^n = \mathcal{R}_i^{n-1}$ for all $i \in I$. Therefore, the computation takes at most $\sum_i 2^{|A_i|} - 2|I|$ steps.

For the algorithm to compute $\mathcal{R}_i^{\downarrow}$, let $\mathcal{R}_i^{\downarrow,0} := \mathcal{A}_i$ for each $i \in I$. For each $i \in I$ and $n \geq 1$, we define $\mathcal{R}_i^{\downarrow,n}$ inductively as follows:

$$\mathcal{R}_i^{\downarrow,n} := \left\{ R_i \in \mathcal{A}_i : \exists \mu_i \in \Delta\left(\Theta \times \mathcal{R}_{-i}^{\downarrow,n-1}\right) \text{ s.t. } R_i \subset \bigcup_{\pi_i \in \Pi_i^{\mu_i}} BR_i(\pi_i) \right\}.$$

Symmetrically to $\mathcal{R}_i^{\uparrow,n}$, $\mathcal{R}_i^{\downarrow,n}$ is decreasing in n, i.e., $\mathcal{R}_i^{\downarrow,n-1} \subset \mathcal{R}_i^{\downarrow,n}$ for every positive integer n.

The next proposition shows that from the primitives (i.e., the fixed game $G = (A_i, u_i)_{i \in I}$), we can obtain $\mathcal{R}_i^{\uparrow}$ (resp. $\mathcal{R}_i^{\downarrow}$) by computing $\mathcal{R}_i^{\uparrow,n}$ (resp. $\mathcal{R}_i^{\downarrow,n}$) in finitely many steps (see Appendix A.2 for the proof). Thus, we will subsequently take $\mathcal{R}_i^{\uparrow}$ and $\mathcal{R}_i^{\downarrow}$ as given.

Proposition 2 *For any $n \geq \sum_i 2^{|A_i|} - 2|I|$, we have (a) $\mathcal{R}_i^{\uparrow,n} = \mathcal{R}_i^{\uparrow}$; (b) $\mathcal{R}_i^{\downarrow,n} = \mathcal{R}_i^{\downarrow}$.*

[13] That is, instead of $\mathcal{R}_i^{\uparrow,n}$, we could have defined $\mathcal{R}_i^{\min,n}$ by letting $\mathcal{R}_i^{\min,0} := \{A_i\}$ and

$$\mathcal{R}_i^{\min,n} := \text{all minimal sets of } \left\{ R_i \in \mathcal{A}_i : \exists \mu_i \in \Delta\left(\Theta \times \mathcal{R}_{-i}^{\min,n-1}\right) \text{ s.t. } R_i = \bigcup_{\pi_i \in \Pi_i^{\mu_i}} BR_i(\pi_i) \right\}.$$

4.2 The Local Upper ICR Collection and Characterizations of Selections and Robust Refinements

Fix a finite model (T, κ). Say that a conjecture $\pi_i \in \Delta(\Theta \times A_{-i})$ is *consistent with* $\mu_i \in \Delta\left(\Theta \times T_{-i} \times \mathcal{R}^{\uparrow}_{-i}\right)$ if there exists a function $\varphi_i \colon \Theta \times T_{-i} \times \mathcal{R}^{\uparrow}_{-i} \to \Delta(A_{-i})$ such that

$$\varphi_i(\theta, t_{-i}, R_{-i})[a_{-i}] > 0 \Rightarrow a_{-i} \in R_{-i}; \tag{4}$$

$$\pi_i[\theta, a_{-i}] = \sum_{t_{-i}, R_{-i}} \mu_i\left[\theta, t_{-i}, R_{-i}\right] \varphi_{-i}(\theta, t_{-i}, R_{-i})[a_{-i}]. \tag{5}$$

For a given $\mu_i \in \Delta\left(\Theta \times T_{-i} \times \mathcal{R}^{\uparrow}_{-i}\right)$, with slight abuse of notations, we also denote by $\Pi_i^{\mu_i}$ the set of all conjectures that are consistent with μ_i.[14]

In order to characterize all selections, for each type $t_i \in T_i^*$, we define the *local upper ICR collection* $\mathcal{S}_i^*\left(t_i\right)$ *for* t_i as follows:

$$\mathcal{S}_i^*\left(t_i\right) := \left\{R_i \in \mathcal{A}_i : \exists \left\{t_{i,m}\right\}_{m=0}^{\infty} \subset T_i^* \text{ s.t. } t_{i,m} \to t_i \text{ and } R_i \supset ICR_i\left(t_{i,m}\right), \forall m\right\}.$$

Then, characterizing actions that can be selected for t_i amounts to determining the singletons in $\mathcal{S}_i^*\left(t_i\right)$. We now define an algorithm that can be used to "solve" $\mathcal{S}_i^*\left(t_i\right)$ in finitely many steps.

For each $i \in I$ and $t_i \in T_i$, let $\mathcal{S}_i^0\left(t_i\right) := \mathcal{R}_i^{\uparrow}$, and for each $n \geq 1$, define

$$\mathcal{S}_i^n\left(t_i\right) := \left\{ R_i \in \mathcal{A}_i : \begin{array}{l} \forall \varepsilon \in (0,1], \exists \left(\mu_i, \mu_i'\right) \in \Delta\left(\Theta \times T_{-i} \times \mathcal{R}^{\uparrow}_{-i}\right) \times \Delta\left(\Theta \times \mathcal{R}^{\uparrow}_{-i}\right) \text{ s.t.} \\ \text{(i) } \operatorname{marg}_{\Theta \times T_{-i}} \mu_i = \kappa_{t_i}; \\ \text{(ii) } \mu_i\left[\left\{\left(\theta, t_{-i}, R_{-i}\right) : R_{-i} \in \mathcal{S}_{-i}^{n-1}\left(t_{-i}\right)\right\}\right] = 1; \\ \text{(iii) } R_i \supset \bigcup_{\left(\pi_i, \pi_i'\right) \in \Pi_i^{\mu_i} \times \Pi_i^{\mu_i'}} BR_i\left(\left(1-\varepsilon\right)\pi_i + \varepsilon \pi_i'\right) \end{array} \right\}. \tag{6}$$

Note that each step is a semi-algebraic problem, i.e., a problem based on finitely many variables and polynomial equations and inequalities. Also, it is without loss of generality that μ_i and μ_i' put positive probabilities only on minimal sets in $S_{-i}^{n-1}(t_{-i})$ and in $\mathcal{R}^{\uparrow}_{-i}$ (i.e., minimal ICR sets), respectively. Moreover, $\mathcal{S}_i^n\left(t_i\right)$ is decreasing, and reaches its limit, denoted by $\mathcal{S}_i\left(t_i\right)$, in at most $\sum_i \left(|\mathcal{R}_i^{\uparrow}| - 1\right)|T_i|$ steps. Put differently, $\mathcal{S}_i\left(t_i\right)$ is the largest

[14]In particular, we will use $\Pi_i^{\mu_i}$ for both $\mu_i \in \Delta\left(\Theta \times \mathcal{A}_{-i}\right)$ and $\mu_i \in \Delta\left(\Theta \times T_{-i} \times \mathcal{R}^{\uparrow}_{-i}\right)$.

profile of sub-collections of $\mathcal{R}_i^{\uparrow}$ that satisfies the following fixed-point property:

$$\mathcal{S}_i(t_i) = \left\{ R_i \in \mathcal{A}_i : \begin{array}{l} \forall \varepsilon \in (0,1], \exists (\mu_i, \mu_i') \in \Delta\left(\Theta \times T_{-i} \times \mathcal{R}_{-i}^{\uparrow}\right) \times \Delta\left(\Theta \times \mathcal{R}_{-i}^{\uparrow}\right) \text{ s.t.} \\ \text{(i) } \operatorname{marg}_{\Theta \times T_{-i}} \mu_i = \kappa_{t_i}; \\ \text{(ii) } \mu_i\left[\{(\theta, t_{-i}, R_{-i}) : R_{-i} \in \mathcal{S}_{-i}(t_{-i})\}\right] = 1; \\ \text{(iii) } R_i \supset \bigcup_{(\pi_i, \pi_i') \in \Pi_i^{\mu_i} \times \Pi_i^{\mu_i'}} BR_i\left((1-\varepsilon)\pi_i + \varepsilon \pi_i'\right) \end{array} \right\}. \tag{7}$$

Formally, we obtain the following result

Theorem 1 *$\mathcal{S}_i(t_i) = \mathcal{S}_i^*(t_i)$ for any finite type t_i.*

See Appendix A.3 for the proof. We prove one direction $\mathcal{S}_i(t_i) \subset \mathcal{S}_i^*(t_i)$ by exploiting the fixed point property of $\mathcal{S}_i(t_i)$ in (7) and for each $R_i \in \mathcal{S}_i(t_i)$, constructing types $\{t_{i,m}\}$ with $t_{i,m} \to t_i$ and $R_i \supset ICR_i(t_{i,m})$. The other direction $\mathcal{S}_i^*(t_i) \subset \mathcal{S}_i(t_i)$ follows from establishing that $\mathcal{S}_i^*(t_i)$ also satisfies the same fixed point property as $\mathcal{S}_i(t_i)$.[15] Intuitively speaking, the iteration in $\mathcal{S}_i^n(t_i)$ is to match $t_{i,m}$ with the limit type t_i up to the n-th order, and ε in (6) and (7) corresponds to perturbations in beliefs at each order.

Recall that an action can be selected for a type t_i if there is a sequence of types $\{t_{i,m}\}_{m=0}^{\infty} \subset T_i^*$ such that $t_{i,m} \to t_i$ and $ICR_i(t_{i,m}) = \{a_i\}$ for every m. Also recall from Proposition 1 that a refinement ψ_i is robust for type t_i if and only if there exists an open set $E_i \subset T_i^*$ such that $t_i \in E_i$ and $\psi_i(t_i) \cap ICR_i(s_i) \neq \varnothing$ for every $s_i \in E_i$.

The following theorems, which follow immediately from Theorem 1, show that $\mathcal{S}_i(t_i)$ contains enough information to characterize all selections and robust refinement for t_i.

Theorem 2 *Action a_i can be selected for finite type t_i if and only if $\{a_i\} \in \mathcal{S}_i(t_i)$.*

Specifically, Theorem 2 characterizes the set of actions which can be selected for a finite type t_i in terms of the singleton sets in the collection $\mathcal{S}_i(t_i)$. We will make use of Theorem 2 to obtain a series of corollaries in the next subsection which characterize the structure theorem and generic uniqueness.

[15] We will employ the fixed-point property to extend our characterization of the local upper ICR collection of infinite types in Section 6.

Theorem 3 *A refinement ψ_i is robust for finite type t_i if and only if $\psi_i(t_i) \cap R_i \neq \varnothing$ for every $R_i \in \mathcal{S}_i(t_i)$.*

Theorem 3 exemplifies another useful aspect of Theorem 1. In particular, Theorem 3 shows that in order for a refinement to be robust for finite type t_i, it is necessary and sufficient that it prescribes a set of actions for t_i which overlaps which each set in the collection $\mathcal{S}_i(t_i)$. We will make use of Theorem 3 to obtain a novel robust refinement in Subsection 4.4 and apply it to a first-price auction example in Subsection 5.2.

4.3 The Structure Theorem and Generic Uniqueness

Based upon our characterization of the selections, this subsection fully characterizes the structure theorem as well as generic uniqueness. Unlike the existing papers, our characterizations will be stated in terms of the primitives, and do not presuppose the existence of dominant actions or any richness condition. The characterizations will thus delineate an exact boundary of the WY critique.

In order to present the characterization, let R_i^{u} (where superscript u stands for uniqueness) be the set of all actions that are uniquely rationalizable for some type:

$$R_i^{\mathrm{u}} := \left\{ a_i \in A_i \mid \{a_i\} \in \mathcal{R}_i^{\uparrow} \right\}.$$

Given a finite model (T, κ), for each $i \in I$ and $t_i \in T_i$, let $S_i^{\mathrm{u},0}(t_i) := ICR_i(t_i) \cap R_i^{\mathrm{u}}$, and for each $n \geq 1$, define

$$S_i^{\mathrm{u},n}(t_i) := \left\{ a_i \in R_i^{\mathrm{u}} : \begin{array}{l} \exists \mu_i^{\mathrm{u}} \in \Delta\left(\Theta \times T_{-i} \times R_{-i}^{\mathrm{u}}\right) \text{ s.t.} \\ \text{(i) } \operatorname{marg}_{\Theta \times T_{-i}} \mu_i^{\mathrm{u}} = \kappa_{t_i}; \\ \text{(ii) } \mu_i^{\mathrm{u}}\left[\left\{(\theta, t_{-i}, a_{-i}) : a_{-i} \in S_{-i}^{\mathrm{u},n-1}(t_{-i})\right\}\right] = 1; \\ \text{(iii) } a_i \in BR_i\left(\operatorname{marg}_{\Theta \times A_{-i}} \mu_i^{\mathrm{u}}\right) \end{array} \right\}. \tag{8}$$

We have $R_i^{\mathrm{u}} = S_i^{\mathrm{u},0}(t_i) \supset S_i^{\mathrm{u},1}(t_i) \supset \cdots$, which reaches its limit $S_i^{\mathrm{u}}(t_i)$ in finitely many steps.[16] The following corollary will be used to provide a characterization for the structure theorem and the generic uniqueness in Corollaries 2 and 3.

[16]Note that $S_i^{\mathrm{u}}(t_i)$ can be empty. More precisely, $S_i^{\mathrm{u}}(t_i) = \varnothing$ if and only if $ICR_j(t_j) \cap R_j^{\mathrm{u}} = \varnothing$ for some t_j in the smallest belief-closed type space containing t_i. In this case, Corollary 1 is vacuously true, and we should instead apply Theorem 2.

Corollary 1 *Action a_i can be selected for finite type t_i if $a_i \in S_i^{\mathrm{u}}(t_i)$.*

Proof By Theorem 2, it suffices to show that $a_i \in S_i^{\mathrm{u}}(t_i)$ implies $\{a_i\} \in \mathcal{S}_i(t_i)$. We prove by induction that $a_i \in S_i^{\mathrm{u},n}(t_i)$ implies $\{a_i\} \in \mathcal{S}_i^n(t_i)$. The case for $n = 0$ holds by definition. Now suppose that $a_i \in S_i^{\mathrm{u},n-1}(t_i)$ implies $\{a_i\} \in \mathcal{S}_i^{n-1}(t_i)$ for any $i \in I$ and $t_i \in T_i$. Let $a_i \in S_i^{\mathrm{u},n}(t_i)$ and we show that $\{a_i\} \in \mathcal{S}_i^n(t_i)$. Since $a_i \in S_i^{\mathrm{u},n}(t_i)$, there exists $\mu_i^{\mathrm{u}} \in \Delta\left(\Theta \times T_{-i} \times R_{-i}^{\mathrm{u}}\right)$ that satisfies (i)-(iii) in (8). Moreover, since $a_i \in R_i^{\mathrm{u}}$, there exists $t_i' \in T_i^*$ such that $\{a_i\} = ICR_i(t_i')$. By (1), we have $\{a_i\} = BR_i\left(\mathrm{marg}_{\Theta\times A_{-i}}\,\nu_i'\right)$ for any valid conjecture $\nu_i' \in \Delta(\Theta \times T_{-i}^* \times A_{-i})$ for t_i'. Define $(\mu_i, \mu_i') \in \Delta\left(\Theta \times T_{-i} \times \mathcal{R}_{-i}^{\uparrow}\right) \times \Delta\left(\Theta \times \mathcal{R}_{-i}^{\uparrow}\right)$ such that

$$\mu_i[\theta, t_{-i}, \{a_{-i}\}] = \mu_i^{\mathrm{u}}[\theta, t_{-i}, a_{-i}];$$
$$\mu_i'[\theta, R_{-i}] = \kappa_{t_i'}^*[\{(\theta, s_{-i}) : ICR_{-i}(s_{-i}) = R_{-i}\}]$$

for each $(\theta, t_{-i}, a_{-i}, R_{-i}) \in \Theta \times T_{-i} \times A_{-i} \times \mathcal{R}_{-i}^{\uparrow}$. Then, μ_i satisfies (i) and (ii) in (6) because μ_i^{u} satisfies (i) and (ii) in (8) and we assume the induction hypothesis. It follows from (iii) in (8) and $\{a_i\} = BR_i\left(\mathrm{marg}_{\Theta\times A_{-i}}\,\nu_i'\right)$ for any valid conjecture ν_i' for t_i' that $\{a_i\} = BR_i\left((1-\varepsilon)\pi_i + \varepsilon\pi_i'\right)$ for every $(\pi_i, \pi_i') \in \Pi_i^{\mu_i} \times \Pi_i^{\mu_i'}$. Thus, $\{a_i\} \in \mathcal{S}_i^n(t_i)$. ■

Observe that under the richness condition, $R_i^{\mathrm{u}} = A_i$, and therefore $S_i^{\mathrm{u},n}(t_i) = ICR_i^n(t_i)$ for every n and $S_i^{\mathrm{u}}(t_i) = ICR_i(t_i)$. Thus, Corollary 1 immediately reproduces WY's result that every ICR action can be selected for every finite type, under the richness condition.

We now characterize the structure theorem.

Corollary 2 *The following two conditions are equivalent:*

1. *for any type $t_i \in T_i^*$, any action in $ICR_i(t_i)$ can be selected for t_i;*
2. *for any $i \in I$ and $R_i \in \mathcal{R}_i^{\downarrow}$, we have $R_i \subset R_i^{\mathrm{u}}$.*

Proof For "1 ⇒ 2" for any $i \in I$ and $R_i \in \mathcal{R}_i^{\downarrow}$, by Lemma 1, there exists a finite type t_i such that $R_i \subset ICR_i(t_i)$. Thus we have $R_i \subset ICR_i(t_i) \subset R_i^{\mathrm{u}}$.

For "2 $\Rightarrow$ 1" by Corollary 1, it suffices to show that $ICR_i(t_i) \subset S_i^{\mathrm{u}}(t_i)$ for any finite type t_i. We fix any finite model (T, κ), and prove by induction that $ICR_i(t_i) \subset S_i^{\mathrm{u},n}(t_i)$ for any $i \in I$ and $t_i \in T_i$. The case of $n = 0$ is obvious. Now suppose that $ICR_i(t_i) \subset S_i^{\mathrm{u},n-1}(t_i)$ for any $i \in I$ and $t_i \in T_i$. Given any $i \in I$ and $t_i \in T_i$, consider any valid conjecture $\nu_i \in \Delta(\Theta \times T_{-i} \times A_{-i})$ for t_i. Then, $\mu_i^{\mathrm{u}} = \nu_i$ satisfies (i) and (ii) in (8) because ν_i is valid for t_i and we assume the induction hypothesis. Thus $BR_i\left(\mathrm{marg}_{\Theta \times A_{-i}} \nu_i\right) \subset S_i^{\mathrm{u},n}(t_i)$. By (1), we have $ICR_i(t_i) \subset S_i^{\mathrm{u},n}(t_i)$. ■

This corollary reproduces Chen (2012, Theorem 1). In words, a necessary and sufficient condition for every rationalizable action to be selected for any (finite) type (i.e., the structure theorem) is that every rationalizable action is uniquely rationalizable for some type. This condition is called richness in uniquely rationalizable actions (RURA) in Chen (2012). Note that the RURA condition is not imposed on the primitives directly, but our algorithms to compute $\mathcal{R}_i^{\uparrow}$ and $\mathcal{R}_i^{\downarrow}$ provide a way to decide whether the RURA condition holds from the primitives.

We then turn to characterize generic uniqueness.

Corollary 3 *The following two conditions are equivalent:*

1. *for any $i \in I$, $\{t_i \in T_i^* : |ICR_i(t_i)| = 1\}$ is open and dense in T_i^*;*
2. *for any $i \in I$ and $R_i \in \mathcal{R}_i^{\uparrow}$, we have $R_i \cap R_i^{\mathrm{u}} \neq \varnothing$.*

Proof For the "1 $\Rightarrow$ 2" direction, for any $i \in I$ and $R_i \in \mathcal{R}_i^{\uparrow}$, by Lemma 1, there exists a finite type $\bar{t}_i$ such that $R_i \supset ICR_i(\bar{t}_i)$. Since $\{t_i \in T_i^* : |ICR_i(t_i)| = 1\}$ is dense in T_i^* and $ICR_i(\cdot)$ is upper hemicontinuous, we have $ICR_i(\bar{t}_i) \cap R_i^{\mathrm{u}} \neq \varnothing$, and hence $R_i \cap R_i^{\mathrm{u}} \neq \varnothing$.

For the "2 $\Rightarrow$ 1" direction, $\{t_i \in T_i^* : |ICR_i(t_i)| = 1\}$ is open in T_i^* since $ICR_i(\cdot)$ is upper hemicontinuous. To show that $\{t_i \in T_i^* : |ICR_i(t_i)| = 1\}$ is dense in T_i^*, by Lemma 1 and Corollary 1, it suffices to show that $ICR_i(t_i) \cap S_i^{\mathrm{u}}(t_i) \neq \varnothing$ for any finite type t_i. We fix any finite model (T, κ), and prove by induction that $ICR_i(t_i) \cap S_i^{\mathrm{u},n}(t_i) \neq \varnothing$ for any $i \in I$ and $t_i \in T_i$. The case of $n = 0$ is obvious. Now suppose that $ICR_i(t_i) \cap S_i^{\mathrm{u},n-1}(t_i) \neq \varnothing$ for any $i \in I$ and $t_i \in T_i$. Then, given any $i \in I$ and $t_i \in T_i$, there exists $\nu_i \in \Delta(\Theta \times T_{-i} \times A_{-i})$

such that $\text{marg}_{\Theta \times T_{-i}} \nu_i = \kappa_{t_i}$ and $\nu_i \left[a_{-i} \in ICR_{-i}(t_{-i}) \cap S_{-i}^{\text{u},n-1}(t_{-i})\right] = 1$. Since ν_i is a valid conjecture for t_i and $\mu_i^{\text{u}} = \nu_i$ satisfies (i) and (ii) in (8), by (1), we have $ICR_i(t_i) \cap S_i^{\text{u},n}(t_i) \supset BR_i\left(\text{marg}_{\Theta \times A_{-i}} \nu_i\right) \neq \varnothing$. ■

Corollary 3 shows that a necessary and sufficient condition for types with uniquely rationalizable actions to be generic in the universal type space (i.e., generic uniqueness) is that every set in the upper ICR collection contains some action that can be identified with a singleton in the upper ICR collection, i.e., every minimal ICR set is a singleton. Note that generic uniqueness (Condition 1 in Corollary 3) is a weaker statement than the structure theorem (Conditions 1 and 2 in Corollary 2): the former requires that *some* rationalizable action be selected for each type, whereas the latter only requires that *every* rationalizable action be selected for each type. In particular, Corollary 3 can apply to games with weakly dominated actions in any state (see the next Subsection). In WY, the richness condition implies both results and renders their distinction moot.

4.4 A Robust Refinement: ICR_W

When the structure theorem holds, the only robust refinement of rationalizability is the trivial one, i.e., $ICR_i(\cdot)$ itself. An immediate question is when the structure theorem does not hold, is it possible to offer a non-trivial robust refinement for rationalizability? We exemplify one such robust refinement which we denote by ICR_W. The solution concept ICR_W amounts to iterated deletion of (globally) weakly dominated actions, followed by iterated deletion of interim strictly dominated actions. We provide a formal definition of ICR_W and establish its robustness in the rest of the section. In the next section, we provide an example to illustrate how ICR_W strictly refines rationalizability and pins down a unique outcome.

We say that an action $a_i \in A_i$ is *(globally) weakly dominated against* $B_{-i} \subset A_{-i}$ if there exists $\alpha_i \in \Delta(A_i)$ such that

$$u_i(a_i, a_{-i}, \theta) \leq u_i(\alpha_i, a_{-i}, \theta)$$

for any $(a_{-i}, \theta) \in B_{-i} \times \Theta$ with strict inequality for some $(a_{-i}, \theta) \in B_{-i} \times \Theta$. Let $W_i^0 = A_i$. For $n \geq 1$, W_i^n is the set of actions $a_i \in A_i$ that are not globally weakly dominated against

W_{-i}^{n-1}. Let $W_i = \bigcap_n W_i^n$.

We define $ICR_{i,W}(t_i)$ as the ICR actions of t_i for the game whose action set is restricted to W. ICR_W resembles the well known Dekel-Fudenberg procedure, i.e., one round deletion of weakly dominated strategies followed by iterated deletion of strictly dominated strategies in complete-information games.[17] There are two important differences. First, ICR_W performs iterated weak dominance as opposed to one round deletion of weakly dominated strategies. As a result, ICR_W prescribes sharper refinement, while we will show that it remains robust. Second, the iterated (global) weak dominance in ICR_W applies directly to the payoff function with no reference to the type space (or complete information in particular) which is used only in performing the iterated deletion of interim strictly dominated actions. The feature is important for the robustness of ICR_W. Indeed, when the richness assumption is satisfied, ICR_W is equivalent to ICR. The following proposition shows that ICR_W is indeed a robust refinement of rationalizability.

Proposition 3 *$ICR_{i,W}$ is a robust refinement.*

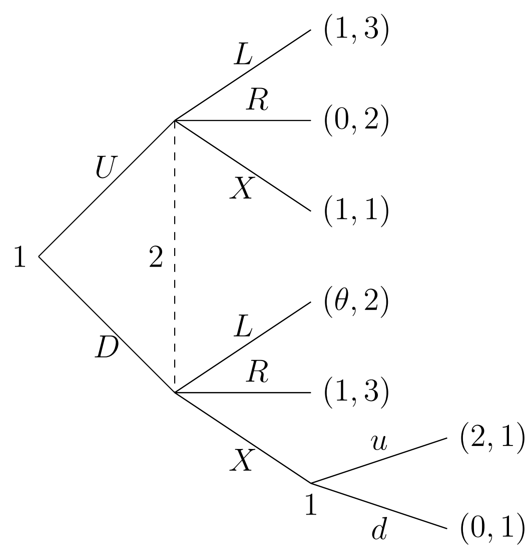
5 Examples

In this section, we present three examples to illustrate our main results. The first example shows that the simplified condition in Corollary 1 is only sufficient but not necessary for an action to be selected. In the next two economic examples of incomplete-information games, no player has a dominant action at any state, and thus WY's analysis cannot be applied. Nevertheless, we can "endogenize" the richness condition by identifying a large set of actions that are uniquely rationalizable for some type.

[17]See Frick and Romm (2015) for a generalization of the Dekel-Fudenberg procedure to incomplete-information games.

5.1 A Game with "Twins"

Modifying Morris, Takahashi, and Tercieux (2012, Example 2), consider the following extensive-form game [insert a game tree]



and its reduced normal form

θ :		L	R	X
	U	$1, 3$	$0, 2$	$1, 1$
	Du	$\theta, 2$	$1, 3$	$2, 1$
	Dd	$\theta, 2$	$1, 3$	$0, 1$

with $\theta \in \Theta = \{0, 2\}$. (The following argument is insensitive to small payoff perturbations on terminal nodes in the extensive form.) Let $\tau_{i,0}$ be the type of player i with complete information about $\theta = 0$. Then we have $\mathcal{R}_1^{\uparrow} = \mathcal{S}_1(\tau_{1,0}) = \{\{Du, Dd\}, \{U, Du, Dd\}\}$ and $\mathcal{R}_2^{\uparrow} = \mathcal{S}_2(\tau_{2,0}) = \{\{R\}, \{L, R\}, \{R, X\}, \{L, R, X\}\}$. Thus, by Theorem 2, R can be selected for $\tau_{2,0}$. On the other hand, we have $R_1^{\mathrm{u}} = \varnothing$ and $R_2^{\mathrm{u}} = \{R\}$, and hence $S_1^{\mathrm{u}}(\tau_{1,0}) = S_2^{\mathrm{u}}(\tau_{2,0}) = \varnothing$. The example shows that while $a_i \in S_i^{\mathrm{u}}(t_i)$ is a sufficient condition for a_i to be selected for t_i, it is not necessary and misses cases where some selection is possible. Note that the state space is too small to satisfy the extensive-form richness condition in Chen (2012).

5.2 First-Price Auction with Discrete Bids

Consider a sealed-bid first-price auction with $|I| \geq 3$, where bidders submit their bids $b_1, \ldots, b_{|I|} \in \{0, 1, \ldots, 9, 10\}$ simultaneously. Break tie with a fair coin toss. Each bidder's value for the object is in $\{0, 1, \ldots, 9, 10\}$, i.e., $\Theta = \{0, 1, \ldots, 9, 10\}^I$. Observe that no bidder (regardless of his value) has a strictly dominant bid and thus WY's richness condition does not hold. We will identify which complete-information type has a non-trivial robust refinement.

Let $\tau_{i,v}$ be the type with complete information that all bidders have values v. Specifically, we show that for $v \leq 9$, the minimal robust refinement for $\tau_{i,v}$ is $\{0, ..., v\}$; in contrast, for $\tau_{i,10}$, the minimal robust refinement is $\{9\}$.

First, for $v \leq 9$, we have $ICR_i(\tau_{i,v}) = \{0, ..., v\}$. Moreover, we also have $\{b\} \in \mathcal{R}_{-i}^{\uparrow}$ for each $b = 0, ..., v$. To see this, note that $\{0\} \in \mathcal{R}_{-i}^{\uparrow}$ since $ICR_i(\overline{\tau}_{i,0}) = \{0\}$; moreover, for any $b \geq 1$, we also have (inductively) $\{b\} \in \mathcal{R}_{-i}^{\uparrow}$ since b is the only best reply to any belief consistent with $\mu_i \in \Delta(\Theta \times \mathcal{A}_{-i})$ which assigns probability one to $\theta = b + 1$ and $\{b\}$. Since $\{b\} \in \mathcal{R}_{-i}^{\uparrow}$ for each $b = 0, ..., v$, it follows that $\mathcal{S}_i(\tau_{i,v}) = \{\{0\}, \{1\}, ..., \{v\}\}$. Hence, the minimal robust refinement is $\{0, ..., v\}$ which is reminiscent of the structure theorem applied to the complete-information model where everyone has value v.

Second, consider $v = 10$. We have $ICR_i(\tau_{i,10}) = \{0, 1, \ldots, 9, 10\}$ since every bid is a best reply if the other bidders all bid 10. We now show that $\{9\}$ is the minimal robust refinement for $\tau_{i,10}$. By Proposition 3, it suffices to show that $ICR_{i,W}(\tau_{i,10}) = \{9\}$. To see this, observe that bidding 10 is globally weakly dominated by bidding 0. Moreover, bidding any b with $1 \leq b \leq 9$ is a strict best response when everyone has value 10 and every opponent bids $b - 1$. Also bidding 0 is a strict best response when everyone has value 0 and every opponent bids 0. Hence, $W_i = \{0, 1, ..., 9\}$. Now suppose that for some $k \geq 0$ (and $k \leq 7$), we have $k' \notin ICR_{i,W}(\tau_{i,10})$ for every $k' \leq k$ and we show that $k + 1 \notin ICR_{i,W}(\tau_{i,10})$. Indeed, if a bidder assigns a positive probability that the other bidders all bid $k + 1$, then bidding $k + 2$ is strictly better than bidding $k + 1$ (since $10/|I| < 9$); if bidder assigns probability zero that the other bidders all bid $k + 1$, then bidding 9 to win with a positive probability is strictly better than bidding $k + 1$.

5.3 Cournot Oligopoly with Uncertainty in Demand

Consider the Cournot oligopoly game, where the inverse demand is linear in the form of $P(Q,\theta) = \theta - Q$ with $Q = \sum_i q_i$ and parameter $\theta > 0$, and marginal costs are constant and normalized to be 0.[18] Assume that firm i can produce any nonnegative output q_i. Thus firm i's profit is given by $u_i(q_1, \dots, q_{|I|}, \theta) = \left(\theta - \sum_j q_j\right) q_i$.

Under complete information about θ, it is well known that the Cournot oligopoly game is dominance-solvable (i.e., has a uniquely rationalizable action) if and only if $|I| = 2$ (Bernheim, 1984). Moreover, if $|I| = 2$, then the dominance solvability result extends to the case with incomplete information (Weinstein and Yildiz, 2007a, Proposition 1).[19] We will thus analyze the case where $|I| \geq 3$ and firms have incomplete information about θ. For simplicity, we assume that θ takes two possible values, θ_H and θ_L with $\theta_H > \theta_L > 0$.[20]

We denote by $\mathbb{E}_{t_i^1}(\theta)$ the expected value of θ with respect to the first-order belief t_i^1 of type t_i.

Proposition 4 *Consider the Cournot oligopoly game with $|I| \geq 3$ firms and uncertainty in demand.*

(a) Suppose that $\theta_H/\theta_L > (|I| - 1)/2$. Then action q is uniquely rationalizable for some type in T_i^* if and only if $q \in [0, \theta_H/2]$.

(b) Suppose that $\theta_H/\theta_L \leq (|I| - 1)/2$. Then we have $ICR_i(t_i) = \left[0, \mathbb{E}_{t_i^1}(\theta)/2\right]$ for any $t_i \in T_i^*$; in particular, no type has a uniquely rationalizable action.

[18] We allow for negative prices which only mean that the demand function is linear in prices even below marginal costs.

[19] Weinstein and Yildiz (2011) study the sensitivity of *equilibrium* behavior to higher-order beliefs in Cournot games. In contrast, here we focus on selecting rationalizable actions as uniquely rationalizable actions. Moreover, Weinstein and Yildiz (2011) analyze the equilibrium behavior of nearby types whose lower-order beliefs are all the same as (as opposed to only weak*-close to) those in the original type. Hence, the notion of proximity of types which Weinstein and Yildiz (2011) consider is also slightly stronger than the notion which WY and we consider.

[20] A similar exercise can be done with uncertainty in cost functions.

To see how the condition on θ_H/θ_L is used in the proof of part (a), consider a type $\tau_{i,1,H}$ who is certain that "$\theta = \theta_H$ and each opponent $j \neq i$ is certain about $\theta = \theta_L$." Since $\tau_{i,1,H}$ believes that each $j \neq i$ plays an action of at most $\theta_L/2$, the action that $\tau_{i,1,H}$ can rationalize is at least

$$\frac{1}{2}\left(\theta_H - (|I|-1)\frac{\theta_L}{2}\right),$$

which is strictly positive since $\theta_H/\theta_L > (|I|-1)/2$. Similarly, we consider the type $\tau_{i,2,L}$ who is certain that "$\theta = \theta_L$ and each opponent $j \neq i$ is of type $\tau_{j,1,H}$." Then the action that $\tau_{i,2,L}$ can rationalize is at most

$$\frac{1}{2}\left(\theta_L - (|I|-1)\frac{1}{2}\left(\theta_H - (|I|-1)\frac{\theta_L}{2}\right)\right),$$

which is strictly below $\theta_L/2$. Continuing these processes alternatingly sufficiently many times, we can construct a type for which action 0 is uniquely rationalizable. Then the final step of the proof is to extend this result to any action in $[0, \theta_H/2]$. See Appendix A.5 for a more formal proof.

Proposition 4 exhibits a sharp discontinuity: (a) if θ_H/θ_L is large, then any action that is rationalizable for some type is uniquely rationalizable for some other type; (b) if θ_H/θ_L is small, then no type has a uniquely rationalizable action. In particular, if $|I| = 3$, with an arbitrarily small amount of uncertainty in demand, we have $\theta_H/\theta_L > 1 = (|I|-1)/2$, and Proposition 4(a) applies. This is in contrast with the case under complete information, where the Cournot oligopoly game is not dominant-solvable.

Note that Proposition 4 continues to hold for finely discretized action spaces. For example, suppose that firms can produce outputs only in $d\mathbb{N}$, the set of nonnegative integer multiples of $d > 0$. Assume $\theta_L/2 \in d\mathbb{N}$ for simplicity. Then, (a) if $\theta_H/\theta_L > (|I|-1)/2 + d/\theta_L$, then any action in $[0, (\theta_H + d)/2] \cap d\mathbb{N}$ is uniquely rationalizable for some type in T_i^*; (b) if $\theta_H/\theta_L \leq (|I|-1)/2 + d/\theta_L$, then we have $ICR_i(t_i) = \left[0, \mathbb{E}_{t_i^1}(\theta + d)/2\right] \cap d\mathbb{N}$.

The Cournot example in Subsection 5.3 has infinitely many actions. There are two ways to apply our results to this example. One is to discretize the action space as specified at the end of Subsection 5.3. The other is to analyze the infinite game directly. To do so, observe that the proof of $\mathcal{S}_i(t_i) \subset \mathcal{S}_i^*(t_i)$ in Theorem 1 and also the proof of Corollaries 1 and 2 do not depend on the finiteness assumption of A_i. Thus, it follows from Proposition 4 and Corollary 2 that (a) if $\theta_H/\theta_L > (|I|-1)/2$, we can select every $q \in [0, \theta_H/2]$ for every type t_i, whereas (b) if $\theta_H/\theta_L \leq (|I|-1)/2$, no type has a uniquely rationalizable action. Therefore, the sharp discontinuity between the two cases remains regarding the selections and the structure theorem.

Note that for $\theta_H/\theta_L > (|I|-1)/2$, this structure theorem without discretization is slightly different from WY's original one that requires the openness of the set of types for which a given action is uniquely rationalizable (Weinstein and Yildiz, 2007b, p. 372). Indeed, when the action set is infinite, even though the ICR correspondence remains to be upper hemicontinuous (Weinstein and Yildiz, 2012, Proposition 3), $\{t_i \in T_i^* : |ICR_i(t_i)| = 1\}$ need not be open. Nonetheless, $\{t_i \in T_i^* : |ICR_i(t_i)| = 1\}$ is still a countable intersection of $\{t_i \in T_i^* : \text{diameter of } ICR_i(t_i) < 1/n\}$, each of which is open (because $ICR(\cdot)$ is upper hemicontinuous) and dense (because every $q \in [0, \theta_H/2]$ can be selected for every type t_i). Therefore, the generic uniqueness holds in a slightly weaker sense, i.e., $\{t_i \in T_i^* : |ICR_i(t_i)| = 1\}$ is a residual set in T_i^*.

6 Infinite Types

This section extends our results to infinite types. The key to such an extension is a measurability requirement. To see this, suppose instead that we adopt the same definition of $\mathcal{S}_i^n(t_i)$ as in (6) for finite types. Since $\mathcal{S}_i^n(t_i)$ is specified on a type-by-type basis, we may not be able to find (μ_i, μ_i') that depends on (t_i, R_i) measurably, which is an indispensable step in the proof of Theorem 1. To circumvent this problem, we introduce a fixed-point counterpart of $\mathcal{S}_i^n(t_i)$ that already incorporates the measurability of (μ_i, μ_i') as a part of definition.

Formally, fix any (possibly infinite) model (T, κ). A profile $\left(\widetilde{\mathcal{S}}_i\right)_{i\in I}$ of measurable mappings $\widetilde{\mathcal{S}}_i \colon T_i \to 2^{\mathcal{R}_i^{\uparrow}} \setminus \{\varnothing\}$ is called an $\mathcal{R}^{\uparrow}$-*perturbed curb collection* on (T, κ) if for every $i \in I$ and $\varepsilon \in (0, 1]$, there exists a measurable mapping

$$(\mu, \mu') \colon T_i \times \mathcal{R}_i^{\uparrow} \to \Delta\left(\Theta \times T_{-i} \times \mathcal{R}_{-i}^{\uparrow}\right) \times \Delta\left(\Theta \times \mathcal{R}_{-i}^{\uparrow}\right)$$

such that for each $t_i \in T_i$ and $R_i \in \widetilde{\mathcal{S}}_i(t_i)$,

(i) $\operatorname{marg}_{\Theta \times T_{-i}} \mu_{t_i, R_i} = \kappa_{t_i}$;

(ii) $\mu_{t_i,R_i}\left[\left\{(\theta, t_{-i}, R_{-i}) : R_{-i} \in \widetilde{\mathcal{S}}_{-i}(t_{-i})\right\}\right] = 1$;

(iii) $R_i \supset \bigcup_{(\pi_i,\pi'_i)\in\Pi_i^{\mu_{t_i,R_i}}\times\Pi_i^{\mu'_{t_i,R_i}}} BR_i\left((1-\varepsilon)\pi_i + \varepsilon\pi'_i\right)$,

where $\Pi_i^{\mu_{t_i,R_i}}$ and $\Pi_i^{\mu'_{t_i,R_i}}$ are the sets of π_i satisfying (2)-(3) and (4)-(5), respectively, with the additional measurability requirement on φ_i. Note that $\mathcal{R}^{\uparrow}$-perturbed curb collections are defined on each model (T,κ), which may be infinite, but much smaller than the universal model (T^*,κ^*).

The following is a generalization of Theorem 1 to infinite types. The proof is in Appendix A.6.

Proposition 5 *For any model (T,κ), $(\mathcal{S}_i^*|_{T_i})_{i\in I}$ is the largest $\mathcal{R}^{\uparrow}$-perturbed curb collection on (T,κ).*

By Proposition 5, we can characterize all selections and robust refinement in terms of $\mathcal{R}^{\uparrow}$-perturbed curb collections.

Theorem 4 *Fix a model (T,κ). Action a_i can be selected for type $t_i \in T_i$ if and only if $\{a_i\} \in \widetilde{\mathcal{S}}_i(t_i)$ for some $\mathcal{R}^{\uparrow}$-perturbed curb collection $\left(\widetilde{\mathcal{S}}_j\right)_{j\in I}$ on (T,κ).*

Theorem 5 *Fix a model (T,κ). A refinement ψ_i is robust for type $t_i \in T_i$ if and only if $\psi_i(t_i) \cap R_i \neq \varnothing$ for any $\mathcal{R}^{\uparrow}$-perturbed curb collection $\left(\widetilde{\mathcal{S}}_j\right)_{j\in I}$ on (T,κ) and any $R_i \in \widetilde{\mathcal{S}}_i(t_i)$.*

7 Conclusion

In this paper, we derive for each finite game a global-upper ICR collection $\mathcal{R}_i^{\uparrow}$ (resp. global-lower ICR collection $\mathcal{R}_i^{\downarrow}$), which is the collection of all action sets that contain (resp. are contained in) the rationalizable action set for some type. Based on $\mathcal{R}_i^{\uparrow}$, we also derive for each finite type t_i a local upper ICR collection $\mathcal{S}_i(t_i)$, which is the collection of all action sets that contain the rationalizable action set for some sequence of types converging to t_i.

Finally, we show that the local upper ICR collection $\mathcal{S}_i(t_i)$ fully characterizes the selections of rationalizable actions as well as the robust refinement for any finite type t_i. By making use of the characterization, we delineate the boundary of the WY critique on the global-game equilibrium refinement approach. Our results strengthen and unify the existing results in Weinstein and Yildiz (2007b), Penta (2013), and Chen (2012). To exemplify a robust refinement, we also identify the solution concept ICR_W and demonstrate, by means of economic examples, how ICR_W can prescribe a unique robust refinement.

Our approach can also be useful for alternative robustness exercises. For instance, we may instead call our notion of robust refinement a UHC-robust refinement and define an LHC-robust refinement as a refinement which is lower hemicontinuous. Then, it also follows from Theorem 1 that a refinement φ_i is LHC-robust for finite type t_i if and only if the intersection of sets in $\mathcal{S}_i(t_i)$ is nonempty and $\varphi_i(t_i)$ is contained in the intersection.

A Appendix

A.1 Proof of Proposition 1

We first prove the "only if" part. Suppose that $\psi_i : T_i^* \to 2^{A_i} \setminus \{\varnothing\}$ is a robust refinement for t_i with $\psi_i(t_i) = P_i$. By Definition 3, ψ_i is upper hemicontinuous at t_i, which implies that there exists an open neighborhood $E_i \subset T_i^*$ of t_i such that $\psi_i(s_i) \subset P_i$ for every $s_i \in E_i$. Since $\psi_i(s_i) \subset ICR_i(s_i)$, we thus have

$$\varnothing \neq \psi_i(s_i) = P_i \cap \psi_i(s_i) \subset P_i \cap ICR_i(s_i), \forall s_i \in E_i.$$

We now prove the "if" part. Suppose that there is an open set $E_i \subset T_i^*$ and $P_i \subset ICR_i(t_i)$ such that $t_i \in E_i$ and $P_i \cap ICR_i(s_i) \neq \varnothing$ for every $s_i \in E_i$. Consider the refinement ψ_i defined as follows:

$$\psi_i(s_i) = \begin{cases} P_i \cap ICR_i(s_i) & \text{if } s_i \in E_i; \\ ICR_i(s_i), & \text{if } s_i \notin E_i. \end{cases}$$

In particular, $\psi_i(t_i) = P_i$. We now show that ψ_i is robust for t_i. By upper hemicontinuity of ICR, there exists an open neighborhood $E_i' \subset T_i^*$ of t_i such that

$$t_i \in E_i' \subset E_i \text{ and } ICR_i(s_i) \subset ICR_i(t_i), \forall s_i \in E_i'.$$

Hence, for any $s_i \in E_i'$,

$$\psi_i(s_i) = P_i \cap ICR_i(s_i) \subset P_i \cap ICR_i(t_i) = \psi_i(t_i).$$

In other words, ψ_i is upper hemicontinuous at t_i, or equivalently, ψ_i is a robust refinement for t_i.

A.2 Proof of Proposition 2

We first prove the following lemma.

Lemma 2 *For any $n \geq 0$, we have (a) $\mathcal{R}_i^{\uparrow,n} = \{R_i \in \mathcal{A}_i : \exists t_i \in T_i^* \text{ s.t. } R_i \supset ICR_i^n(t_i)\}$; (b) $\mathcal{R}_i^{\downarrow,n} = \{R_i \in \mathcal{A}_i : \exists t_i \in T_i^* \text{ s.t. } R_i \subset ICR_i^n(t_i)\}$.*

Proof The proof of (b) is similar to the proof of (a) and thus omitted. We prove (a) by induction. The case for $n = 0$ is obvious. Suppose that the claim holds for $n - 1$ and we prove the case for n.

For "$\supset$", suppose that $R_i \supset ICR_i^n(t_i)$ for some $t_i \in T_i^*$. Define $\mu_i \in \Delta(\Theta \times \mathcal{A}_{-i})$ such that

$$\mu_i[\theta, R_{-i}] = \kappa_{t_i}^* \left[\left\{(\theta, t_{-i}) : ICR_{-i}^{n-1}(t_{-i}) = R_{-i}\right\}\right] \tag{9}$$

for every $(\theta, R_{-i}) \in \Theta \times \mathcal{A}_{-i}$.[21] By the induction hypothesis, $\mu_i \in \Delta\left(\Theta \times \mathcal{R}_{-i}^{\uparrow,n-1}\right)$. We prove that $R_i \supset BR_i(\pi_i)$ for every $\pi_i \in \Pi_i^{\mu_i}$ to conclude $R_i \in \mathcal{R}_i^{\uparrow,n}$. Pick any $\pi_i \in \Pi_i^{\mu_i}$. Then there exists a function $\varphi_i \colon \Theta \times \mathcal{A}_{-i} \to \Delta(A_{-i})$ such that (2) and (3) hold. Define $\nu_i \in \Delta\left(\Theta \times T_{-i}^* \times A_{-i}\right)$ such that

$$\begin{aligned} &\nu_i[\{\theta\} \times E_{-i} \times \{a_{-i}\}] \\ = &\sum_{R_{-i} \in \mathcal{A}_{-i}} \kappa_{t_i}^* \left[\left\{(\theta, t_{-i}) : t_{-i} \in E_{-i} \text{ and } ICR_{-i}^{n-1}(t_{-i}) = R_{-i}\right\}\right] \varphi_i(\theta, R_{-i})[a_{-i}] \end{aligned} \tag{10}$$

[21] By Dekel, Fudenberg, and Morris (2007, Lemma 1), $ICR_j^{n-1}(\cdot)$ is upper hemicontinuous when T_j^* is endowed with the product topology. Thus, $\left\{(\theta, t_{-i}) : ICR_{-i}^{n-1}(t_{-i}) = R_{-i}\right\}$ is measurable.

for every measurable $E_{-i} \subset T^*_{-i}$ and $(\theta, a_{-i}) \in \Theta \times A_{-i}$. It then follows that $\operatorname{marg}_{\Theta \times T_{-i}} \nu_i = \kappa^*_{t_i}$; $\nu_i \left[a_{-i} \in ICR^{n-1}_{-i}(t_{-i})\right] = 1$ by (2); $\operatorname{marg}_{\Theta \times A_{-i}} \nu_i = \pi_i$ because

$$\begin{aligned}
\operatorname{marg}_{\Theta \times A_{-i}} \nu_i[\theta, a_{-i}] &= \sum_{R_{-i} \in \mathcal{A}_{-i}} \kappa^*_{t_i} \left[\left\{(\theta, t_{-i}) : ICR^{n-1}_{-i}(t_{-i}) = R_{-i}\right\}\right] \varphi_i(\theta, R_{-i})[a_{-i}] \\
&= \sum_{R_{-i} \in \mathcal{A}_{-i}} \mu_i \left[\theta, R_{-i}\right] \varphi_i(\theta, R_{-i})[a_{-i}] \\
&= \pi_i[\theta, a_{-i}]
\end{aligned}$$

for every $(\theta, a_{-i}) \in \Theta \times A_{-i}$, where the three equalities follow from (10), (9), and (3), respectively. Thus, we have $R_i \supset ICR^n_i(t_i) \supset BR_i(\pi_i)$.

For "$\subset$", suppose that $R_i \in \mathcal{R}^{\uparrow,n}_i$. Then, there exists $\mu_i \in \Delta(\Theta \times \mathcal{R}^{\uparrow,n-1}_{-i})$ such that $R_i \supset BR_i(\pi_i)$ for every $\pi_i \in \Pi^{\mu_i}_i$. By the induction hypothesis, for every $R_{-i} \in \mathcal{R}^{\uparrow,n-1}_{-i}$, there exists $\tau_{-i,R_{-i}} \in T^*_{-i}$ such that $R_{-i} \supset ICR^{n-1}_{-i}\left(\tau_{-i,R_{-i}}\right)$. Define $t_i \in T^*_i$ with $\kappa^*_{t_i}$ having a finite support such that

$$\kappa^*_{t_i}\left[\theta, t_{-i}\right] = \mu_i \left[\left\{(\theta, R_{-i}) : \tau_{-i,R_{-i}} = t_{-i}\right\}\right].$$

We now show $R_i \supset ICR^n_i(t_i)$. Pick any conjecture $\nu_i \in \Delta\left(\Theta \times T^*_{-i} \times A_{-i}\right)$ such that $\operatorname{marg}_{\Theta \times T^*_{-i}} \nu_i = \kappa^*_{t_i}$ and $\nu_i \left[a_{-i} \in ICR^{n-1}_{-i}(t_{-i})\right] = 1$. Let $\pi_i = \operatorname{marg}_{\Theta \times A_{-i}} \nu_i$. Define φ_i as the conditional probability of ν_i on each $\left(\theta, \tau_{-i,R_{-i}}\right)$, i.e., $\varphi_i\left(\theta, R_{-i}\right)[a_{-i}] = \nu_i \left[a_{-i} \mid \theta, \tau_{-i,R_{-i}}\right]$. (If $\kappa^*_{t_i}\left[\theta, \tau_{-i,R_{-i}}\right] = 0$, then pick $\varphi_i(\theta, R_{-i}) \in \Delta(R_{-i})$ arbitrarily.) Then, (2) holds because $\nu_i \left[a_{-i} \in ICR^{n-1}_{-i}(t_{-i})\right] = 1$ and $R_{-i} \supset ICR^{n-1}_{-i}\left(\tau_{-i,R_{-i}}\right)$ for every $R_{-i} \in \mathcal{R}^{\uparrow,n-1}_{-i}$; (3) holds because

$$\begin{aligned}
\pi_i[\theta, a_{-i}] &= \operatorname{marg}_{\Theta \times A_{-i}} \nu_i[\theta, a_{-i}] \\
&= \sum_{t_{-i} \in T^*_{-i}} \kappa^*_{t_i}[\theta, t_{-i}] \nu_i \left[a_{-i} \mid \theta, t_{-i}\right] \\
&= \sum_{t_{-i} \in T^*_{-i}} \sum_{R_{-i} \in \mathcal{A}_{-i} : \tau_{-i,R_{-i}} = t_{-i}} \mu_i[\theta, R_{-i}] \varphi(\theta, R_{-i})[a_{-i}] \\
&= \sum_{R_{-i} \in \mathcal{A}_{-i}} \mu_i[\theta, R_{-i}] \varphi(\theta, R_{-i})[a_{-i}]
\end{aligned}$$

for every $(\theta, a_{-i}) \in \Theta \times A_{-i}$. Thus, we have $\pi_i \in \Pi^{\mu_i}_i$, and hence $R_i \supset BR_i(\pi_i)$. Therefore, we have $R_i \supset ICR^n_i(t_i)$. ■

We now turn to prove Proposition 2.

Proof of Proposition 2 (a) For "$\subset$", suppose that $R_i \in \mathcal{R}_i^{\uparrow,n}$ for some n. By Lemma 2(a), there exists $t_i \in T_i^*$ such that $R_i \supset ICR_i^n(t_i)$. Since $R_i \supset ICR_i^n(t_i) \supset ICR_i(t_i)$, we have $R_i \in \mathcal{R}_i^{\uparrow}$.

For "$\supset$", suppose that $R_i \in \mathcal{R}_i^{\uparrow}$. Then there exist $t_i \in T_i^*$ and m such that $R_i \supset ICR_i(t_i) = ICR_i^m(t_i)$. By Lemma 2(a), we have $R_i \in \mathcal{R}_i^{\uparrow,m} \subset \mathcal{R}_i^{\uparrow,n}$ for any $n \geq \sum_i 2^{|A_i|} - 2|I|$.

(b) For "$\subset$", suppose that $R_i \in \mathcal{R}_i^{\downarrow,n}$ for some $n \geq \sum_i 2^{|A_i|} - 2|I|$. For each m, since $R_i \in \mathcal{R}_i^{\downarrow,n} \subset \mathcal{R}_i^{\downarrow,m}$, by Lemma 2(b), there exists $t_{i,m} \in T_i^*$ such that $R_i \subset ICR_i^m(t_{i,m})$. Since T_i^* is a compact metric space, $\{t_{i,m}\}$ admits a convergent subsequence $\{t_{i,m_k}\}$. We denote its limit by t_i. For any m and $m_k \geq m$, we have $R_i \subset ICR_i^{m_k}(t_{i,m_k}) \subset ICR_i^m(t_{i,m_k})$. Since $t_{i,m_k} \to t_i$ as $k \to \infty$ and $ICR_i^m(\cdot)$ is upper hemicontinuous, we have $R_i \subset ICR_i^m(t_i)$. Since m is arbitrary, we have $R_i \subset ICR_i(t_i)$, and hence $R_i \in \mathcal{R}_i^{\downarrow}$.

For "$\supset$", suppose that $R_i \in \mathcal{R}_i^{\downarrow}$. Then there exists $t_i \in T_i^*$ such that $R_i \subset ICR_i(t_i) \subset ICR_i^n(t_i)$ for any n. By Lemma 2(b), we have $R_i \in \mathcal{R}_i^{\downarrow,n}$. ■

A.3 Proof of Theorem 1

We prove Theorem 1 in the following two lemmas.

Lemma 3 *$\mathcal{S}_i(t_i) \subset \mathcal{S}_i^*(t_i)$ for finite type t_i.*

Proof Define $\mathcal{S}_i^{*,0}(t_i) := \mathcal{R}_i^{\uparrow}$ and

$$\mathcal{S}_i^{*,n}(t_i) := \left\{R_i \in \mathcal{A}_i : \exists \{t_{i,m}\}_{m=0}^{\infty} \subset T_i^* \text{ s.t. } t_{i,m}^n \to t_i^n \text{ as } m \to \infty \text{ and } R_i \supset ICR_i(t_{i,m}), \forall m\right\}$$

for each $n \geq 1$. We show that $\mathcal{S}_i(t_i) \subset \mathcal{S}_i^{*,n}(t_i)$, and thus $\mathcal{S}_i(t_i) \subset \mathcal{S}_i^*(t_i)$ by taking a diagonal sequence. We fix a finite model (T, κ), and prove by induction that $\mathcal{S}_i(t_i) \subset \mathcal{S}_i^{*,n}(t_i)$. For $n = 0$, we have $\mathcal{S}_i(t_i) \subset \mathcal{R}_i^{\uparrow} = \mathcal{S}_i^{*,0}(t_i)$. Suppose that $\mathcal{S}_i(t_i) \subset \mathcal{S}_i^{*,n-1}(t_i)$ for any $i \in I$ and $t_i \in T_i$, and we prove $\mathcal{S}_i(t_i) \subset \mathcal{S}_i^{*,n}(t_i)$ for any $i \in I$ and $t_i \in T_i$. Let $i \in I$, $t_i \in T_i$, and $R_i \in \mathcal{S}_i(t_i)$. By the fixed-point property of $\mathcal{S}_i(\cdot)$, for each m, there exists $(\mu_{i,m}, \mu'_{i,m}) \in \Delta\left(\Theta \times T_{-i} \times \mathcal{R}_{-i}^{\uparrow}\right) \times \Delta\left(\Theta \times \mathcal{R}_{-i}^{\uparrow}\right)$ such that (i)-(iii) in (7) with $\varepsilon = \frac{1}{m+1}$ holds. First, for each $R_{-i} \in \mathcal{R}^{\uparrow}$, there exists $\tau_{-i,R_{-i}} \in T_{-i}^*$ such that $R_{-i} \supset ICR_{-i}\left(\tau_{-i,R_{-i}}\right)$. Also, for each

$t_{-i} \in T_{-i}$ and $R_{-i} \in \mathcal{S}_{-i}(t_{-i})$, by the induction hypothesis, there is some sequence of types $\left\{\tau_{t_{-i},R_{-i},m}\right\}_{m=0}^{\infty} \subset T^*_{-i}$ such that $\tau^{n-1}_{t_{-i},R_{-i},m} \to t^{n-1}_{-i}$ as $m \to \infty$ and $R_{-i} \supset ICR_{-i}\left(\tau_{t_{-i},R_{-i},m}\right)$ for every m. (If $n = 1$, we set $\tau_{t_{-i},R_{-i},m} = \tau_{-i,R_{-i}}$.) Define $t_{i,m} \in T^*_i$ with $\kappa^*_{t_{i,m}}$ having a finite support such that

$$\kappa^*_{t_{i,m}}[\theta, s_{-i}] = \frac{m}{m+1}\mu_{i,m}\left[\left\{(\theta, t_{-i}, R_{-i}) : \tau_{t_{-i},R_{-i},m} = s_{-i}\right\}\right] + \frac{1}{m+1}\mu'_{i,m}\left[\left\{(\theta, R_{-i}) : \tau_{-i,R_{-i}} = s_{-i}\right\}\right] \tag{11}$$

for every $(\theta, s_{-i}) \in \Theta \times T^*_{-i}$. Since $\tau^{n-1}_{t_{-i},R_{-i},m} \to t^{n-1}_{-i}$ as $m \to \infty$ and $\operatorname{marg}_{\Theta\times T_{-i}} \mu_{i,m} = \kappa_{t_i}$ for every m, it follows that $t^n_{i,m} \to t^n_i$ as $m \to \infty$.

Finally, we show that $R_i \supset ICR_i(t_{i,m})$ for every m. Pick any $a_i \in ICR_i(t_{i,m})$ and we show $a_i \in R_i$. Since $a_i \in ICR_i(t_{i,m})$, by (1), there is a valid conjecture $\nu_{i,m} \in \Delta\left(\Theta \times T^*_{-i} \times A_{-i}\right)$ for $t_{i,m}$ such that $a_i \in BR_i\left(\operatorname{marg}_{\Theta\times A_{-i}} \nu_{i,m}\right)$. Fix $\alpha_i \in \Delta(A_i \setminus \{a_i\})$. For each $(\theta, R_{-i}) \in \Theta \times \mathcal{R}^{\uparrow}_{-i}$, let $\psi^{\alpha_i}_{-i}(\theta, R_{-i}) \in R_{-i}$ be one of the action profiles of player i's opponents that favor action a_i most relative to α_i, i.e.,

$$\psi^{\alpha_i}_{-i}(\theta, R_{-i}) \in \arg\max_{a_{-i}\in R_{-i}} \left[u_i(\theta, a_i, a_{-i}) - u_i(\theta, \alpha_i, a_{-i})\right]. \tag{12}$$

Since $\operatorname{marg}_{\Theta\times T^*_{-i}} \nu_{i,m} = \kappa^*_{t_{i,m}}$, $\nu_{i,m}[a_{-i} \in ICR_{-i}(t_{-i})] = 1$, and $a_i \in BR_i\left(\operatorname{marg}_{\Theta\times A_{-i}} \nu_{i,m}\right)$, it follows that a_i is no worse than α_i against $\pi^*_{i,m}$, where

$$\pi^*_{i,m}[\theta, a_{-i}] = \kappa^*_{t_{i,m}}\left[\left\{(\theta, s_{-i}) : \psi^{\alpha_i}_{-i}(\theta, ICR_{-i}(s_{-i})) = a_{-i}\right\}\right] \tag{13}$$

for every $(\theta, a_{-i}) \in \Theta \times A_{-i}$. Let

$$\pi_{i,m}[\theta, a_{-i}] = \mu_{i,m}\left[\left\{(\theta, t_{-i}, R_{-i}) : \psi^{\alpha_i}_{-i}(\theta, ICR_{-i}(\tau_{t_{-i},R_{-i},m})) = a_{-i}\right\}\right], \tag{14}$$

$$\pi'_{i,m}[\theta, a_{-i}] = \mu'_{i,m}\left[\left\{(\theta, R_{-i}) : \psi^{\alpha_i}_{-i}(\theta, ICR_{-i}(\tau_{-i,R_{-i}})) = a_{-i}\right\}\right] \tag{15}$$

for every $(\theta, a_{-i}) \in \Theta \times A_{-i}$. Observe that $(\pi_{i,m}, \pi'_{i,m}) \in \Pi^{\mu_{i,m}}_i \times \Pi^{\mu'_{i,m}}_i$. Moreover,

$$\begin{aligned}
\pi^*_{i,m}[\theta, a_{-i}] &= \kappa^*_{t_{i,m}}\left[\left\{(\theta, s_{-i}) : \psi^{\alpha_i}_{-i}(\theta, ICR_{-i}(s_{-i})) = a_{-i}\right\}\right] \\
&= \frac{m}{m+1}\mu_{i,m}\left[\left\{(\theta, t_{-i}, R_{-i}) : \psi^{\alpha_i}_{-i}(\theta, ICR_{-i}(\tau_{t_{-i},R_{-i},m})) = a_{-i}\right\}\right] \\
&\quad + \frac{1}{m+1}\mu'_{i,m}\left[\left\{(\theta, R_{-i}) : \psi^{\alpha_i}_{-i}(\theta, ICR_{-i}(\tau_{-i,R_{-i}})) = a_{-i}\right\}\right] \\
&= \frac{m}{m+1}\pi_{i,m}[\theta, a_{-i}] + \frac{1}{m+1}\pi'_{i,m}[\theta, a_{-i}].
\end{aligned}$$

where the first equality follows from (13); the second follows from (11); the third follows from (14) and (15). Therefore, for each $\alpha_i \in \Delta(A_i \setminus \{a_i\})$, a_i is no worse than α_i against $\frac{m}{m+1}\pi_{i,m} + \frac{1}{m+1}\pi'_{i,m}$. By the usual duality argument, $a_i \in BR_i\left(\frac{m}{m+1}\hat{\pi}_{i,m} + \frac{1}{m+1}\hat{\pi}'_{i,m}\right)$ for some $(\hat{\pi}_{i,m}, \hat{\pi}'_{i,m}) \in \Pi_i^{\mu_{i,m}} \times \Pi_i^{\mu'_{i,m}}$. It then follows from (iii) in (7) that $a_i \in R_i$. ∎

Lemma 4 *$\mathcal{S}_i^*(t_i) \subset \mathcal{S}_i(t_i)$ for finite type t_i.*

Proof We fix a finite model (T, κ). We assume without loss of generality that (T, κ) is embedded in the universal type space (T^*, κ^*). We prove the claim by showing that for each $i \in I$, $t_i \in T_i$, $R_i \in \mathcal{S}_i^*(t_i)$, and $\varepsilon \in (0, 1]$, there exists $(\mu_i, \mu'_i) \in \Delta\left(\Theta \times T_{-i} \times \mathcal{R}_{-i}^{\uparrow}\right) \times \Delta\left(\Theta \times \mathcal{R}_{-i}^{\uparrow}\right)$ such that

(i) $\operatorname{marg}_{\Theta \times T_{-i}} \mu_i = \kappa_{t_i}$;

(ii) $\mu_i\left[\left\{(\theta, t_{-i}, R_{-i}) : R_{-i} \in \mathcal{S}_{-i}^*(t_{-i})\right\}\right] = 1$;

(iii) $R_i \supset \bigcup_{(\pi_i, \pi'_i) \in \Pi_i^{\mu_i} \times \Pi_i^{\mu'_i}} BR_i\left((1-\varepsilon)\pi_i + \varepsilon \pi'_i\right)$.

Consequently, by (7), we have $\mathcal{S}_i^*(t_i) \subset \mathcal{S}_i(t_i)$.

First, since $R_i \in \mathcal{S}_i^*(t_i)$, there exist $\{t_{i,m}\}_{m=0}^{\infty} \subset T_i^*$ such that $t_{i,m} \to t_i$ and $R_i \supset ICR_i(t_{i,m})$ for every m. For each m, we define $\mu_{i,m} \in \Delta\left(\Theta \times T_{-i}^* \times \mathcal{R}_{-i}^{\uparrow}\right)$ by

$$\mu_{i,m}\left[\{\theta\} \times E_{-i} \times \{R_{-i}\}\right] = \kappa_{t_{i,m}}^*\left[\{(\theta, s_{-i}) : s_{-i} \in E_{-i} \text{ and } ICR_{-i}(s_{-i}) = R_{-i}\}\right] \tag{16}$$

for every measurable $E_{-i} \subset T_{-i}^*$ and $(\theta, R_{-i}) \in \Theta \times \mathcal{R}_{-i}^{\uparrow}$. Since $\Delta\left(\Theta \times T_{-i}^* \times \mathcal{R}_{-i}^{\uparrow}\right)$ is a weak* compact metric space, $\{\mu_{i,m}\}_{m=0}^{\infty}$ admits a convergent subsequence $\{\mu_{i,m_k}\}_{k=0}^{\infty}$. We denote its limit by μ_i. Second, we show that μ_i satisfies (i) and (ii). By the definition of $\mu_{i,m}$, we know that $\operatorname{marg}_{\Theta \times T_{-i}^*} \mu_{i,m} = \kappa_{t_{i,m}}$. Since $\mu_{i,m_k} \to \mu_i$ as $k \to \infty$ and $t_{i,m} \to t_i$ as $m \to \infty$, it follows that $\operatorname{marg}_{\Theta \times T_{-i}^*} \mu_i = \kappa_{t_i}$, i.e., (i) holds. In particular, we have $\mu_i \in \Delta\left(\Theta \times T_{-i} \times \mathcal{R}_{-i}^{\uparrow}\right)$.

To prove (ii), for each $\ell \in \mathbb{N}$, let

$$F_\ell = \operatorname{cl}\left\{(\theta, s_{-i}, R_{-i}) : \exists s'_{-i} \in T_{-i}^* \text{ s.t. } d_{-i}(s'_{-i}, s_{-i}) \leq \frac{1}{\ell} \text{ and } ICR_{-i}\left(s'_{-i}\right) = R_{-i}\right\},$$

$$F_\infty = (\Theta \times T_{-i} \times \mathcal{R}_{-i}^{\uparrow}) \cap \bigcap_{\ell \in \mathbb{N}} F_\ell,$$

where d_{-i} is the metric on T^*_{-i}. Note that

$$F_\ell \supset \{(\theta, t_{-i}, R_{-i}) : ICR_{-i}(t_{-i}) = R_{-i}\}, \forall \ell, \tag{17}$$

$$F_\infty \subset \left\{(\theta, t_{-i}, R_{-i}) : t_{-i} \in T_{-i} \text{ and } R_{-i} \in \mathcal{S}^*_{-i}(t_{-i})\right\}. \tag{18}$$

Hence, (16) and (17) imply that $\mu_{i,m}[F_\ell] \geq \mu_{i,m}[ICR_{-i}(t_{-i}) = R_{-i}] = 1$, i.e., $\mu_{i,m}[F_\ell] = 1$ for all ℓ. Since F_ℓ is closed and $\mu_{i,m_k} \to \mu_i$ as $k \to \infty$, we have $\mu_i[F_\ell] = 1$ for all ℓ. As a result, $\mu_i[\bigcap_{\ell \in \mathbb{N}} F_\ell] = 1$. Combining this with $\mu_i\left[\Theta \times T_{-i} \times \mathcal{R}^{\uparrow}_{-i}\right] = 1$, we have $\mu_i[F_\infty] = 1$, which, together with (18), implies (ii).

Finally, we prove (iii). First, let

$$\mu'_{i,m} = \frac{1}{\varepsilon}\left(\mathrm{marg}_{\Theta \times \mathcal{R}^{\uparrow}_{-i}} \mu_{i,m} - (1-\varepsilon)\, \mathrm{marg}_{\Theta \times \mathcal{R}^{\uparrow}_{-i}} \mu_i\right). \tag{19}$$

Since $\mu_{i,m_k} \to \mu_i$, pick k sufficiently large so that $\mu'_{i,m_k}[\theta, R_{-i}] \geq 0$ for every $(\theta, R_{-i}) \in \Theta \times \mathcal{R}^{\uparrow}_{-i}$, and hence $\mu'_{i,m_k} \in \Delta\left(\Theta \times \mathcal{R}^{\uparrow}_{-i}\right)$. Now fix any $a_i \in A_i$ such that $a_i \in BR_i((1-\varepsilon)\pi_i + \varepsilon\pi'_i)$ for some $(\pi_i, \pi'_i) \in \Pi_i^{\mu_i} \times \Pi_i^{\mu'_{i,m_k}}$, and we show $a_i \in R_i$. Fix $\alpha_i \in \Delta(A_i \setminus \{a_i\})$. For each $(\theta, R_{-i}) \in \Theta \times \mathcal{R}^{\uparrow}_{-i}$, define $\psi^{\alpha_i}_{-i}(\theta, R_{-i})$ as in (12). Then since $a_i \in BR_i((1-\varepsilon)\pi_i + \varepsilon\pi'_i)$ for some $(\pi_i, \pi'_i) \in \Pi_i^{\mu_i} \times \Pi_i^{\mu'_{i,m_k}}$, it follows that

$$\int_{\Theta \times \mathcal{R}^{\uparrow}_{-i}} [u_i(\theta, a_i, \psi^{\alpha_i}_{-i}(\theta, R_{-i})) - u_i(\theta, \alpha_i, \psi^{\alpha_i}_{-i}(\theta, R_{-i}))]d\left((1-\varepsilon)\,\mathrm{marg}_{\Theta \times \mathcal{R}^{\uparrow}_{-i}} \mu_i + \varepsilon\mu'_{i,m_k}\right) \geq 0. \tag{20}$$

Let $\nu_{i,m_k} \in \Delta\left(\Theta \times T^*_{-i} \times A_{-i}\right)$ be such that

$$\nu_{i,m_k}[\{\theta\} \times E_{-i} \times \{a_{-i}\}] = \kappa^*_{t_{i,m_k}}[\{(\theta, t_{-i}) : t_{-i} \in E_{-i} \text{ and } \psi^{\alpha_i}_i(\theta, ICR_{-i}(t_{-i})) = a_{-i}\}] \tag{21}$$

for every measurable $E_{-i} \subset T^*_{-i}$ and $(\theta, a_{-i}) \in \Theta \times A_{-i}$. Since $\psi^{\alpha_i}_i(\theta, ICR_{-i}(t_{-i})) \in$

$ICR_{-i}(t_{-i})$, ν_{i,m_k} is a valid conjecture. We then have

$$\begin{aligned}
&\int_{\Theta\times T^*_{-i}\times A_{-i}} [u_i(\theta, a_i, a_{-i}) - u_i(\theta, \alpha_i, a_{-i})] d\nu_{i,m_k} \\
&= \int_{\Theta\times T^*_{-i}} [u_i(\theta, a_i, \psi^{\alpha_i}_{-i}(\theta, ICR_{-i}(t_{-i}))) - u_i(\theta, \alpha_i, \psi^{\alpha_i}_{-i}(\theta, ICR_{-i}(t_{-i})))] d\kappa^*_{t_{i,m_k}} \\
&= \int_{\Theta\times T^*_{-i}\times \mathcal{R}^{\uparrow}_{-i}} [u_i(\theta, a_i, \psi^{\alpha_i}_{-i}(\theta, R_{-i})) - u_i(\theta, \alpha_i, \psi^{\alpha_i}_{-i}(\theta, R_{-i}))] d\mu_{i,m_k} \\
&= \int_{\Theta\times \mathcal{R}^{\uparrow}_{-i}} [u_i(\theta, a_i, \psi^{\alpha_i}_{-i}(\theta, R_{-i})) - u_i(\theta, \alpha_i, \psi^{\alpha_i}_{-i}(\theta, R_{-i}))] d\left((1-\varepsilon)\operatorname{marg}_{\Theta\times\mathcal{R}^{\uparrow}_{-i}} \mu_i + \varepsilon\mu'_{i,m_k}\right) \\
&\geq 0,
\end{aligned}$$

where the three equalities follow from (21), (16), and (19), respectively, and the inequality follows from (20). Therefore, for each $\alpha_i \in \Delta(A_i \setminus \{a_i\})$, there exists a valid conjecture ν_{i,m_k} for t_{i,m_k} against which a_i is no worse than α_i. Then it follows from the usual duality argument that we can find a valid conjecture for t_{i,m_k}, independent of α_i, against which a_i is a best reply. By (1), we have $a_i \in ICR_i(t_{i,m_k}) \subset R_i$. ■

A.4 Proof of Proposition 3

To prove Proposition 3, we need the following lemma.

Lemma 5 *$W_i \cap R_i \neq \emptyset$ for every $i \in I$ and $R_i \in \mathcal{R}^{\uparrow}_i$. Moreover, $W_i \cap BR_i(\pi_i) \neq \emptyset$ for every $\pi_i \in \Delta(\Theta \times W_{-i})$.*

Proof We will show

$$\forall n \in \mathbb{N}, \forall i \in I, R_i \in \mathcal{R}^{\uparrow,n}_i, W^n_i \cap R_i \neq \emptyset$$

by induction on n. Lemma 5 will then follow as $n \to \infty$. The case with $n = 0$ is obvious.

Assume the case with $n-1$. Pick any $i \in I$ and $R_i \in \mathcal{R}^{\uparrow,n}_i$. By the definition of $\mathcal{R}^{\uparrow,n}_i$, there exists $\mu_i \in \Delta(\Theta \times \mathcal{R}^{\uparrow,n-1}_{-i})$ such that $R_i \supset \bigcup_{\pi_i\in\Pi^{\mu_i}_i} BR_i(\pi_i)$. By the induction hypothesis, we have $\Pi_{i,W^{n-1}} \neq \emptyset$. By the construction and nonemptiness of $\Pi^{\mu_i}_{i,W^{n-1}}$, we have $W_i^n \cap \bigcup_{\pi_i \in \Pi^{\mu_i}_{i,W^{n-1}}} BR_i(\pi_i) \neq \emptyset$. Thus, we have

$$W_i^n \cap R_i \supset W_i^n \cap \bigcup_{\pi_i \in \Pi_i^{\mu_i}} BR_i(\pi_i) \supset W_i^n \cap \bigcup_{\pi_i \in \Pi^{\mu_i}_{i,W^{n-1}}} BR_i(\pi_i) \neq \emptyset.$$

■

We now turn to prove Proposition 3. In the proof, for any $i \in I$, $\mu_i \in \Delta(\Theta \times T_{-i} \times \mathcal{R}^{\uparrow}_{-i})$, we define $\Pi^{\mu_i}_{i,W}$ as the set of all conjectures $\pi_i \in \Delta(\Theta \times W_{-i})$ such that

$$\varphi_i(\theta, t_{-i}, R_{-i})[a_{-i}] > 0 \Rightarrow a_{-i} \in ICR^{n-1}_{-i,W}(t_{-i}) \cap R_{-i};$$

$$\pi_i[\theta, a_{-i}] = \sum_{t_{-i}, R_{-i} : R_{-i} \in \mathcal{S}^{n-1}_{-i}(t_{-i})} \mu_i\left[\theta, t_{-i}, R_{-i}\right] \varphi_{-i}(\theta, t_{-i}, R_{-i})[a_{-i}]$$

for some $\varphi_i \colon \Theta \times T_{-i} \times \mathcal{R}^{\uparrow}_{-i} \to \Delta(W_{-i})$.

Proof of Proposition 3 We will show

$$\forall n \in \mathbb{N}, \forall i \in I, \forall t_i \in T_i, \forall R_i \in \mathcal{S}_i^n(t_i),\ ICR^n_{i,W}(t_i) \cap R_i \neq \emptyset$$

by induction on n. Proposition 3 will then follow as $n \to \infty$.

The case with $n = 0$ follows from Lemma 5.

Assume the case with $n - 1$. Pick any $i \in I$, $t_i \in T_i$, and $R_i \in \mathcal{S}_i^n(t_i)$. Pick sufficiently small $\varepsilon > 0$. By the definition of $\mathcal{S}_i^n(t_i)$, there exists $(\mu_i, \mu_i') \in \Delta(\Theta \times T_{-i} \times \mathcal{R}^{\uparrow}_{-i}) \times \Delta(\Theta \times \mathcal{R}^{\uparrow}_{-i})$ such that $\mathrm{marg}_{\Theta \times T_{-i}} \mu_i = \kappa_{t_i}$, $\mu_i[\{(\theta, t_{-i}, R_{-i}) : R_{-i} \in \mathcal{S}^{n-1}_{-i}(t_{-i})\}] = 1$, and

$$R_i \supset \bigcup_{(\pi_i, \pi_i') \in \Pi_i^{\mu_i} \times \Pi_i^{\mu_i'}} BR_i((1 - \varepsilon)\pi_i + \varepsilon \pi_i'). \tag{22}$$

By the induction hypothesis, we have $\Pi^{\mu_i}_{i,W} \neq \emptyset$. Similarly, by Lemma 5, we have $\Pi^{\mu_i'}_{i,W} \neq \emptyset$. Also, by the construction of $\Pi^{\mu_i}_{i,W}$, we have

$$ICR^n_{i,W}(t_i) \supset W_i \cap \bigcup_{\pi_i \in \Pi^{\mu_i}_{i,W}} BR_i(\pi_i).$$

Since ε is sufficiently small, by the upper hemicontinuity of the best response correspondence and the compactness of $\Pi^{\mu_i}_{i,W}$ and $\Pi^{\mu_i'}_{i,W}$, we have

$$ICR^n_{i,W}(t_i) \supset W_i \cap \bigcup_{(\pi_i, \pi_i') \in \Pi^{\mu_i}_{i,W} \times \Pi^{\mu_i'}_{i,W}} BR_i((1 - \varepsilon)\pi_i + \varepsilon \pi_i') \neq \emptyset, \tag{23}$$

where the nonemptiness follows from Lemma 5 and the construction and nonemptiness of $\Pi^{\mu_i}_{i,W}$ and $\Pi^{\mu'_i}_{i,W}$.

Thus, we have

$$\begin{aligned}
ICR^n_{i,W}(t_i) \cap R_i &\supset ICR^n_{i,W}(t_i) \cap \bigcup_{(\pi_i,\pi'_i)\in\Pi^{\mu_i}_i\times\Pi^{\mu'_i}_i} BR_i((1-\varepsilon)\pi_i + \varepsilon\pi'_i) \\
&\supset ICR^n_{i,W}(t_i) \cap \bigcup_{(\pi_i,\pi'_i)\in\Pi^{\mu_i}_{i,W}\times\Pi^{\mu'_i}_{i,W}} BR_i((1-\varepsilon)\pi_i + \varepsilon\pi'_i) \\
&= W_i \cap \bigcup_{(\pi_i,\pi'_i)\in\Pi^{\mu_i}_{i,W}\times\Pi^{\mu'_i}_{i,W}} BR_i((1-\varepsilon)\pi_i + \varepsilon\pi'_i) \\
&\neq \emptyset,
\end{aligned}$$

where the first set inclusion follows from (22), the second set inclusion follows from $\Pi^{\mu_i}_i \supset \Pi^{\mu_i}_{i,W}$ and $\Pi^{\mu'_i}_i \supset \Pi^{\mu'_i}_{i,W}$, and the rest follows from (23). ■

A.5 Proof of Proposition 4

Proof of Proposition 4 (a) The "only if" direction is obvious. To show the "if" direction, let $r = (|I|-1)/2 \geq 1$ and $x = (\theta_H - r\theta_L)/2 > 0$.

Claim 1 *For any $m \geq 1$, there exist $\tau_{i,m,L}, \tau_{i,m,H} \in T^*_i$ such that*

$$\begin{aligned}
ICR_i(\tau_{i,1,L}) &\subset \left[0, \frac{\theta_L}{2}\right], \\
ICR_i(\tau_{i,m,L}) &\subset \left[0, \max\left(0, \frac{\theta_L}{2} - (r + r^3 + \cdots + r^{2m-3})x\right)\right], \forall m \geq 2 \\
ICR_i(\tau_{i,m,H}) &\subset \left[\min\left(\frac{\theta_H}{2}, (1 + r^2 + \cdots + r^{2m-2})x\right), \frac{\theta_H}{2}\right], \forall m \geq 1.
\end{aligned}$$

Proof of Claim 1 We construct desired types inductively. For $m = 1$, we can take type $\tau_{i,1,L}$ to be any type whose first-order belief puts probability 1 on $\theta = \theta_L$. Then we have $ICR_i(\tau_{i,1,L}) \subset [0, \theta_L/2]$.

For any $m \geq 1$, we take type $\tau_{i,m,H}$ to be the type who puts probability 1 on $\theta = \theta_H$ and $t_j = \tau_{j,m,L}$ for any $j \neq i$. By (1) and the induction hypothesis, any action that is rationalizable for $\tau_{i,m,H}$ is bounded from below by

$$\frac{1}{2}\left(\theta_H - (|I|-1)\max\left(0, \frac{\theta_L}{2} - (r + r^3 + \cdots + r^{2m-3})x\right)\right) = \min\left(\frac{\theta_H}{2}, (1 + r^2 + \cdots + r^{2m-2})x\right).$$

Thus we have $ICR_i(\tau_{i,m,H}) \subset [\min(\theta_H/2, (1 + r^2 + \cdots + r^{2m-2})x), \theta_H/2]$.

Similarly, for any $m \geq 2$, we take type $\tau_{i,m,L}$ to be the type who puts probability 1 on $\theta = \theta_L$ and $t_j = \tau_{j,m-1,H}$ for any $j \neq i$. By (1) and the induction hypothesis, any action that is rationalizable for $\tau_{i,m,L}$ is bounded from above by

$$\begin{aligned}
&\max\left(0, \frac{1}{2}\left(\theta_L - (|I|-1)\min\left(\frac{\theta_H}{2}, (1 + r^2 + \cdots + r^{2m-4})x\right)\right)\right) \\
&= \max\left(0, \frac{\theta_L}{2} - (r + r^3 + \cdots + r^{2m-3})x\right).
\end{aligned}$$

Thus we have $ICR_i(\tau_{i,m,L}) \subset [0, \max(0, \theta_L/2 - (r + r^3 + \cdots + r^{2m-3})x)]$. ■

Claim 2 *For any $n \geq 0$ and any*

$$q \in \left[0, \min\left(\frac{\theta_H}{2}, (1 + r^2 + \cdots + r^{2n})x\right)\right] \cup \left[\max\left(0, \frac{\theta_L}{2} - (r + r^3 + \cdots + r^{2n+1})x\right), \frac{\theta_H}{2}\right],$$

there exists $\tau_{i,q} \in T_i^*$ such that $ICR_i(\tau_{i,q}) = \{q\}$.

Proof of Claim 2 We construct desired types inductively. For $n = 0$, we take $\tau_{i,0} = \tau_{i,m,L}$ in Claim 1 with sufficiently large m. Then we have $ICR_i(\tau_{i,0}) = \{0\}$.

Also, for $n = 0$ and any $q \in [\theta_L/2, \theta_H/2]$, we take $\tau_{i,q}$ to be the type who puts probability $(2q - \theta_L)/(\theta_H - \theta_L)$ on $\theta = \theta_H$ and $t_j = \tau_{j,0}$ for any $j \neq i$, and probability $(\theta_H - 2q)/(\theta_H - \theta_L)$ on $\theta = \theta_L$ and $t_j = \tau_{j,0}$ for any $j \neq i$. By (1), we have $ICR_i(\tau_{i,q}) = \{q\}$.

For any $n \geq 1$ and any $q \in [0, \min(\theta_H/2, (1 + r^2 + \cdots + r^{2n})x)]$, let $q' = (\theta_H - 2q)/(|I| - 1)$. Since

$$\begin{aligned}
q' &\in \left[\frac{\theta_H - 2\min(\theta_H/2, (1 + r^2 + \cdots + r^{2n})x)}{|I|-1}, \frac{\theta_H}{|I|-1}\right] \\
&\subset \left[\max\left(0, \frac{\theta_L}{2} - (r + r^3 + \cdots + r^{2n+1})x\right), \frac{\theta_H}{2}\right]
\end{aligned}$$

by the induction hypothesis, there exists $\tau_{j,q'} \in T_i^*$ such that $ICR_i(\tau_{i,q'}) = \{q'\}$. We take $\tau_{i,q}$ to be the type who puts probability 1 on $\theta = \theta_H$ and $t_j = \tau_{j,q'}$ for any $j \neq i$. By (1), we have $ICR_i(\tau_{i,q}) = \{q\}$.

Similarly, for any $n \geq 1$ and any $q \in [\max(0, \theta_L/2 - (r + r^3 + \cdots + r^{2n+1})x), \theta_H/2]$, if $q \geq \theta_L/2$, then the desired $\tau_{i,q}$ is already constructed in the case of $n = 0$. If $q < \theta_L/2$, then let $q'' = (\theta_L - 2q)/(|I| - 1)$. Since

$$q'' \in \left(0, \frac{\theta_L - 2\max(0, \theta_L/2 - (r + r^3 + \cdots + r^{2n+1})x)}{|I| - 1}\right] \subset \left[0, \min\left(\frac{\theta_H}{2}, (1 + r^2 + \cdots + r^{2n})x\right)\right]$$

by the induction hypothesis, there exists $\tau_{j,q''} \in T_i^*$ such that $ICR_i(\tau_{i,q''}) = \{q''\}$. We take $\tau_{i,q}$ to be the type who puts probability 1 on $\theta = \theta_L$ and $t_j = \tau_{j,q''}$ for any $j \neq i$. By (1), we have $ICR_i(\tau_{i,q}) = \{q\}$. ■

By taking $n \to \infty$ in Claim 2, we can construct $\tau_{i,q} \in T_i^*$ for any $q \in [0, \theta_H/2]$.

(b) For each $t_i \in T_i^*$, we have $ICR_i^1(t_i) = \left[0, \mathbb{E}_{t_i^1}(\theta)/2\right]$. For each $t_i \in T_i^*$ and $q \in \left[0, \mathbb{E}_{t_i^1}(\theta)/2\right]$, let $q(t_i) = \left(\mathbb{E}_{t_i^1}(\theta) - 2q\right)/(|I| - 1)$. Then q is a best response to the conjecture ν_i such that $\text{marg}_{\Theta \times T_{-i}^*} \nu_i = \kappa_{t_i}^*$ and $\nu_i[a_{-i} = q(t_i)] = 1$. Also,

$$q(t_i) \in \left[0, \frac{\mathbb{E}_{t_i^1}(\theta)}{|I| - 1}\right] \subset \left[0, \frac{\theta_H}{|I| - 1}\right] \subset \left[0, \frac{\theta_L}{2}\right] \subset \left[0, \frac{\mathbb{E}_{t_i^1}(\theta)}{2}\right]$$

for any $t_{-i} \in T_i^*$. Thus we have $ICR_i(t_i) = \left[0, \mathbb{E}_{t_i^1}(\theta)/2\right]$. ■

A.6 Proof of Proposition 5

First, suppose that $\left(\widetilde{\mathcal{S}}_i\right)_{i \in I}$ is an $\mathcal{R}^{\uparrow}$-perturbed curb collection. Then, the fact that $\widetilde{\mathcal{S}}_i(t_i) \subset \mathcal{S}_i^*(t_i)$ follows from the proof of Lemma 3 in Appendix A.3 by noting that $\left(\widetilde{\mathcal{S}}_i\right)_{i \in I}$ satisfies the same fixed-point property as $(\mathcal{S}_i)_{i \in I}$ in (7); moreover, the measurability of $\tau_{t_{-i}, R_{-i}, m}$ on T_{-i}^* is ensured by the measurability of μ_{t_j, R_i} on t_j for every j.

Second, the fact that $(\mathcal{S}_i^*|_{T_i})_{i \in I}$ is an $\mathcal{R}^{\uparrow}$-perturbed curb collection follows from the proof of Lemma 4 in Appendix A.3 by adding the following step to ensure the measurability of μ: For each $t_i \in T_i$ and each $R_i \in \mathcal{S}_i^*(t_i)$, let Δ_{t_i, R_i} be the set of weak* limits of all $\mu_{t_{i,m}} \in \Delta\left(\Theta \times T_{-i}^* \times \mathcal{A}_{-i}\right)$ such that $\{t_{i,m}\} \to t_i$ and $S_i^{\infty}(t_{i,m}) = R_i$ for all m, where $\mu_{t_{i,m}}$ is defined as $\mu_{i,m}$ in (16). By the compactness of $\Delta(\Theta \times T_{-i}^* \times \mathcal{A}_{-i})$, we have $\Delta_{t_i,R_i} \neq \varnothing$. Also Δ_{t_i,R_i} depends on (t_i, R_i) upper hemicontinuously. Thus, it follows from the Kuratowski–Ryll-Nardzewski selection theorem that we have a measurable function $\mu : T_i \times \mathcal{A}_i \to \Delta\left(\Theta \times T_{-i} \times \mathcal{R}_{-i}^{\uparrow}\right)$ such that $\mu_{t_i,R_i} \in \Delta_{t_i,R_i}$ whenever $R_i \in \mathcal{S}_i^*(t_i)$.

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