Abstract

An important question in mechanism design is whether there is any theoretical foundation for the use of dominant-strategy mechanisms. Following Chung and Ely (2007) who study maxmin and Bayesian foundations of dominant-strategy mechanisms in the single unit auction setting, we revisit these foundations in general social choice environments with quasi-linear preferences and private values. Based on a graph-theoretic formulation of incentive constraints, we propose a condition called uniform graph that, under regularity, ensures foundations of dominant-strategy mechanisms. We show that the uniform graph condition is satisfied in several classical environments including single unit auction, public good, bilateral trade as well as some multi-dimensional environments. When the uniform graph condition is not satisfied, we show that, by means of a bilateral trade model with ex ante unidentified traders, maxmin/ Bayesian foundations might not exist.

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1 Introduction

Wilson (1987) criticizes applied game theory’s reliance on common-knowledge assumptions. This has motivated the search for detail-free mechanisms in mechanism design. In practice, this has meant to use simple mechanisms such as dominant-strategy mechanisms. A dominant-strategy mechanism does not rely on any assumptions of agents’ beliefs and is robust to changes in agents’ beliefs. However, dominant-strategy mechanisms constitute just one special class of detail-free mechanisms and a fundamental issue is to justify the leap from detail-free mechanisms in general to dominant-strategy mechanisms in particular. In an important paper, Chung and Ely (2007) study maxmin and Bayesian foundations for using dominant-strategy mechanisms in the context of an expected revenue maximizing auctioneer. In this paper, we revisit this question in general social choice environments with quasi-linear preferences and private values. Our objective is to explore in some generality the class of environments for which such foundations of dominant-strategy mechanisms exist. This will help expose the underlying logic of the existence of such foundations in the single unit auction setting, as well as extend the argument to cases where it was hitherto unknown.

Suppose that the mechanism designer has an estimate of the distribution of the agents’ payoff types, but he does not have any reliable information about the agents’ beliefs (including their beliefs about one another’s payoff types, their beliefs about these beliefs, etc.), as these are arguably difficult to observe. Furthermore, the mechanism designer does not want to make any restrictive assumptions about such beliefs. The mechanism designer is a maxmin decision maker. That is, he ranks mechanisms according to their worst-case performance - the minimum expected revenue - where the minimum is taken over all possible agents’ beliefs. The use of dominant-strategy mechanisms has a maxmin foundation if the mechanism designer finds it optimal to use a dominant-strategy mechanism.

The notion of maxmin foundation is best illustrated graphically. Consider Figure 1 excerpted from Chung and Ely (2007), where the horizontal axis represents different assumptions about (distributions of) agents’ beliefs, the vertical axis represents the mechanism designer’s expected utility and different lines represent different mechanisms. Dominant-

\[1\text{While this issue arises for mechanism designers with various objectives (for example, revenue maximization or implementation of a certain social choice function), this paper is mainly concerned with revenue maximization. See Section 6.2 for a brief discussion when the mechanism design is interested in the implementation of a certain social choice function.}\]
strategy mechanisms (including the optimal dominant-strategy mechanism) always secure a fixed expected utility for the mechanism designer, being independent of agents’ beliefs. Therefore, the optimal dominant-strategy mechanism is represented by the straight line. Graphically, maximin foundation means that the graph of an arbitrary detail-free mechanism must dip below (or intersect) the graph of the optimal dominant-strategy mechanism at some point.

Figure 1: The graph of any detail-free mechanism dips below (or intersects) the graph of the optimal dominant-strategy mechanism at some point.

A closely related notion is Bayesian foundation. We say that the use of dominant-strategy mechanisms has a Bayesian foundation if there is a particular assumption about (the distribution of) agents’ beliefs, against which the optimal dominant-strategy mechanism achieves the highest expected utility among all detail-free mechanisms. If there exists such an assumption, then the worse case expected utility of an arbitrary detail-free mechanism cannot exceed its expected utility against this particular assumption, which in turn cannot exceed the (fixed) expected utility of the optimal dominant-strategy mechanism. Therefore, Bayesian foundation is a stronger notion than maxmin foundation.

In the context of an expected revenue maximizing auctioneer, Chung and Ely (2007) show that a regularity condition on the distribution of the bidders’ valuations ensures the maxmin/Bayesian foundations for dominant-strategy mechanisms. It is well known that the revenue maximizing dominant-strategy auction can be found by solving a "relaxed" problem
which, roughly speaking, corresponds to an assumption that only the "downward" incentive constraints for bidders are binding. More importantly, the set of binding incentive constraints is robust to changes in the decision rule or other bidders’ reports. As it turns out, this embedded structure in the single unit auction setting is the key for Chung and Ely’s results.

Based on a graph-theoretic formulation of incentive constraints, we formulate such a structure and examine its applicability in many important economic environments. Consider the optimal dominant-strategy mechanism design problem. Fix a decision rule $p$, we can find the optimal transfer scheme $t^*$. Fix a decision rule $p$ (and its corresponding transfer scheme $t^*$) and other agents’ reports $v_{-i}$, we say payoff type $v'_i$ is a best deviation of payoff type $v_i$ if agent $i$ is indifferent between reporting truthfully and reporting $v'_i$ when his payoff type is $v_i$. Best deviation serves as a building block for the graph. Best deviation and the graph are closely related with the set of binding incentive constraints and can be considered as an order-based interpretation of incentive constraints.\footnote{Such interpretation of incentive constraints has been used before by Rochet (1987), Border and Sobel (1987), Rochet and Stole (2003), Heydenreich, Müller, Uetz, and Vohra (2009) and Vohra (2011).}

2 Say there is uniform graph if for each agent $i$, the graph is independent of the decision rule $p$ or other agents’ report $v_{-i}$. We show that under regularity, uniform graph ensures the maxmin/ Bayesian foundations for dominant-strategy mechanisms.

As in Chung and Ely (2007), for the exposition of our theorem, we assume the mechanism designer maximizes expected revenue. The main contribution of this paper is, for expected revenue maximizing mechanism designers, we identify a sufficient condition for the maxmin/ Bayesian foundations of dominant-strategy mechanisms. But it is clear from our proof that our result holds as long as the mechanism designer’s utility function is increasing with respect to the agents’ transfers, for example, revenue/ profit maximization, welfare maximization with budget balance, etc.

Probably less apparently, our result also has implications if the mechanism designer is interested in the implementation of a social choice function. Consider the standard social choice environment with linear utilities and one-dimensional types, suppose a social choice function $p$ is dsIC with a transfer scheme $t$, we can increase the agents’ transfers such that only the adjacent downward incentive constraints are binding (and no constraints are violated). Hence, the modified transfer scheme also implements the social choice function. A useful observation is that, this property still holds even if we are restricted to use mechanisms that
satisfy ex post budget balance (in the weak sense), since such modification of transfer scheme only increases the transfers from the agents. This observation connects our result to the equivalence result of Bergemann and Morris (2005) (see Section 6.2).

Uniform graph is naturally satisfied in environments with linear utilities and one-dimensional types. This fits many classical applications of mechanism design, including single unit auction (e.g., Myerson (1981)), public good (e.g., Mailath and Postlewaite (1990)) and standard bilateral trade (e.g., Myerson and Satterthwaite (1983)). Our results also hold for multi-unit auctions with homogeneous or heterogeneous goods, combinatorial auctions and the like, as long as the agents’ private values are one-dimensional and utilities are linear. In this case, the payoff types are completely ordered via a single path. Uniform graph can also be satisfied in some multi-dimensional environments. For multi-unit auctions with capacitated bidders (a representative multi-dimensional environment), the agent’s payoff types are located on different paths and are only partially ordered.

When uniform graph is not satisfied, maxmin/ Bayesian foundations might not exist. We consider a bilateral trade model with ex ante unidentified traders. In this important economic environment, we construct an example whereby uniform graph is not satisfied and we explicitly construct a single Bayesian mechanism that does strictly better than the optimal dominant-strategy mechanism, regardless of the assumption about (the distribution of) agents’ beliefs. In other words, there is no maxmin/ Bayesian foundation.\(^3\)

We conclude with several discussions. We discuss a closely related paper, Yamashita and Zhu (2014) that study "digital-goods" auctions in the interdependent setting. We also show that in the standard social choice environment with linear utilities and one-dimensional types, we can still obtain the equivalence result in Bergemann and Morris (2005) in the quasi-linear private value environment with weak budget balance.

The rest of the paper is organized as follows. The reminder of this introduction discusses some related literature. Section 2 presents the notations, concepts and the model. Section 3 focuses on the standard social choice environment with linear utilities and one-dimensional types. In this leading environment whereby the uniform graph condition is satisfied, we present our results and also apply our results to the classical single unit auction, public

\(^3\) (Chung and Ely, 2007, Proposition 2) present an example whereby their regularity condition is violated and for each assumption about the agents’ beliefs, there is a Bayesian mechanism that does strictly better than the optimal dominant-strategy mechanism. Thus, while the example shows that a Bayesian foundation does not exist, it leaves open the existence of a maxmin foundation in a single unit auction setting.
good provision and bilateral trade. Section 4 formulates the notion of uniform graph and presents our main result in the general setting. In Section 5, we consider a prominent bilateral trade model with ex ante unidentified traders and illustrate that when uniform graph is not satisfied, the maxmin/ Bayesian foundations might not exist. Section 6 concludes with several discussions. The appendix contains proofs omitted from the main body of the paper.

1.1 Related literature

In a seminal paper, Bergemann and Morris (2005) study the implementability of a given social choice correspondence (SCC) and ask when ex post implementation (equivalent to dominant-strategy implementation in private value settings) is equivalent to interim (or Bayesian) implementation for all possible type spaces. They focus exclusively on mechanisms in which the outcome can depend only on payoff-relevant data and these mechanisms are naturally suited to study efficient design. In contrast, Chung and Ely (2007) and this paper are interested in revenue maximization for the mechanism designer and the optimal mechanism will almost always depend not just on the payoff types, but also on payoff-irrelevant data such as beliefs and higher order beliefs.

Nevertheless, we highlight the following similarity. Under a separability condition, Bergemann and Morris (2005) show that if a SCC cannot be implemented ex post, it cannot be interim implemented for all type spaces. In other words, Bayesian mechanisms cannot do better (in terms of implementing the SCC) than the ex post incentive compatible mechanisms at some type space. This is reminiscent of the notion of maxmin foundation in our setting, but the mechanism designer in our setting has a different objective - revenue maximization.

This paper joins a growing literature exploring mechanism design with worst case objectives. This includes the seminal work of Hurwicz and Shapiro (1978), Bergemann and Morris (2005), Chung and Ely (2007) and more recently, Carroll (forthcoming), Yamashita (2014) and work such as Yamashita (forthcoming) that studies the mechanism design problem of guaranteeing desirable performances whenever agents are rational in the sense of not playing weakly dominated strategies.

Another contribution of this paper is to add to the family of results where Bayesian mechanism robustly improves over dominant-strategy mechanisms. Bergemann and Morris (2005) points out that in non-separable environments, dominant-strategy implementability may be a stronger requirement than Bayesian implementability on all type spaces. In
particular, in the prominent quasilinear environment with budget balance, once there are
more than two agents and at least one agent has at least three types, a SCC can be interim
implementable on all type spaces without being ex post implementable. In this paper, we
highlight the role of uniform graph in the foundation of dominant-strategy mechanisms and
illustrate in Section 5 that, in environments where uniform graph is not satisfied, a single
Bayesian mechanism can robustly achieve strictly higher expected revenue than the optimal
dominant-strategy mechanism.

Lastly, a recent line of literature studies the equivalence of Bayesian and dominant-
strategy implementation. Manelli and Vincent (2010) show that in the context of single
unit, independent private value auction, for any Bayesian incentive compatible auction, there
exists an equivalent dominant-strategy incentive compatible auction that yields the same
interim expected utilities for all agents. Gershkov, Goeree, Kushnir, Moldovanu, and Shi
(2013) extend this equivalence result to social choice environments with linear utilities and
independent, one-dimensional, private types. Goeree and Kushnir (2013) provide a simpler
proof by showing that the support functions of the sets of the interim values under Bayesian
and dominant-strategy incentive compatibility are identical. These results are valuable, as
they imply that w.l.o.g. the mechanism designer could restrict attention to dominant-strategy
mechanisms. Our paper differs significantly in that, the mechanism designer does not make
any assumptions about the agents’ beliefs, let alone independent types.

2 Preliminaries

2.1 Notation

There is a finite set $\mathcal{I} = \{1, 2, ..., I\}$ of risk neutral agents and a finite set $\mathcal{K} = \{1, 2, ..., K\}$
of social alternatives. Agent $i$’s payoff type $v_i \in \mathbb{R}^K$ represents agent $i$’s gross utility under

\[\text{\footnotesize \textsuperscript{4}}\text{More literature along this direction include Meyer-Ter-Vehn and Morris (2011) and Yamashita (forthcoming).}\]

\[\text{\footnotesize \textsuperscript{5}}\text{Börgers (2013) criticizes the notion of the maxmin foundation. For every dominant-strategy mechanism,}
Börgers constructs another mechanism which never yields lower revenue and yields strictly higher revenue
sometimes. In contrast, the mechanism we construct achieves strictly higher expected revenue for every
assumption of the agents’ beliefs.}\]
all $K$ alternatives.\footnote{We may use different ways to represent agent $i$’s payoff types, when it is more convenient to do so. For example, in Section 3 when we consider one-dimensional payoff types, agent $i$’s payoff type $v_i \in \mathbb{R}$.} The set of possible payoff types of agent $i$ is a finite set $V_i \subset \mathbb{R}^K$. We denote a payoff type profile by $v = (v_1, v_2, ..., v_I)$. The set of all possible payoff type profiles is $V \equiv V_1 \times V_2 \times \cdots \times V_I$. We write $v_{-i}$ for a payoff type profile of agent $i$’s opponents, i.e., $v_{-i} \in V_{-i} = \times_{j \neq i} V_j$. If $Y$ is a measurable set, then $\Delta Y$ is the set of all probability measures on $Y$. If $Y$ is a metric space, then we treat it as a measurable space with its Borel $\sigma$-algebra.

2.2 Types

Agents’ information is modelled using a type space. A type space, denoted $\Omega = (\Omega_i, f_i, g_i)_{i \in I}$ is defined by a measurable space of types $\Omega_i$ for each agent, and a pair of measurable mappings $f_i : \Omega_i \rightarrow V_i$, defining the payoff type of each type, and $g_i : \Omega_i \rightarrow \Delta(\Omega_{-i})$, defining each type’s belief about the types of the other agents.

A type space encodes in a parsimonious way the beliefs and all higher-order beliefs of the agents. One simple kind of type space is the naive type space generated by a payoff type distribution $\pi \in \Delta(V)$. In the naive type space, each agent believes that all agents’ payoff types are drawn from the distribution $\pi$, and this is common knowledge. Formally, a naive type space associated with $\pi$ is a type space $\Omega_\pi = (\Omega_i, f_i, g_i)_{i \in I}$ such that $\Omega_i = V_i$, $f_i(v_i) = v_i$, and $g_i(v_i)[v_{-i}] = \pi(v_{-i}|v_i)$ for every $v_i$ and $v_{-i}$. The naive type space is used almost without exception in auction theory and mechanism design. The cost of this parsimonious model is that it implicitly embeds some strong assumptions about the agents’ beliefs, and these assumptions are not innocuous. For example, if the agents’ payoff types are independent under $\pi$, then in the naive type space, the agents’ beliefs are common knowledge. On the other hand, for a generic $\pi$, it is common knowledge that there is a one-to-one correspondence between payoff types and beliefs. The spirit of the Wilson Doctrine is to avoid making such assumptions.

To implement the Wilson Doctrine, the common approach is to maintain the naive type space, but try to diminish its adverse effect by imposing stronger solution concepts. To provide foundations for this methodology, we have to return to the fundamentals. Formally, weaker assumptions about the agents’ beliefs are captured by larger type spaces. Indeed, we can remove these assumptions altogether by allowing for every conceivable hierarchy of higher-order beliefs. By the results of Mertens and Zamir (1985), there exists a universal
type space, $\Omega^* = (\Omega^*_i, f^*_i, g^*_i)_{i \in I}$, with the property that, for every payoff type $v_i$ and every infinite hierarchy of beliefs $\hat{h}_i$, there is a type of player $i, \omega_i$, with payoff type $v_i$ and whose hierarchy is $\hat{h}_i$. Moreover, each $\Omega^*_i$ is a compact topological space.

When we start with the universal type space, we remove any implicit assumptions about the agents’ beliefs. We can now explicitly model any such assumption as a probability distribution over the agents’ universal types. Specifically, an assumption for the auctioneer is a distribution $\mu$ over $\Omega^*$.

2.3 Mechanisms

A mechanism consists of a set of messages $M_i$ for each agent $i$, a decision rule $p : M \to \Delta K$, and payment functions $t_i : M \to \mathbb{R}$. Each agent $i$ selects a message from $M_i$, and based on the resulting profile of messages $m$, the decision rule $p$ specifies the outcome from $\Delta K$ (lotteries are allowed) and the payment function $t_i$ specifies the transfer of agent $i$ to the mechanism designer. Agent $i$ obtains utility $p \cdot v_i - t_i$.

The mechanism defines a game form, which together with the type space constitutes a game of incomplete information. The mechanism design problem is to fix a solution concept and search for the mechanism that delivers the maximum expected revenue for the mechanism designer in some outcome consistent with the solution concept. To implement the Wilson Doctrine and minimize the role of assumptions built into the naive type space, the common approach is to adopt a strong solution concept which does not rely on these assumptions. In practice, the often-used solution concept for this purpose is dominant-strategy equilibrium. The revelation principle holds, and we can restrict attention to direct mechanisms.

Definition 1. A direct-revelation mechanism $\Gamma$ for type space $\Omega$ is dominant-strategy incentive compatible (dsIC) if for each agent $i$ and type profile $\omega \in \Omega$,

\[
p(\omega) \cdot f_i(\omega_i) - t_i(\omega) \geq 0, \text{ and }  
p(\omega) \cdot f_i(\omega_i) - t_i(\omega) \geq p(\omega'_i, \omega_{-i}) \cdot f_i(\omega_i) - t_i(\omega'_i, \omega_{-i}),
\]

for any alternative type $\omega'_i \in \Omega_i$.

Definition 2. A dominant-strategy mechanism is a dsIC direct-revelation mechanism for the naive type space $\Omega^\pi$. We denote by $\Phi$ the class of all dominant-strategy mechanisms.
To provide a foundation for using dominant-strategy mechanisms, we shall compare it to the route of completely eliminating common knowledge assumptions about beliefs. We maintain the standard solution concept of Bayesian equilibrium, but now we enlarge the type space all the way to the universal type space. By revelation principle, we restrict attention to direct mechanisms.

**Definition 3.** A direct-revelation mechanism $\Gamma$ for type space $\Omega = (\Omega_i, f_i, g_i)$ is Bayesian incentive compatible (BIC) if for each agent $i$ and type $\omega_i \in \Omega_i$,

$$
\int_{\Omega_{-i}} (p(\omega) \cdot f_i(\omega_i) - t_i(\omega)) g_i(\omega_i) d\omega_{-i} \geq 0, \quad \text{and} \\
\int_{\Omega_{-i}} (p(\omega) \cdot f_i(\omega_i) - t_i(\omega)) g_i(\omega_i) d\omega_{-i} \geq \int_{\Omega_{-i}} (p(\omega'_i, \omega_{-i}) \cdot f_i(\omega_i) - t_i(\omega'_i, \omega_{-i})) g_i(\omega_i) d\omega_{-i}
$$

for any alternative type $\omega'_i \in \Omega_i$.

**Definition 4.** Let $\Psi$ be the class of all BIC direct-revelation mechanism for the universal type space. We say that such a mechanism is detail free.

### 2.4 The mechanism designer as a maxmin decision maker

The mechanism designer has an estimate of the distribution of the agents’ payoff types, $\pi$. Following Chung and Ely (2007), we assume that $\pi$ has full support. An assumption $\mu$ about the distribution of the payoff types and beliefs of the agents is consistent with this estimate if the induced marginal distribution on $V$ is $\pi$. Let $\mathcal{M}(\pi)$ denote the compact subset of such assumptions. For any mechanism $\Gamma$, the $\mu$-expected revenue of $\Gamma$ is

$$
R_\mu(\Gamma) = \int_{\Omega_{-i}} \sum_{i \in I} t_i(\omega) d\mu(\omega).
$$

We do not assume that the mechanism designer has confidence in the naive type space as his model of agents’ beliefs. Rather he considers other assumptions within the set $\mathcal{M}(\pi)$ as possible as well. The mechanism designer who chooses a mechanism that maximizes the worst case performance solves the maxmin problem of

$$
\sup_{\Gamma \in \Psi} \inf_{\mu \in \mathcal{M}(\pi)} R_\mu(\Gamma).
$$

If the mechanism designer uses a dominant-strategy mechanism, then his maximum revenue would be

$$
\Pi^D(\pi) = \sup_{\Gamma \in \Phi} R_\pi(\Gamma),
$$
where
\[ R_\pi(\Gamma) = \sum_{v \in V} \pi(v) \sum_{i \in I} t_i(v) \]
for any dominant-strategy mechanism \( \Gamma \in \Phi \).

**Definition 5.** The use of dominant-strategy mechanisms has a maxmin foundation if
\[
\sup_{\Gamma \in \Psi} \inf_{\mu \in \mathcal{M}(\pi)} R_\mu(\Gamma) = \Pi^D(\pi).
\]

**Definition 6.** The use of dominant-strategy mechanisms has a Bayesian foundation if for some belief \( \mu^* \in \mathcal{M}(\pi) \),
\[
\Pi^D(\pi) = \sup_{\Gamma \in \Psi} R_{\mu^*}(\Gamma).
\]

Bayesian foundation is a stronger notion than maxmin foundation. If there is a particular assumption about (the distribution of) agents’ beliefs, against which the optimal dominant-strategy mechanism achieves the highest expected revenue among all detail-free mechanisms, then the worse case expected utility of an arbitrary detail-free mechanism cannot exceed its expected utility against this particular assumption, which in turn cannot exceed the (fixed) expected utility of the optimal dominant-strategy mechanism. We record this observation as the following proposition (the proof is straightforward and omitted).

**Proposition 1.** Bayesian foundation is a stronger notion than maxmin foundation. That is, if for some belief \( \mu^* \in \mathcal{M}(\pi) \), \( \Pi^D(\pi) = \sup_{\Gamma \in \Psi} R_{\mu^*}(\Gamma) \), then \( \sup_{\Gamma \in \Psi} \inf_{\mu \in \mathcal{M}(\pi)} R_\mu(\Gamma) = \Pi^D(\pi) \).

3 Linear utilities and one-dimensional payoff types

To fix ideas, consider a standard social choice environment with linear utilities and one-dimensional payoff types.\(^7\) This fits many classical applications of mechanism design, including single unit auction (e.g., Myerson (1981)), public good (e.g., Mailath and Postlewaite (1990)) and standard bilateral trade (e.g., Myerson and Satterthwaite (1983)). We then abstract from this environment the uniform graph property, and study the foundations of dominant-strategy mechanisms in a general setting in Section 4.

There is a finite set \( \mathcal{I} = \{1, 2, \ldots, I\} \) of risk neutral agents and a finite set \( \mathcal{K} = \{1, 2, \ldots, K\} \) of social alternatives. Agent \( i \)'s gross utility in alternative \( k \) equals \( u^k_i(v_i) = a^k_i v_i \).

\(^7\)This set-up covers the environment studied in (Gershkov, Goeree, Kushnir, Moldovanu, and Shi, 2013, Page 199, Section 2).
where \( v_i \in \mathbb{R} \) is agent \( i \)'s payoff type, \( a_i^k \in \mathbb{R} \) are constants and \( a_i^k \geq 0 \) for all \( k \).\(^8\) Agent \( i \) obtains utility

\[
p(v) \cdot A_i v - t_i(v)
\]

for decision rule \( p \in \Delta K \) and transfer \( t_i \), where \( A_i = (a_i^1, a_i^2, \ldots, a_i^K) \). For notational simplicity, we assume that each agent has \( M \) possible payoff types and that the set \( V_i \) is the same for each agent: \( V_i = \{v^1, v^2, \ldots, v^M\} \), where \( v^m - v^{m-1} = \gamma \) for each \( m \) and some \( \gamma > 0 \).

In what follows, we first prove several preliminary results in Section 3.1. In the context of a revenue maximizing mechanism designer, we establish the maxmin/Bayesian foundations of dominant-strategy mechanisms in Section 3.2. We apply our results in several classical environments in Section 3.3.

### 3.1 Preliminary results

The dominant-strategy mechanism must satisfy the following incentive constraints: \( \forall i \in I, \forall m, l = 1, 2, \ldots, M, \forall v_{-i} \in V_{-i}, \)

\[
p(v^m, v_{-i}) \cdot A_i v^m - t_i(v^m, v_{-i}) \geq 0, \quad \langle \text{DIR}^m_i \rangle \\
p(v^m, v_{-i}) \cdot A_i v^m - t_i(v^m, v_{-i}) \geq p(v^l, v_{-i}) \cdot A_i v^m - t_i(v^l, v_{-i}). \quad \langle \text{DIC}^{m-l}_i \rangle
\]

In the environment with linear utilities and one-dimensional payoff types, say that a decision rule \( p \) is dsIC if there exists transfer scheme \( t \) such that the mechanism \((p, t)\) satisfies the constraints \( \langle \text{DIR}^m_i \rangle \) and \( \langle \text{DIC}^{m-l}_i \rangle \). For such decision rule \( p \), we prove the following preliminary results.

**Lemma 1.** \( p \) must be monotonic. That is, for \( m \geq l \), we have \( p(v^m, v_{-i}) \cdot A_i \geq p(v^l, v_{-i}) \cdot A_i \).

If the mechanism designer’s objective function is increasing with respect to the agents’ transfers, for example, in the context of a revenue maximizing mechanism designer, we can establish the familiar property that at an optimal solution of the maximization problem, the adjacent downward constraints are binding.

**Proposition 2.** At an optimal solution of the maximization problem, the adjacent downward constraints bind. That is,

\[
p(v^m, v_{-i}) \cdot A_i v^m - t_i^*(v^m, v_{-i}) = p(v^{m-1}, v_{-i}) \cdot A_i v^{m-1} - t_i^*(v^{m-1}, v_{-i}) \text{ for } m = 2, 3, \ldots, M;
\]

\[
p(v^1, v_{-i}) \cdot A_i v^1 - t_i^*(v^1, v_{-i}) = 0.
\]

\(^8\)For the case where \( a_i^k \leq 0 \) for all \( k \), the treatment is symmetric.
An immediate application of Proposition 2 is that, we can express the net utility (rent) and transfer of any payoff type in terms of the decision rule only. This is akin to a discretized version of the well known payoff equivalence result.

Lemma 2.

\[ t_i^*(v^m, v_{-i}) = p(v^m, v_{-i}) \cdot A_i v^m - \gamma \sum_{m'=1}^{m-1} p(v^{m'}, v_{-i}) \cdot A_i. \]

3.2 Foundations of dominant-strategy mechanisms

In the context of a revenue maximizing mechanism designer, we can formulate the optimal dominant-strategy mechanism design problem as follows:

\[
\max_{p(\cdot), t(\cdot)} \sum_{v \in V} \pi(v) \sum_{i \in I} t_i(v) \\
\text{subject to } \forall i \in I, \forall m, l = 1, 2, ..., M, \forall v_{-i} \in V_{-i}, \\
p(v^m, v_{-i}) \cdot A_i v^m - t_i(v^m, v_{-i}) \geq 0, \\
p(v^m, v_{-i}) \cdot A_i v^m - t_i(v^m, v_{-i}) \geq p(v^l, v_{-i}) \cdot A_i v^m - t_i(v^l, v_{-i}).
\]

It follows from the Lemma 1 and Lemma 2 that an equivalent formulation of the problem is:

\[
\max_{p(\cdot)} \sum_{i \in I} \sum_{m=1}^{M} \sum_{v_{-i} \in V_{-i}} \pi(v^m, v_{-i}) \left[ p(v^m, v_{-i}) \cdot A_i v^m - \gamma \sum_{m'=1}^{m-1} p(v^{m'}, v_{-i}) \cdot A_i \right] \\
\text{subject to } p(v^m, v_{-i}) \cdot A_i \geq p(v^l, v_{-i}) \cdot A_i, \text{ for } m \geq l.
\]

Let \( F_i(v_i, v_{-i}) = \sum_{\hat{v}_i \leq v_i} \pi(\hat{v}_i, v_{-i}) \) denote the cumulative distribution function of \( i \)'s valuation conditional on the other agents having payoff type profile \( v_{-i} \). Define the virtual valuation of agent \( i \) as

\[ r_i(v) = v_i - \gamma \frac{1 - F_i(v)}{\pi(v)}, \]

and rewrite the objective function as

\[
\sum_{v \in V} \pi(v)p(v) \cdot A_i r_i(v) \\
= \sum_{v \in V} \pi(v)p(v) \cdot \sum_{i \in I} A_i r_i(v).
\]
For each alternative \( k \), let \( K_{i}^{k,\text{inf}} = \{ k' \in K : a_{i}^{k'} < a_{i}^{k} \} \). That is, \( K_{i}^{k,\text{inf}} \) is the collection of alternatives that agent \( i \) considers inferior than alternative \( k \).

**Definition 7.** We say that \( \pi \) is regular if the virtual valuations satisfy the following condition: for each \( v \in V, j \in I \),

\[
  k \in \arg \max_{k} \sum_{i \in I} a_{i}^{k} \gamma_{i}(v) \Rightarrow K_{j}^{k,\text{inf}} \cap \arg \max_{k} \sum_{i \in I} a_{i}^{k} \gamma_{i}(\hat{v}_{j}, v_{-j}) = \emptyset
\]

for every \( \hat{v}_{j} > v_{j} \).

**Remark 1.** As we illustrate in Section 3.3, our regularity condition reduces to (is an immediate implication of) the single crossing condition in several classical applications of mechanism design, and when \( \pi \) is independent, this condition further reduces to (is an immediate implication of) the standard regularity condition.

**Theorem 1.** If \( \pi \) is regular, then the use of dominant-strategy mechanisms has a Bayesian/maxmin foundation.

### 3.3 Several applications

#### 3.3.1 Single unit auction

In the context of single unit auction with \( I \) agents, the set of social alternatives is \( K = \{ 1, 2, \ldots, I, I+1 \} \), where alternative \( i \) means the auctioneer allocates the good to bidder \( i, i = 1, 2, \ldots, I \) and alternative \( I+1 \) means the auctioneer keeps the good. That is, agent \( i \)’s gross utility equals \( u_{i}^{i}(v_{i}) = v_{i} \) in alternative \( i \) and 0 otherwise. \( K_{i}^{i,\text{inf}} = K - \{ i \} \) and \( K_{i}^{k,\text{inf}} = \emptyset \) for \( k \neq i \).

Chung and Ely (2007) define \( \pi \) to be regular if the virtual valuations satisfy the single crossing condition: for each \( v, i \in I \) and \( j \in \{ 0 \} \cup I, j \neq i \),

\[
  \gamma_{i}(v) \geq \gamma_{j}(v) \Rightarrow \gamma_{i}(\hat{v}_{i}, v_{-i}) > \gamma_{j}(\hat{v}_{i}, v_{-i})
\]

for every \( \hat{v}_{i} > v_{i} \), where \( \gamma_{i}(\cdot) \equiv 0 \) denotes the auctioneer’s value for the object. This condition extends Myerson’s (1981) regularity condition to correlated \( \pi \), and reduces to his original condition when \( \pi \) is independent.\(^9\) It is not hard to see that in the single unit auction setting,

\(^9\)To see this, note that if \( \pi \) is independent, then the virtual valuation of bidder \( j \) depends only on \( v_{j} \). Thus, the single-crossing condition reduces to the requirement that the virtual valuation of each bidder \( i \) is increasing.
(3) is sufficient for our regularity condition.\footnote{Consider agent \(i\), note that our regularity condition has no bite when \(i \notin \arg \max_i \sum_{j \in \mathcal{I}} a^i_j \gamma_j(v)\). When \(i \in \arg \max_k \sum_{j \in \mathcal{I}} a^k_j \gamma_j(v), \gamma_i(v) \geq \gamma_{i'}(v)\) for any \(i' \in \{0\} \cup \mathcal{I}, i' \neq i\) and by (3), it must be the case that \(i = \arg \max_k \sum_{j \in \mathcal{I}} a^k_j \gamma_j(v)\) for every \(\hat{v}_i > v_i\).}

**Corollary 1.** If \(\pi\) is regular, then the use of dominant-strategy mechanisms has a Bayesian/maxmin foundation.

### 3.3.2 Public good

In the context of public good with \(I\) agents, the set of social alternatives is \(\mathcal{K} = \{1, 2\}\), where alternative 1 means providing the good and alternative 2 means not providing the good. The cost of providing the good is assumed to be 0. That is, agent \(i\)'s gross utility equals \(u^1_i(v_i) = v_i\) in alternative 1 and 0 otherwise. \(K^1_{i,\inf} = \{2\}\) and \(K^2_{i,\inf} = \emptyset\) for all \(i \in \mathcal{I}\).

A sufficient condition for our regularity condition is that the virtual valuations satisfy the single crossing condition: for each \(v \in V, j \in \mathcal{I}\),

\[
\sum_{i \in \mathcal{I}} \gamma_i(v) \geq 0 \Rightarrow \sum_{i \in \mathcal{I}} \gamma_i(\hat{v}_j, v_{-j}) > 0
\]

for every \(\hat{v}_j > v_j\). When \(v\) is independent, (3) is implied by the standard Myerson (1981)'s regularity condition.

**Corollary 2.** If \(\pi\) is regular, then the use of dominant-strategy mechanisms has a Bayesian/maxmin foundation.

### 3.3.3 Bilateral trade

In the context of bilateral trade with the set of agents \(\mathcal{I} = \{B, S\}\), the set of social alternatives is \(\mathcal{K} = \{1, 2\}\), where alternative 1 means trade and alternative 2 means no trade. That is, the buyer’s gross utility equals \(u^1_B(v_i) = v_i\) in alternative 1 and 0 otherwise and the seller’s gross utility equals \(u^1_S(v_i) = -v_i\) in alternative 1 and 0 otherwise. \(K^1_{B,\inf} = \{2\}\), \(K^2_{B,\inf} = \emptyset\), \(K^1_{S,\inf} = \emptyset\), \(K^2_{S,\inf} = \{1\}\).

We have for the buyer

\[
t_B(v^m_B, v^m_S) = p(v^m_B, v^m_S) \cdot A_B v^m_B - \gamma \sum_{m' = 1}^{m-1} p(v^{m'}, v_{-i}) \cdot A_B.
\]
For the seller, following similar reasoning, we can show:

\[ t_S(v_B, v^n) = p(v_B, v^n) \cdot A_S v^n + \gamma \sum_{n'=n+1}^M p(v_{B}, v^{n'}) \cdot A_S. \]

Therefore, the reduced objective function is

\[
\sum_{i \in I} \sum_{m=1}^M \sum_{n=1}^M \pi(v^m, v^n) \left[ p(v^m, v^n) \cdot A_B v^m - \gamma \sum_{m'=1}^{m-1} p(v^{m'}, v_{-i}) \cdot A_B + p(v^m, v^n) \cdot A_S v^n + \gamma \sum_{n'=n+1}^M p(v_{B}, v^{n'}) \cdot A_S \right].
\]

The virtual valuation of the buyer is the same as before

\[ r_B(v_B, v_S) = v_B - \frac{1 - F_B(v_B, v_S)}{\pi(v_B, v_S)} \]

and we define the virtual valuation of the seller as follows

\[ r_S(v_B, v_S) = v_S + \frac{F_S(v_B, v_S)}{\pi(v_B, v_S)}. \]

We can rewrite the objective function as

\[
\sum_{v \in V} \pi(v) p(v) \cdot [A_B \gamma_B(v) + A_S \gamma_S(v)]
\]

\[
= \sum_{v \in V} \pi(v) p(v) \cdot A_B [\gamma_B(v) - \gamma_S(v)]
\]

A sufficient condition for our regularity condition is that the virtual valuations satisfy the single crossing condition: for each \( v \),

\[ \gamma_B(v_B, v_S) - \gamma_S(v_B, v_S) \geq 0 \Rightarrow \gamma_B(\hat{v}_B, v_S) - \gamma_S(\hat{v}_B, v_S) > 0 \quad (5) \]

for every \( \hat{v}_B > v_B \) and

\[ \gamma_B(v_B, v_S) - \gamma_S(v_B, v_S) \geq 0 \Rightarrow \gamma_B(v_B, \hat{v}_S) - \gamma_S(v_B, \hat{v}_S) > 0 \quad (6) \]

for every \( \hat{v}_S < v_S \). When \( v \) is independent, \( 5 \) and \( 6 \) are implied by the standard Myerson (1981)’s regularity condition.

**Corollary 3.** If \( \pi \) is regular, then the use of dominant-strategy mechanisms has a Bayesian/maxmin foundation.
4 General setting

We can formulate the optimal dominant-strategy mechanism design problem as follows:

\[
\max_{p\cdot t} \sum_{v \in V} \pi(v) \sum_{i \in I} t_i(v) \tag{(B.1)}
\]

subject to \(\forall i \in I, \forall v_i, v'_i \in V_i, \forall v_{-i} \in V_{-i},\)

\[
p(v_i, v_{-i}) \cdot v_i - t_i(v_i, v_{-i}) \geq 0, \quad \langle \text{DIR}^v_i \rangle
\]

\[
p(v_i, v_{-i}) \cdot v_i - t_i(v_i, v_{-i}) \geq p(v'_i, v_{-i}) \cdot v_i - t_i(v'_i, v_{-i}). \quad \langle \text{DIC}^v_i \rightarrow v'_i \rangle
\]

Say that a decision rule \(p\) is dsIC if there exists transfer scheme \(t\) such that the mechanism \((p, t)\) satisfies the constraints \(\langle \text{DIR}^v_i \rangle\) and \(\langle \text{DIC}^v_i \rightarrow v'_i \rangle\). We omit the proof of the following standard lemma. For a detailed explanation and proof of the lemma, please see Rochet (1987).

**Lemma 3.** A necessary and sufficient condition for a decision rule \(p\) to be dsIC is the following cyclical monotonicity condition: \(\forall i \in I, \forall v_{-i} \in V_{-i}\) and every sequence of payoff types of agent \(i, (v_{i,1}, v_{i,2}, ..., v_{i,k})\) with \(v_{i,k} = v_{i,1}\), we have

\[
\sum_{\kappa=1}^{k-1} [p(v_{i,\kappa}, v_{-i}) \cdot v_{i,\kappa+1} - p(v_{i,\kappa}, v_{-i}) \cdot v_{i,\kappa}] \leq 0. \tag{7}
\]

4.1 Uniform graph

Consider the optimal dominant-strategy mechanism design problem \((B.1)\). For any decision rule \(p\) that is dsIC, we can find the optimal transfer scheme \(t^*\). To formally define uniform graph, we first introduce what we call "best derivation". This can be viewed as an order-based interpretation of incentive constraints.

**Definition 8.** Fix a decision rule \(p\) and other agents’ reports \(v_{-i}\). For \(v_i, v'_i \in V_i\), we say \(v'_i\) is a best deviation of \(v_i\) if

\[
p(v_i, v_{-i}) \cdot v_i - t^*_i(v_i, v_{-i}) = p(v'_i, v_{-i}) \cdot v_i - t^*_i(v'_i, v_{-i}).
\]

That is, agent \(i\) is indifferent between reporting truthfully and reporting \(v'_i\) when his payoff type is \(v_i\).

\(^{11}\)In the remainder of this section, whenever we fix a decision rule \(p\), we mean that we are fixing a decision rule \(p\) that is dsIC.
Definition 9. Fix a decision rule $p$ and other agents’ reports $v_{-i}$.

i) The set of nodes for agent $i$ is $V_i$.

ii) For $v_i, v_i' \in V_i$, $v_i \rightarrow v_i'$ is a directed edge for agent $i$ if $v_i'$ is a best deviation of $v_i$.

iii) For $v_i, v_i' \in V_i$, a path from $v_i$ to $v_i'$ is a sequence of payoff types of agent $i$, $(v_{i,1}, v_{i,2}, ..., v_{i,R})$ with $v_{i,1} = v_i, v_{i,R} = v_i'$ and $v_{i,r-1} \rightarrow v_{i,r}, \forall r = 2, ..., R$.

Denote by $V_i^T = \{v_i \in V_i : p(v_i, v_{-i}) \cdot v_i - t_i^*(v_i, v_{-i}) = 0\}$. That is, $V_i^T$ is the collection of agent $i$’s payoff types such that agent $i$ is indifferent between truth telling and not participating in the mechanism. For any payoff type $v_i$ of agent $i$, we are particularly interested in the path from $v_i$ to some $v_i' \in V_i^T$, as this gives a systematic way to calculate the rent of payoff type $v_i$. Since payoff type $v_i$ may have multiple best deviations, for each $v_i \in V_i$, we will select a particular best deviation from the set of best deviations.

Definition 10. Fix a decision rule $p$ and other agents’ reports $v_{-i}$. A graph is a collection of paths from each payoff type $v_i$ to some $v_i' \in V_i^T$, such that i) for each payoff type, such path is uniquely selected; and ii) if $v_i'$ belongs to the path from some payoff type to some $v_i \in V_i^T$, the truncation of the path from $v_i'$ to $v_i$ defines the path from $v_i'$ to $v_i$.\(^{12}\)

Condition i) says there is a path from any payoff type $v_i$ to some $v_i' \in V_i^T$; Condition ii) says that we are selecting this best deviation in a consistent way. A graph induces an order on the agents’ payoff types. For a typical path $(v_{i,1}, v_{i,2}, ..., v_{i,R})$ of the graph, we write $v_{i,1} \succ_{i}^{p,v_{-i}} ... \succ_{i}^{p,v_{-i}} v_{i,R}$. We denote the best deviation of payoff type $v_i$ selected in the graph by $v_i^-$.\(^{12}\)

Definition 11. There is uniform graph if for each agent $i \in I$, there is the same graph for all decision rules $p$ and other agents’ reports $v_{-i}$.

We drop the superscript $p, v_{-i}$ and denote the uniform graph of agent $i$ by $\succ_i$ and its transitive closure by $\succ_i^+$. For notational convenience, write $v_i' \succeq_{i}^{p,v_{-i}} v_i$ if $v_i' \succ_{i}^{p,v_{-i}} v_i$ or $v_i' = v_i$. Also write $V_i^I = \{v_i \in V_i : \text{there is no } v_i' \text{ such that } v_i' \succ_i v_i\}$.

With uniform graph, the rent of any payoff type is easily calculated and all incentive constraints can be replaced by the cyclical monotonicity constraints on the decision rule. We record this as the following proposition.

\(^{12}\)Our formulation of graph is closely related to that of Rochet and Stole (2003). To be self-contained, we show that graph is well defined. Appendix A.4 presents the technical details.
Proposition 3. With uniform graph $\succ_i$, the maximization problem in (B.1) is equivalent to

$$
\max_{p(\cdot)} \sum_{i \in \mathcal{I}} \sum_{v_i \in V_i} \sum_{v_{i-} \in V_{i-}} \pi(v_i, v_{i-}) \left[ p(v_i, v_{i-}) \cdot v_i - \sum_{v_i' \preceq v_i} p((v_i')^-, v_{i-}) \cdot [v_i' - (v_i')] \right].
$$

subject to $p(\cdot)$ satisfies the cyclical monotonicity constraint (7).

Proof. We first calculate the rent of the payoff types as follows:

$$
U_i(v) = p(v_i, v_{i-}) \cdot v_i - t_i^*(v_i, v_{i-}) = 0 \text{ for } v_i \in V_i^T;
$$

$$
U_i(v) = p(v_i, v_{i-}) \cdot v_i - t_i^*(v_i, v_{i-})
= p(v_i^-, v_{i-}) \cdot v_i - t_i^*(v_i^-, v_{i-})
= p(v_i^-, v_{i-}) \cdot v_i^* - t_i^*(v_i^-, v_{i-}) + p(v_i^-, v_{i-}) \cdot (v_i - v_i^-)
= U_i(v_i^-, v_{i-}) + p(v_i^-, v_{i-}) \cdot (v_i - v_i^-) \text{ for } v_i \in V_i - V_i^T.
$$

By induction,

$$
U_i(v) = \sum_{v_i' \preceq v_i} p((v_i')^-, v_{i-}) \cdot [v_i' - (v_i')]^{13}
$$

Therefore,

$$
t_i^*(v) = p(v_i, v_{i-}) \cdot v_i - U_i(v)
= p(v_i, v_{i-}) \cdot v_i - \sum_{v_i' \preceq v_i} p((v_i')^-, v_{i-}) \cdot [v_i' - (v_i')]^{13}.
$$

The maximization problem in (B.1) is equivalent to

$$
\max_{p(\cdot)} \sum_{i \in \mathcal{I}} \sum_{v_i \in V_i} \sum_{v_{i-} \in V_{i-}} \pi(v_i, v_{i-}) \left[ p(v_i, v_{i-}) \cdot v_i - \sum_{v_i' \preceq v_i} p((v_i')^-, v_{i-}) \cdot [v_i' - (v_i')] \right].
$$

Lemma 3 applies and $p(\cdot)$ is subject to the cyclical monotonicity constraint (7). \qed

Definition 12. Say $\pi$ is regular if cyclical monotonicity constraint (7) is automatically satisfied for all $p^*$ that maximizes the reduced objective function (8).

To the best of our knowledge, there is no formal definition of regularity in the general environments. Our definition is not in terms of primitives, but can be simplified considerably and becomes a primitive condition if additional structure is imposed. As we illustrate in

$^{13}$For simplicity of exposition, write $v_i^- = v_i$ for $v_i \in V_i^T$. 

19
Section 3.2, it reduces to (or is implied by) familiar regularity condition in environments such as single unit auction, public good and bilateral trade. Furthermore, our regularity is defined in a way that is consistent with its usage in the literature. That is, to ensure that the cyclical monotonicity constraint is automatically satisfied for the optimal $p^*$. A technical aspect of this definition lies in that we require the monotonicity constraints to be automatically satisfied for all $p^*$ that maximizes (8). This ensures that for singular $\pi$, we can find a sequence $\pi_n$ converging to $\pi$ and $\pi_n$ is regular and nonsingular for $n$ sufficiently large.

**Theorem 2.** If $\pi$ is regular and there is the uniform graph, then the use of dominant-strategy mechanisms has a Bayesian/ maxmin foundation.

### 4.2 An illustrative example

Imagine that you and your colleagues are buying a coffee maker to be kept at work. While everyone is in favor, different people might have different preferences as to which model to buy. We consider the problem of selecting a public good among mutually exclusive choices. While all agents are in favor of providing a public good, different agents might have different preferences as to which public good to provide.

There are two agents and two public goods. The set of social alternatives is $\mathcal{K} = \{1, 2\}$, where alternative 1 means providing public good 1 and alternative 2 means providing public good 2. Each agent is either indifferent between the two public goods, or prefers a particular one. Each agent is described by a pair $v^1 = (1, 1), v^2 = (2, 1)$ or $v^3 = (1, 2)$, where the first number denotes the agent’s valuation for public good 1 and the second number denotes his valuation for public good 2. The valuations of the public goods are privately known to the agents and the mechanism designer chooses a mechanism that maximizes the expected profit. The costs of the public goods are assumed to be 0.

**Claim 1.** $v^2 \succ_i v^1$ and $v^3 \succ_i v^1$.

### 4.3 Multi-unit auction with capacity-constrained bidders

Besides environments with linear utilities and one-dimensional payoff types, the uniform graph condition is also satisfied in some multi-dimensional environments. Solving for the optimal mechanism in a multi-dimensional environment is in general a daunting task.\(^{14}\)

\(^{14}\)See Rochet and Stole (2003) for a survey of multi-dimensional screening.
this subsection, we examine a specific case where the multi-dimensional analysis can be simplified.

Consider the problem of finding the revenue maximizing auction when bidders have constant marginal valuations as well as capacity constraints. Both the marginal values and capacity constraints are private information to the bidders. A bidder’s payoff type is represented by \((a, b)\), where \(a\) is the maximum amount he is willing to pay for each unit and \(b\) is the largest number of units he seeks. Units beyond the \(b^{th}\) unit are worthless. Let the range of \(a\) be \(A = \{1, 2, ..., A\}\) and the range of \(b\) be \(B = \{1, 2, ..., B\}\). The seller has \(Q\) units to sell.

A crucial assumption is that no bidder can inflate his capacity but can shade it down. In other words, the auctioneer can verify, partially, the claims made by a bidder. Although this assumption seems odd in the selling context, it is natural in a procurement setting. Consider a procurement auction where the auctioneer wishes to procure \(Q\) units from bidders with constant marginal costs and limited capacity. No bidder will inflate his capacity when bidding because of the huge penalties associated with not being able to fulfill the order. Equivalently, we may suppose that the mechanism designer can verify that claims that exceed capacity are false.

**Proposition 4.** \((i, j) \succ_i (i - 1, j) \succ_i \ldots \succ_i (1, j) \succ_i (1, j - 1) \succ_i \ldots \succ_i (1, 1)\).

The proof is similar to (Vohra, 2011, pp150-159) and omitted.

\(^{15}(Vohra, 2011, \text{pp150-159})\) studies the optimal Bayesian mechanism in such an environment.
5 Bilateral trade with ex ante unidentified traders

In this section, we study bilateral trade with ex ante unidentified traders. In the context of this important economic environment, this example illustrates that, when the uniform graph condition is not satisfied, maxmin/ Bayesian foundations might not exist. Section 5.1 presents the basics of the model, Section 5.2 calculates the maximum expected revenue that could be achieved by a dominant-strategy mechanism, and Section 5.3 explicitly constructs a single Bayesian mechanism that achieves a strictly higher expected revenue, regardless of the agents’ beliefs. It should be obvious from the exposition below that this example is robust to small perturbations in the agents’ valuations or the mechanism designer’s estimate $\pi$.

5.1 Setup

We consider the problem of designing a rule to determine the terms of trade between two agents. Each agent is endowed with $\frac{1}{2}$ unit of a good to be traded and has private information about his valuation for the good. Agent 1’s valuation for the good could be either 18 or 38. Agent 2’s valuation for the good could be either 10 or 30. The mechanism designer has the following estimate of the distribution of the agents’ valuations:

\[ (i, j) \succ_i (i-1, j) \succ_i \ldots \succ_i (1, j) \succ_i (1, j-1) \succ_i \ldots \succ_i (1, 1). \]

Figure 3: $(i, j) \succ_i (i-1, j) \succ_i \ldots \succ_i (1, j) \succ_i (1, j-1) \succ_i \ldots \succ_i (1, 1)$.

---

16See Cramton, Gibbons, and Klemperer (1987) and Lu and Robert (2001) for the motivation and detailed exposition of such bilateral trade models.

17It helps to draw a comparison with the standard bilateral trade model. In the standard bilateral trade model, we have shown that there is uniform graph and hence we can construct an assumption about (the distribution of) bidders’ beliefs, against which the dominant-strategy mechanism achieves the highest expected revenue.
Each agent may be either the buyer or the seller, depending on the realization of the privately observed information and the choice of the mechanism. In other words, whether an agent is the buyer or the seller is endogenously determined by the agents’ reported valuations and cannot be identified prior to trade. The buyer’s utility from purchasing $p$ unit of the good and paying a transfer $t_B$ is $pv_B - t_B$ and the seller’s utility from selling $p$ unit of the good and paying a transfer $t_S$ is $-pv_S - t_S$, where $0 \leq p \leq \frac{1}{2}$. The mechanism designer chooses a mechanism that maximizes the expected revenue.

5.2 Optimal dominant-strategy mechanism

We show that the maximum expected revenue the mechanism designer can generate from a dominant-strategy mechanism is 3. To see this, we focus on the case that an agent is assigned to be the buyer if and only if the agent reports a higher valuation, where $B$ (resp. $S$) means that agent 1 is assigned to be the buyer (resp. seller).

\[
\begin{array}{c|c|c}
& v_1 = 18 & v_1 = 38 \\
\hline
v_2 = 10 & \frac{3}{8} & \frac{1}{8} \\
\hline
v_2 = 30 & \frac{1}{8} & \frac{3}{8} \\
\end{array}
\]

(9)

We use $p(v_1, v_2)$ to denote the expected trading amount when agent 1 reports $v_1$ and agent 2 reports $v_2$. For example, in the assignment in (10), $p(18, 10)$ denotes the expected amount of good that agent 1 buys from agent 2 while $p(18, 30)$ denotes the expected amount of good that agent 1 sells to agent 2. We can formulate the optimal dominant-strategy mechanism design problem as follows:

\[
\begin{align*}
&\max 3\left[\frac{1}{8} [t_1(18, 10) + t_2(18, 10)] + \frac{1}{8} [t_1(38, 10) + t_2(38, 10)]
\right. \\
&\left. + \frac{1}{8} [t_1(18, 30) + t_2(18, 30)] + \frac{3}{8} [t_1(38, 30) + t_2(38, 30)]\right]
\end{align*}
\]

\[
\begin{array}{c|c|c}
& v_1 = 18 & v_1 = 38 \\
\hline
v_2 = 10 & B & B \\
\hline
v_2 = 30 & S & B \\
\end{array}
\]

(10)

\(^{18}\)Hence, this model does not belong to the class of environments with linear utilities and one-dimensional payoff types. See Section 3.
by choosing
\[ p(18, 10), p(38, 10), p(18, 30), p(38, 30) \in \left[ 0, \frac{1}{2} \right], t_i (v_1, v_2) \in \mathbb{R}, \]
subject to the following incentive constraints:

\[ 18p(18, 10) - t_1(18, 10) \geq \max \{ 0, 18p(38, 10) - t_1(38, 10) \}; \quad (11) \]
\[ 38p(38, 10) - t_1(38, 10) \geq \max \{ 0, 38p(18, 10) - t_1(18, 10) \}; \quad (12) \]
\[ -18p(18, 30) - t_1(18, 30) \geq \max \{ 0, 18p(38, 30) - t_1(38, 30) \}; \quad (13) \]
\[ 38p(38, 30) - t_1(38, 30) \geq \max \{ 0, -38p(18, 30) - t_1(18, 30) \}; \quad (14) \]
\[ -10p(18, 10) - t_2(18, 10) \geq \max \{ 0, 10p(18, 30) - t_2(28, 30) \}; \]
\[ 30p(18, 30) - t_2(18, 30) \geq \max \{ 0, -30p(18, 10) - t_2(18, 10) \}; \]
\[ -10p(38, 10) - t_2(38, 10) \geq \max \{ 0, -10p(38, 30) - t_2(38, 30) \}; \]
\[ -30p(38, 30) - t_2(38, 30) \geq \max \{ 0, -30p(38, 10) - t_2(38, 10) \}. \]

Consider agent 1’s incentives constraints when agent 2 reports \( v_2 = 10 \). That is, inequalities (11) and (12). Agent 1 is assigned to be the buyer regardless of his report. It follows from standard arguments that \( p \) must satisfy the monotonicity condition
\[ p(38, 10) - p(18, 10) \geq 0 \]
and the binding constraints are
\[ 18p(18, 10) - t_1(18, 10) = 0; \]
\[ 38p(38, 10) - t_1(38, 10) = 38p(18, 10) - t_1(18, 10). \]

Consider agent 1’s incentives constraints when agent 2 reports \( v_2 = 30 \). That is, inequalities (13) and (14). Agent 1 is assigned to be the buyer if he reports \( v_1 = 38 \) and seller if he reports \( v_1 = 18 \). In this case, we can separate these two types and achieve full surplus extraction. Formally, the binding constraints are
\[ -18p(18, 30) - t_1(18, 30) = 0; \]
\[ 38p(38, 30) - t_1(38, 30) = 0. \]
When these two constraints are binding, incentive constraints
\[ -18p(18, 30) - t_1(18, 30) \geq 18p(38, 30) - t_1(38, 30); \]
\[ 38p(38, 30) - t_1(38, 30) \geq -38p(18, 30) - t_1(18, 30) \]
are automatically satisfied.

Following this logic, it is not hard to see that the above maximization problem is equivalent to:

\[
\max \frac{1}{2} p(18, 10) + \frac{7}{2} p(38, 10) + \frac{3}{2} p(18, 30) + \frac{1}{2} p(38, 30)
\]

by choosing

\[
p(18, 10), p(38, 10), p(18, 30), p(38, 30) \in \left[0, \frac{1}{2}\right],
\]

subject to monotonicity constraints:

\[
p(38, 10) - p(18, 10) \geq 0; \quad (15)
\]
\[
p(38, 30) + p(18, 30) \geq 0; \quad (16)
\]
\[
p(18, 30) + p(18, 10) \geq 0; \quad (17)
\]
\[
p(38, 10) - p(38, 30) \geq 0. \quad (18)
\]

Obviously, \(p(18, 10) = p(38, 10) = p(18, 30) = p(38, 30) = 1/2\) solves the unconstrained maximization problem. The monotonicity constraints (15)-(18) are automatically satisfied. The maximum expected revenue is 3.

**Remark 2.** We also need to consider other possible assignment rules. Similar calculations show that the maximum expected revenue for all other cases is less than 3. The detailed calculations can be found in the supplement Chen and Li (2015).

Therefore, the optimal dominant strategy mechanism \(\Gamma\) is as follows, where the first number in each cell indicates the amount of good agent 1 buys from agent 2, the second number is the transfer from agent 1 and the third number is the transfer from agent 2.

\[
\begin{array}{c|c|c}
\text{ } & v_1 = 18 & v_1 = 38 \\
\hline
v_2 = 10 & \frac{1}{2}, 9, -5 & \frac{1}{2}, 9, -15 \\
\hline
v_2 = 30 & -\frac{1}{2}, -9, 15 & \frac{1}{2}, 19, -15
\end{array}
\]

\[
(19)
\]

### 5.3 No maxmin/ Bayesian foundation

We show that there is no maxmin foundation. That is, the mechanism designer could employ a single Bayesian mechanism and achieve a strictly higher expected revenue than he does using the optimal dominant-strategy mechanism, regardless of the agents’ beliefs. To do this,
we first explicitly identify one such mechanism and proceed by verifying i) the mechanism is BIC for the universal type space; and ii) this mechanism achieves a strictly higher expected revenue than 3 regardless of the agents’ beliefs. This also implies that there is no Bayesian foundation.\textsuperscript{19}

Before we present the mechanism, it is helpful to discuss some intuition. It is clear that in the optimal dominant-strategy mechanism (19), the mechanism designer is able to fully extract the surplus except when the agents report \(v_1 = 38\) and \(v_2 = 10\). Hence the construction of the "superior" mechanism exploits the beliefs of the agents and involves betting with the agents. The main difficulty is that such betting must increase the agent’s expected revenue regardless of the agents’ beliefs and the new mechanism (with the bets) must be incentive compatible. We know of two ways to do this. The mechanism we present in this paper exploits the beliefs of agent 2.\textsuperscript{20}

Consider the following mechanism \(\Gamma'\). Following Chung and Ely (2007), we use \(a\) to denote the first-order belief of a low-valuation type of agent 2 that agent 1 has low valuation. In this mechanism, the mechanism designer elicits agent 2’s first-order belief about agent 1’s valuation and offers a bet to agent 2. This increases the transfer from agent 2 when the agents report \(v_1 = 38\) and \(a \in [\frac{1}{2}, 1]\). Also, we implement a different trading rule when agent 2 reports \(a \in [0, \frac{1}{2})\) (from when agent 2 reports \(v_2 = 10\) in the dominant-strategy mechanism). The significance of this modification is that when agent 2 reports \(a \in [0, \frac{1}{2})\), the mechanism designer could separately agent 1’s valuations and achieve full surplus extraction from agent 1. This increases the transfer from agent 1 when the agents report \(v_1 = 38\) and \(a \in [0, \frac{1}{2})\).

<table>
<thead>
<tr>
<th></th>
<th>(v_1 = 18)</th>
<th>(v_1 = 38)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a \in [0, \frac{1}{2}))</td>
<td>(-\frac{1}{2}, -9, 15)</td>
<td>(\frac{1}{2}, 19, -15)</td>
</tr>
<tr>
<td>(a \in [\frac{1}{2}, 1])</td>
<td>(\frac{1}{2}, 9, -5)</td>
<td>(\frac{1}{2}, 9, -5)</td>
</tr>
<tr>
<td>(v_2 = 30)</td>
<td>(-\frac{1}{2}, -9, 15)</td>
<td>(\frac{1}{2}, 19, -15)</td>
</tr>
</tbody>
</table>

To see that \(\Gamma'\) is BIC for the universal type space, note that

1) truth telling continues to be a dominant strategy for agent 1;

2) high-valuation type of agent 2 has a dominant strategy to tell the truth;

\textsuperscript{19}Recall that Bayesian foundation is a stronger notion than maxmin foundation.

\textsuperscript{20}The construction of a mechanism that exploits agent 1’s first-order belief about agent 2’s valuation is analogous.
iii) \( a \in [0, \frac{1}{2}) \) will not announce \( v_2 = 30 \) as utility is unchanged;
iv) \( a \in [\frac{1}{2}, 1] \) will not announce \( v_2 = 30 \) as expected utility is lower; and
v) lastly, a low type of agent 2 will announce \( a \in [\frac{1}{2}, 1] \) if and only if \( a \geq \frac{1}{2} \).

To see that \( \Gamma' \) is expected revenue improving, note that \( \Gamma' \) achieves revenue of at least 4 everywhere and hence the expected revenue is at least 4.

6 Discussion

6.1 Foundations of ex post incentive compatible mechanisms

Our paper focuses exclusively on the private value setting. A natural question to ask is, in the interdependent value setting, whether uniform graph ensures foundations of ex post incentive compatible mechanisms. In a recent work by Yamashita and Zhu (2014), they focus on the so-called "digital-goods" auctions and show that under ordinal invariability, which says each agent has a stable preference ordering over his payoff types, regardless of what payoff type profile the other agents have, the use of ex post incentive compatible mechanisms has a maxmin/ Bayesian foundation. We conjecture that uniform graph has a natural extension in the interdependent value setting, which ensures maxmin/ Bayesian foundations of ex post incentive compatible mechanisms.

Yamashita and Zhu (2014) also show that in the finite-version digital-goods auction, under certain assumptions, the foundations do not exist if such ordinal invariability condition fails. For this result, they restrict attention to a subset of agents’ beliefs - type spaces with full support.

6.2 Implementation - Bergemann and Morris (2005)’s equivalence

Bergemann and Morris (2005) points out that in the quasilinear environment with budget balance, once there are more than two agents and at least one agent has at least three types, a social choice correspondence can be interim implemented (using a single mechanism) on all type spaces whereas ex post implementation is impossible. They conjecture that ex post equivalence results may again be obtained in a general environment only after imposing suitable restrictions on the environment. In this subsection, we consider the environment we studied in Section 3 - the standard social choice environment with linear utilities and
one-dimensional payoff types. We shall also relax the exact budget balance requirement to weak budget balance. We show that the equivalence can still be obtained.

**Definition 13.** A decision rule \( p \) is dominant strategy implementable with weak budget balance (dsIC-BB) if there exists transfer scheme \( t \) such that the mechanism \((p, t)\) satisfies the constraints

\[
\forall i \in \mathcal{I}, \forall m, l = 1, 2, \ldots, M, \forall v_{-i} \in V_{-i},
\]

\[
p(v^m, v_{-i}) \cdot A_i v^m - t_i(v^m, v_{-i}) \geq 0,
\]

\[
p(v^m, v_{-i}) \cdot A_i v^m - t_i(v^m, v_{-i}) \geq p(v^l, v_{-i}) \cdot A_i v^m - t_i(v^l, v_{-i}) \text{ and}
\]

\[
\sum_{i \in \mathcal{I}} t_i(v) \geq 0, \forall v \in V.
\]

**Definition 14.** A decision rule \( p \) is interim implementable with weak budget balance (BIC-BB) on type space \( \Omega = (\Omega_i, f_i, g_i)_{i \in \mathcal{I}} \) if there exists transfer scheme \( t \) such that the mechanism \((p, t)\) satisfies the constraints

\[
\forall i \in \mathcal{I}, \forall \omega_i, \omega'_i \in \Omega_i, \forall \omega_{-i} \in \Omega_{-i},
\]

\[
\int_{\Omega_{-i}} (p(\omega) \cdot f_i(\omega_i) - t_i(\omega)) g_i(\omega_i) d\omega_{-i} \geq 0,
\]

\[
\int_{\Omega_{-i}} (p(\omega) \cdot f_i(\omega_i) - t_i(\omega)) g_i(\omega_i) d\omega_{-i} \geq \int_{\Omega_{-i}} (p(\omega'_i, \omega_{-i}) \cdot f_i(\omega_i) - t_i(\omega'_i, \omega_{-i})) g_i(\omega_i) d\omega_{-i} \text{ and}
\]

\[
\sum_{i \in \mathcal{I}} t_i(\omega) \geq 0, \forall \omega \in \Omega.
\]

**Theorem 3.** In the standard social choice environment with linear utilities and one-dimensional payoff types, the following statements are equivalent:

i) \( p \) is dsIC-BB.

ii) \( p \) is BIC-BB on all type spaces.

\[\text{A Appendix}\]

**A.1 Proof of Lemma 1**

**Proof.** Consider \( v^m \) and \( v^l \) with \( m \geq l \). Incentive compatibility requires

\[
p(v^m, v_{-i}) \cdot A_i v^m - t_i(v^m, v_{-i}) \geq p(v^l, v_{-i}) \cdot A_i v^m - t_i(v^l, v_{-i}) \text{ and}
\]

\[
p(v^m, v_{-i}) \cdot A_i v^l - t_i(v^m, v_{-i}) \leq p(v^l, v_{-i}) \cdot A_i v^l - t_i(v^l, v_{-i}).
\]
Subtracting these two inequalities, we obtain

\[ p(v^m, v_{-i}) \cdot A_i(v^m - v') \geq p(v^l, v_{-i}) \cdot A_i(v^m - v') \iff p(v^m, v_{-i}) \cdot A_i = p(v^l, v_{-i}) \cdot A_i. \]

\[ p(v^m, v_{-i}) \cdot A_i \geq p(v^l, v_{-i}) \cdot A_i. \]

\[ A.2 \text{ Proof of Proposition 2} \]

**Lemma 4.** All constraints are implied by the following:

\[
\begin{align*}
    p(v^m, v_{-i}) \cdot A_i v^m - t_i(v^m, v_{-i}) & \geq p(v^{m-1}, v_{-i}) \cdot A_i v^m - t_i(v^{m-1}, v_{-i}) \text{ for } m = 2, 3, ..., M; \\
p(v^m, v_{-i}) \cdot A_i v^m - t_i(v^m, v_{-i}) & \geq p(v^{m+1}, v_{-i}) \cdot A_i v^m - t_i(v^{m+1}, v_{-i}) \text{ for } m = 1, 2, ..., M - 1; \\
p(v^1, v_{-i}) \cdot A_i v^1 - t_i(v^1, v_{-i}) & \geq 0.
\end{align*}
\]

**Proof.** (i) Downward constraints \( p(v^m, v_{-i}) \cdot A_i v^m - t_i(v^m, v_{-i}) \geq p(v^l, v_{-i}) \cdot A_i v^m - t_i(v^l, v_{-i}) \) for \( m > l \) are implied by the adjacent downward constraints. This is because the following pair of inequalities:

\[
\begin{align*}
    p(v^m, v_{-i}) \cdot A_i v^m - t_i(v^m, v_{-i}) & \geq p(v^{m-1}, v_{-i}) \cdot A_i v^m - t_i(v^{m-1}, v_{-i}); \\
p(v^{m-1}, v_{-i}) \cdot A_i v^{m-1} - t_i(v^{m-1}, v_{-i}) & \geq p(v^{m-2}, v_{-i}) \cdot A_i v^{m-1} - t_i(v^{m-2}, v_{-i})
\end{align*}
\]

imply

\[
\begin{align*}
p(v^m, v_{-i}) \cdot A_i v^m - t_i(v^m, v_{-i}) & \\
\geq & p(v^{m-1}, v_{-i}) \cdot A_i v^m - t_i(v^{m-1}, v_{-i}) \\
= & p(v^{m-1}, v_{-i}) \cdot A_i v^{m-1} - t_i(v^{m-1}, v_{-i}) + p(v^{m-1}, v_{-i}) \cdot A_i \gamma \\
\geq & p(v^{m-2}, v_{-i}) \cdot A_i v^{m-1} - t_i(v^{m-2}, v_{-i}) + p(v^{m-2}, v_{-i}) \cdot A_i \gamma \\
= & p(v^{m-2}, v_{-i}) \cdot A_i v^{m-2} - t_i(v^{m-2}, v_{-i}),
\end{align*}
\]

where the second last line follows from Lemma 1. The rest will follow by induction.

(ii) Similar arguments show that upward constraints can be implied by the adjacent upward constraints.

(iii) Constraints \( p(v^m, v_{-i}) \cdot A_i v^m - t_i(v^m, v_{-i}) \geq 0, m = 2, 3, ..., M \) are implied by
then the corresponding upward constraint is satisfied.

where the second last line follows from Lemma 1. \[ \square \]

**Lemma 5.** If the adjacent downward constraint binds

\[
p(v^m, v_{-i}) \cdot A_i v^m - t_i(v^m, v_{-i}) = p(v^{m-1}, v_{-i}) \cdot A_i v^{m-1} - t_i(v^{m-1}, v_{-i}),
\]

then the corresponding upward constraint

\[
p(v^{m-1}, v_{-i}) \cdot A_i v^{m-1} - t_i(v^{m-1}, v_{-i}) \geq p(v^{m-1}, v_{-i}) \cdot A_i v^{m-1} - t_i(v^{m-1}, v_{-i})
\]

is satisfied.

**Proof.** Suppose

\[
p(v^m, v_{-i}) \cdot A_i v^m - t_i(v^m, v_{-i}) = p(v^{m-1}, v_{-i}) \cdot A_i v^{m-1} - t_i(v^{m-1}, v_{-i}),
\]

then

\[
\begin{align*}
p(v^{m-1}, v_{-i}) \cdot A_i v^{m-1} - t_i(v^{m-1}, v_{-i}) &= p(v^{m-1}, v_{-i}) \cdot A_i v^{m-1} - t_i(v^{m-1}, v_{-i}) - p(v^{m-1}, v_{-i}) \cdot A_i \gamma \\
&= p(v^m, v_{-i}) \cdot A_i v^m - t_i(v^m, v_{-i}) - p(v^{m-1}, v_{-i}) \cdot A_i \gamma \\
&\geq p(v^m, v_{-i}) \cdot A_i v^m - t_i(v^m, v_{-i}) - p(v^m, v_{-i}) \cdot A_i \gamma \\
&= p(v^m, v_{-i}) \cdot A_i v^{m-1} - t_i(v^{m-1}, v_{-i}),
\end{align*}
\]

where the second last line follows from Lemma 1. \[ \square \]

**Proof of Proposition 2.**

It follows from Lemma 4 that only the following constraints are relevant:

\[
\begin{align*}
p(v^m, v_{-i}) \cdot A_i v^m - t_i(v^m, v_{-i}) &\geq p(v^{m-1}, v_{-i}) \cdot A_i v^{m-1} - t_i(v^{m-1}, v_{-i}) \quad \text{for } m = 2, 3, \ldots, M; \\
p(v^m, v_{-i}) \cdot A_i v^m - t_i(v^m, v_{-i}) &\geq p(v^{m+1}, v_{-i}) \cdot A_i v^{m+1} - t_i(v^{m+1}, v_{-i}) \quad \text{for } m = 1, 2, \ldots, M - 1; \\
p(v^1, v_{-i}) \cdot A_i v^1 - t_i(v^1, v_{-i}) &\geq 0.
\end{align*}
\]
At an optimal solution, it must be the case that
\[
p(v^m, v_{-i}) \cdot A_i v^m - t_i^*(v^m, v_{-i}) = p(v^{m-1}, v_{-i}) \cdot A_i v^m - t_i^*(v^{m-1}, v_{-i}) \text{ for } m = 2, 3, ..., M;
p(v^1, v_{-i}) \cdot A_i v^1 - t_i^*(v^1, v_{-i}) = 0.
\]

Suppose not. Choose the largest \( m \) such that the inequality is not binding and let \( \epsilon \) be the slack in this inequality. We can increase \( t_i^*(v^{m'}, v_{-i}) \) by \( \epsilon \) for all \( m' \geq m \) and achieve a higher expected revenue. It follows from Lemma 5 that no constraints are violated. This contradicts with the optimality of \( t^* \).

### A.3 Proof of Lemma 2

**Proof.** It follows from Proposition 2 that, the agents’ rent will be
\[
U_i(v^1, v_{-i}) = p(v^1, v_{-i}) \cdot A_i v^1 - t_i^*(v^1, v_{-i}) = 0;
\]
\[
U_i(v^m, v_{-i}) = p(v^m, v_{-i}) \cdot A_i v^m - t_i^*(v^m, v_{-i})
\]
\[
= p(v^{m-1}, v_{-i}) \cdot A_i v^m - t_i^*(v^{m-1}, v_{-i})
\]
\[
= U_i(v^{m-1}, v_{-i}) + p(v^{m-1}, v_{-i}) \cdot A_i \gamma \text{ for } m = 2, 3, ..., M.
\]

By induction,
\[
U_i(v^m, v_{-i}) = \gamma \sum_{m' = 1}^{m-1} p(v^{m'}, v_{-i}) \cdot A_i \text{ and }
\]
\[
t_i^*(v^m, v_{-i}) = p(v^m, v_{-i}) \cdot A_i v^m - \gamma \sum_{m' = 1}^{m-1} p(v^{m'}, v_{-i}) \cdot A_i.
\]

### A.4 Graph is well defined

The following recursive procedure offers a consistent way to select the best deviation.

**Step 1)** Choose \( v_i \in V_i^T \) and let \( V_i^{T'} = \{v_i\} \).

**Step 2)** If for some \( v_i \in V_i^T - V_i^{T'} \) and \( v_i^- \in V_i^{T'} \), \( p(v_i, v_{-i}) \cdot v_i - t_i^*(v_i, v_{-i}) = p(v_i^-, v_{-i}) \cdot v_i - t_i^*(v_i^-, v_{-i}) \), write \( v_i \sim_i^{p,v_i} v_i^- \) and update \( V_i^{T'} = V_i^{T'} \cup \{v_i\} \). If not, choose some \( v_i \in V_i^T - V_i^{T'} \) and update \( V_i^{T'} = V_i^{T'} \cup \{v_i\} \). Repeat until \( V_i^{T'} = V_i^T \).

**Step 3)** Choose \( v_i \in V_i - V_i' \) such that \( p(v_i, v_{-i}) \cdot v_i - t_i^*(v_i, v_{-i}) = p(v_i^-, v_{-i}) \cdot v_i - t_i^*(v_i^-, v_{-i}) \) for some \( v_i^- \in V_i' \). Write \( v_i \succ_i^{p,v_i} v_i^- \) and update \( V_i' = V_i' \cup \{v_i\} \). Repeat until \( V_i' = V_i \).
We show in a series of lemmas that such recursive procedure is well defined. Lemma 6 says that $V_i^T \neq \emptyset$. Hence, Step 1) is well defined. Step 3) is well defined by Lemma 7. Finally, each agent has finitely many payoff types and the procedure ends after finitely many rounds.

**Lemma 6.** Fix a decision rule $p$ and other agents’ reports $v_{-i}$,

$$p(v_i, v_{-i}) \cdot v_i - t_i^*(v_i, v_{-i}) = 0 \text{ for some } v_i \in V_i.$$

*Proof.* Suppose not. $\forall v_i \in V_i$, we have $p(v_i, v_{-i}) \cdot v_i - t_i^*(v_i, v_{-i}) > 0$. Let

$$\lambda = \min_{v_i \in V_i} \{ p(v_i, v_{-i}) \cdot v_i - t_i^*(v_i, v_{-i}) \} > 0.$$  

The mechanism designer could increase $t_i^*(v_i, v_{-i})$ by $\lambda$ for all $v_i \in V_i$ and achieve a higher expected revenue. This contradicts with the optimality of $t^*$.

**Lemma 7.** Fix a decision rule $p$ and other agents’ reports $v_{-i}$. If $V_i' \supseteq V_i^T$ and $V_i - V_i' \neq \emptyset$, there exists $v_i \in V_i - V_i'$ such that

$$p(v_i, v_{-i}) \cdot v_i - t_i^*(v_i, v_{-i}) = p(v_i, v_{-i}) \cdot v_i - t_i^*(v_i, v_{-i})$$

for some $v_i^- \in V_i'$.

*Proof.* Suppose not.

$$p(v_i, v_{-i}) \cdot v_i - t_i^*(v_i, v_{-i}) > p(v_i', v_{-i}) \cdot v_i - t_i^*(v_i', v_{-i}) \text{ for } v_i \in V_i - V_i', v_i' \in V_i'.$$

Furthermore, the following inequalities hold,

$$p(v_i, v_{-i}) \cdot v_i - t_i^*(v_i, v_{-i}) > 0 \text{ for } v_i \in V_i - V_i'; \quad (20)$$

$$p(v_i, v_{-i}) \cdot v_i - t_i^*(v_i, v_{-i}) \geq p(v_i', v_{-i}) \cdot v_i - t_i^*(v_i', v_{-i}) \text{ for } v_i, v_i' \in V_i - V_i'. \quad (21)$$

If (20) is violated, $p(v_i, v_{-i}) \cdot v_i - t_i^*(v_i, v_{-i}) = 0$ for some $v_i \in V_i - V_i'$, we have a contradiction. (21) is simply the incentive constraint.

Let

$$a = \min_{v_i \in V_i - V_i'} \{ p(v_i, v_{-i}) \cdot v_i - t_i^*(v_i, v_{-i}) \} > 0,$$

$$b = \min_{v_i \in V_i - V_i', v_i' \in V_i} \{ p(v_i, v_{-i}) \cdot v_i - t_i^*(v_i, v_{-i}) - [p(v_i', v_{-i}) \cdot v_i - t_i^*(v_i', v_{-i})] \} > 0,$$

The designer could increase $t_i^*(v_i, v_{-i})$ by $\min\{a, b\}$ for all $v_i \in V_i - V_i'$. The expected revenue is higher and no constraints are violated. This contradicts with the optimality of $t^*$.  

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A.5 Proof of Theorem 2

The logic of the proof is summarized as follows: Step 1) begins by assuming \( \pi \) is regular and satisfies an additional constraint, called non-singularity. Step 1) then explicitly constructs an assumption about (the distribution of) bidders’ beliefs, against which we show in Step 2) and Step 3) that, the optimal Bayesian mechanism design problem reduces to the same objective function as the optimal dominant-strategy mechanism design problem. Regularity condition on \( \pi \) ensures that at the optimal, the cyclical monotonicity constraint is automatically satisfied. That is, uniform graph and regularity ensures that, against the assumption constructed in Step 2), the optimal dominant-strategy mechanism and the optimal Bayesian mechanism deliver the same expected utility for the mechanism designer. Step 4) discusses the case of singular \( \pi \).

**Step 1)** Given \( \pi \), write

\[
\pi_{v_i} = \sum_{v_{-i} \in V_{-i}} \pi(v_i, v_{-i})
\]

for the marginal probability of payoff type \( v_i \) and write

\[
G_i(v_i) = \sum_{v'_{i} \succeq i} v'_i \pi_{v'_i}
\]

for the associated distribution function.

Let \( \sigma_{v_i} = \pi(\cdot|v_i) \) be the conditional distribution over the payoff types of agents \( j \neq i \), conditional on agent \( i \) having payoff type \( v_i \). Say \( \pi \) is nonsingular if the collection of vectors \( \{\sigma_{v_i}\}_{v_i \in V_i} \) is linearly independent. Suppose \( \pi \) is nonsingular.

We define agent \( i \)'s beliefs as follows:

\[
\tau_{v_i} = \frac{1}{G_i(v_i)} \sum_{v'_{i} \succeq_i v_i} \pi_{v'_i} \sigma_{v'_i}.
\]

Note that the collection \( \{\tau_{v_i}\}_{v_i \in V_i} \) is linearly independent by the nonsingularity of \( \pi \). Since

\[
\sum_{v'_{i} \succeq_i v_i} \pi_{v'_i} \sigma_{v'_i} = \sum_{v'_{i} \succ_i v_i} G_i(v'_i) \tau_{v'_i} + \pi_{v_i} \sigma_{v_i},
\]

we have the following equivalent recursive definition of \( \tau_{v_i} \):

\[
\tau_{v_i} = \sigma_{v_i} \text{ for } v_i \in V^I_i; \tag{22}
\]

\[
\tau_{v_i} = \frac{1}{G_i(v_i)} \left[ \sum_{v'_{i} \tau_{v_i}} G_i(v'_i) \tau_{v'_i} + \pi_{v_i} \sigma_{v_i} \right] \text{ for } v_i \in V_i - V^I_i. \tag{23}
\]
Step 2) We argue that certain constraints in the Bayesian problem can be manipulated or even ignored without cost to the mechanism designer in steps 2.1-2.3. To further simplify the notations, let \( \vec{p}^v_i \cdot v_i \) and \( \vec{t}^v_i \) denote respectively the vectors \((p(v_i, \cdot) \cdot v_i)_{v_j \in V_i} \) and \((t_i(v_i, \cdot))_{v_j \in V_i} \). Hence, the constraints in the Bayesian problem can be written as:

\[
\forall i \in \mathcal{I}, \forall v_i, v'_i \in V_i, \quad \tau^v_i \cdot (\vec{p}^v_i \cdot v_i - \vec{t}^v_i) \geq 0, \quad (IR^v_i)
\]

\[
\tau^v_i \cdot (\vec{p}^v_i \cdot v_i - \vec{t}^v_i) \geq \tau^{v_i'} \cdot (\vec{p}^{v_i'} \cdot v_i - \vec{t}^{v_i'}), \quad (IC^{v_i \rightarrow v'_i})
\]

Step 2.1) If \( v_i \) and \( v'_i \) can not be ordered or \( v'_i \succ_i^+ v_i \), the following incentive constraint can be ignored:

\[
\tau^v_i \cdot (\vec{p}^v_i \cdot v_i - \vec{t}^v_i) \geq \tau^{v'_i} \cdot (\vec{p}^{v'_i} \cdot v_i - \vec{t}^{v'_i}) \tag{24}
\]

Since the collection \( \{\tau^v_i\}_{v_i \in V_i} \) is linearly independent, there exists a lottery \( \lambda \) such that \( \tau^{v''}_i \cdot \lambda = 0 \) for \( v''_i \succ_i^+ v'_i \) and \( \tau^{v''}_i \cdot \lambda > 0 \) otherwise. Since \( \sigma^{v_i'} \) is a linear combination of \( \tau^{v_i'} \) and \( \tau^{v''_i} \) for \( v''_i \succ_i v'_i \), we have \( \sigma^{v_i'} \cdot \lambda = 0 \). By adding (sufficiently large scale of) \( \lambda \) to \( \vec{t}^{v_i'} \), the revenue is the same since \( \sigma^{v_i'} \cdot \lambda = 0 \) and no constraint is violated. Therefore, the constraint (24) can be relaxed.

Step 2.2) for any mechanism \((p,t)\) that satisfies the remaining constraints, there exists a mechanism \((p,t')\) which satisfies the constraints \(\langle IR^v_i \rangle\) for \( v_i \in V_i \) and \(\langle IC^{v_i \rightarrow v_i'} \rangle\) for \( v_i \in V_i - V_i^T \) with equality and achieves at least as high a revenue. To prove this, fix any mechanism \((p,t)\) that satisfies the remaining constraints. Suppose \(\langle IC^{v_i \rightarrow v_i'} \rangle\) holds with strict inequality. Let \( \Gamma \) denote the matrix whose rows are the vectors \( \{\tau^v_i\} \) for all \( v_i \in V_i \) and let \( \{\Gamma^{-v_i}, \sigma^{-v}_i\} \) be the matrix obtained by replacing \( \tau^v_i \) by vector \( \sigma^{-v}_i \). Note that the matrix is full rank. We can thus solve the following equation for \( \lambda \):

\[
\tau^{v_i'} \cdot \lambda = 0 \text{ for } v'_i \neq v_i;
\]

\[
\sigma^{-v_i} \cdot \lambda = 1.
\]

Note that because \( \tau^{v_i'} \cdot \lambda = 0 < \sigma^{-v_i} \cdot \lambda = 1 \), and because \( \tau^{v_i'} \) is a convex combination of \( \tau^{v''_i} \) for \( v''_i \succ_i v_i \) and \( \sigma^{-v_i} \), we have \( \tau^{v_i} \cdot \lambda < 0 \). We shall add the vector \( \epsilon \lambda \) to \( \vec{t}^{v_i} \). Because \( \tau^{v_i'} \cdot \lambda = 0 \) for \( v'_i \neq v_i \), no constraints for types \( v'_i \neq v_i \) are affected. The only constraint that could be violated is

\[
\tau^{v_i} \cdot (\vec{p}^{v_i} \cdot v_i - \vec{t}^{v_i}) \geq \tau^{v_i} \cdot (\vec{p}^{v_i'} \cdot v_i - \vec{t}^{v_i'}),
\]

\[
34
\]
and this constraint was slack by assumption. Let $S^v_i = \tau^v_i \cdot (\bar{p}^v_i \cdot v_i - \bar{v}^v_i) - \tau^v_i \cdot (\bar{p}^{v'}_i \cdot v_i - \bar{v}^{v'}_i) > 0$, and we can choose $\epsilon = -S^v_i / (\tau^v_i \cdot \lambda) > 0$ and the inequality becomes binding. Since, $\sigma^{v_i} \cdot \epsilon \lambda > 0$, the auctioneer profits from this modification.

**Step 2.3** each $\langle IR^v_i \rangle$ can be treated as equality without loss of generality. Define $S^v_i = \tau^v_i \cdot (\bar{p}^v_i \cdot v_i - \bar{v}^v_i) \geq 0$ to be the slack in $\langle IR^v_i \rangle$. Construct a lottery $\lambda$ that satisfies $\tau^v_i \cdot \lambda = S^v_i$. By the full rank condition, such a lottery can be found. We will add $\lambda$ to each $\bar{v}^v_i$. No constraints will be affected, but now each IR constraint holds with equality. Revenue is at least the same. Indeed,

$$
\sum_{v_i \in V_i} \pi^v_i \sigma^v_i \cdot \lambda = \sum_{v_i \in V_i} \pi^v_i \tau^v_i \cdot \lambda + \sum_{v_i \in V_i - V_i^f} \left[ G_i(v_i) \tau^v_i - \sum_{v_i' \succ v_i} G_i(v_i') \tau^v_i \right] \cdot \lambda
$$

$$
= \sum_{v_i \in V_i^f} G_i(v_i) \tau^v_i \cdot \lambda
$$

$$
> 0.
$$

**Step 3** We can rewrite the Bayesian maximization problem as follows:

$$
\max_{p(\cdot), t(\cdot)} \sum_{v \in V} \pi(v) \sum_{i \in I} t_i(v)
$$

$$
= \sum_{i \in I} \sum_{v_i \in V_i} \pi^v_i \sigma^v_i \cdot \bar{v}^v_i
$$

subject to

$$
\forall i \in I, \forall v_i \in V_i,
$$

$$
\tau^v_i \cdot (\bar{p}^{v_i} \cdot v_i - \bar{v}^{v_i}) = 0,
$$

$$
\tau^v_i \cdot (\bar{p}^{v_i} \cdot v_i - \bar{v}^{v_i}) = \tau^{v_i} \cdot (\bar{p}^{v_i} \cdot v_i - \bar{v}^{v_i}).
$$

That is,

$$
\tau^v_i \cdot \bar{v}^{v_i} = \tau^{v_i} \cdot \bar{p}^{v_i} \cdot v_i \tag{26}
$$

$$
\tau^{v_i} \cdot \bar{v}^{v_i} = \tau^{v_i} \cdot \bar{p}^{v_i} \cdot v_i. \tag{27}
$$

It follows that for $v_i \in V_i^f$,

$$
\pi^v_i \sigma^v_i \cdot \bar{v}^{v_i} = \pi^v_i \tau^{v_i} \cdot \bar{v}^{v_i} = \pi^v_i \tau^{v_i} \cdot \bar{p}^{v_i} \cdot v_i = \pi^v_i \sigma^v_i \cdot \bar{p}^{v_i} \cdot v_i; \tag{28}
$$

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moreover, for \( v_i \in V_i - V'_i \),

\[
\pi^{v_i} \sigma^{v_i} \cdot \bar{p}^{v_i} = G_i(v_i) \tau^{v_i} \cdot \bar{p}^{v_i} - \sum_{v'_i \succ v_i} G_i(v'_i) \tau^{v'_i} \cdot \bar{p}^{v'_i} \tag{29}
\]

\[
= G_i(v_i) \tau^{v_i} \cdot \bar{p}^{v_i} \cdot v_i - \sum_{v'_i \succ v_i} G_i(v'_i) \tau^{v'_i} \cdot \bar{p}^{v'_i} \cdot v'_i \tag{30}
\]

\[
= \left[ \sum_{v'_i \succ v_i} G_i(v'_i) \tau^{v'_i} + \pi^{v_i} \sigma^{v_i} \right] \cdot \bar{p}^{v_i} \cdot v_i - \sum_{v'_i \succ v_i} G_i(v'_i) \tau^{v'_i} \cdot \bar{p}^{v'_i} \cdot v'_i \tag{31}
\]

\[
= \pi^{v_i} \sigma^{v_i} \cdot \bar{p}^{v_i} \cdot v_i - \sum_{v'_i \succ v_i} G_i(v'_i) \tau^{v'_i} \cdot \bar{p}^{v'_i} \cdot (v'_i - v_i) \tag{32}
\]

where (28) follows from (22), (30) follows from (26) and (27), and (31) follows from (23).

Therefore

\[
\sum_{v_i \in V_i} \pi^{v_i} \sigma^{v_i} \cdot \bar{p}^{v_i} = \sum_{v_i \in V'_i} \pi^{v_i} \sigma^{v_i} \cdot \bar{p}^{v_i} + \sum_{v_i \in V_i - V'_i} \pi^{v_i} \sigma^{v_i} \cdot \bar{p}^{v_i} \tag{29}
\]

\[
= \sum_{v_i \in V'_i} \pi^{v_i} \sigma^{v_i} \cdot \bar{p}^{v_i} \cdot v_i - \sum_{v_i \in V_i - V'_i} \pi^{v_i} \sigma^{v_i} \cdot \bar{p}^{v_i} \cdot v_i - \sum_{v_i \in V_i - V'_i} \sum_{v'_i \succ v_i} G_i(v'_i) \tau^{v'_i} \cdot \bar{p}^{v'_i} \cdot (v'_i - v_i) \tag{30}
\]

\[
= \sum_{v_i \in V_i} \pi^{v_i} \sigma^{v_i} \cdot \bar{p}^{v_i} \cdot v_i - \sum_{v_i \in V_i - V'_i} \sum_{v'_i \succ v_i} \pi^{v_i} \sigma^{v_i} \cdot \bar{p}^{v'_i} \cdot [v'_i - (v'_i)^-] \tag{33}
\]

\[
= \sum_{v_i \in V_i} \left[ \pi^{v_i} \sigma^{v_i} \cdot \bar{p}^{v_i} \cdot v_i - \sum_{v'_i \leq v_i} \pi^{v_i} \sigma^{v_i} \cdot \bar{p}^{v'_i} \cdot [v'_i - (v'_i)^-] \right] \tag{34}
\]

where (33) follows from (28) and (32), and (34) follows from the equality:\(^{21}\)

\[
\sum_{v_i \in V_i - V'_i} \sum_{v'_i \succ v_i} G_i(v'_i) \tau^{v'_i} \cdot \bar{p}^{v'_i} \cdot (v'_i - v_i) \tag{36}
\]

\[
= \sum_{v_i \in V_i} \sum_{v'_i \leq v_i} \pi^{v_i} \sigma^{v_i} \cdot \bar{p}^{v'_i} \cdot [v'_i - (v'_i)^-]. \tag{37}
\]

\(^{21}\)Indeed, the coefficient for any \( \bar{p}^{v_i} \cdot (v'_i - v_i) \) is the same. Indeed, fix any \( \bar{p}^{v_i} \cdot (v'_i - v_i) \), it is easy to see that for (36), \( G_i(v'_i) \tau^{v'_i} \) is the coefficient. And for (34), it is the summation of \( \pi^{v_i} \sigma^{v_i} \) for all \( v_i \geq v'_i \).
It follows from (35) that the objective function of the Bayesian problem is

\[ \sum_{i \in I} \sum_{v_i \in V_i} \pi^{v_i} \sigma^{v_i} \cdot \tilde{t}^{v_i} \]

\[ = \sum_{i \in I} \sum_{v_i \in V_i} \left[ \pi^{v_i} \sigma^{v_i} \cdot \tilde{p}^{v_i} \cdot v_i - \sum_{v_i' \leq v_i} \pi^{v_i} \sigma^{v_i} \cdot \tilde{p}^{(v_i')^-} \cdot [v_i' - (v_i')^-] \right] \]

\[ = \sum_{i \in I} \sum_{v_i \in V_i} \sum_{v_{-i} \in V_{-i}} \pi(v_i, v_{-i}) \left[ p(v_i) \cdot v_i - \sum_{v_i' \leq v_i} p((v_i')^-, v_{-i}) \cdot [v_i' - (v_i')^-] \right] \]

Regularity ensures that the optimal dominant-strategy mechanism and the optimal Bayesian mechanism deliver the same expected utility for the mechanism designer.

**Step 4)** Now consider an arbitrary regular \( v \), not necessarily non-singular. We can find a sequence of \( \pi_n \) converging to \( \pi \) such that each \( \pi_n \) is nonsingular. Furthermore, for \( n \) sufficiently large, the set of maximizers is a subset of the set of maximizers \( p^* \) and hence, \( \pi_n \) is regular. Applying a limiting argument as in Chung and Ely (2007), we can show the Bayesian and maxmin foundations in this case as well.

### A.6 Proof of Theorem 1

In the general environment, we have the following definition of regularity: cyclical monotonicity constraint (7) is automatically satisfied for all \( p^* \) that maximizes the reduced objective function.

\[ \sum_{i \in I} \sum_{v_i \in V_i} \sum_{v_{-i} \in V_{-i}} \pi(v_i, v_{-i}) \left[ p(v_i, v_{-i}) \cdot v_i - \sum_{v_i' \leq v_i} p((v_i')^-, v_{-i}) \cdot [v_i' - (v_i')^-] \right]. \]

By Theorem 2, under this regularity, uniform graph ensures Bayesian/ maxmin foundations of dominant-strategy mechanisms.

In this proof, we restrict attention to the social choice environment with linear utilities and one-dimensional payoff types. Uniform graph follows from Proposition 2. It remains to show that the regularity condition (2) is sufficient for the regularity condition defined for the general environment. As aforementioned, the objective function becomes

\[ \sum_{v \in V} \pi(v)p(v) \cdot \sum_{i \in I} A_i \gamma_i(v). \]
Therefore, the mechanism designer could increase the incentive to report truthfully and voluntarily participate, the mechanism designer must take care of the following constraints:

Consider the optimal dominant-strategy mechanism design problem. To ensure that agent \( i \) has the incentive to report truthfully and voluntarily participate, the mechanism designer must take care of the following constraints:

\[
p(v^m, v_{-i}) \cdot v^m - t_i(v^m, v_{-i}) \geq p(v^l, v_{-i}) \cdot v^m - t_i(v^l, v_{-i}) \quad \text{and} \\
p(v^m, v_{-i}) \cdot v^m - t_i(v^m, v_{-i}) \geq 0, \forall l, m = 1, 2, 3.
\]

A.7 Proof of Claim 1

Fix a decision rule \( p \) and other agents’ report \( v_{-i} \), we show that for the optimal transfer scheme \( t^* \),

\[
p(v^1, v_{-i}) \cdot v^1 - t^*_i(v^1, v_{-i}) = 0; \\
p(v^2, v_{-i}) \cdot v^2 - t^*_i(v^2, v_{-i}) = p(v^1, v_{-i}) \cdot v^2 - t^*_i(v^1, v_{-i}); \\
p(v^3, v_{-i}) \cdot v^3 - t^*_i(v^3, v_{-i}) = p(v^1, v_{-i}) \cdot v^3 - t^*_i(v^1, v_{-i}).
\]

First, it must be the case that \( v^1 \cdot p(v^1, v_{-i}) - t_i(v^1, v_{-i}) = 0. \) Suppose not,

\[
v^1 \cdot p(v^1, v_{-i}) - t_i(v^1, v_{-i}) > 0 \quad \text{and} \\
v^j \cdot p(v^j, v_{-i}) - t_i(v^j, v_{-i}) \geq v^j \cdot p(v^1, v_{-i}) - t_i(v^1, v_{-i})
\]

\[
= (v^j - v^1) \cdot p(v^1, v_{-i}) + v^1 \cdot p(v^1, v_{-i}) - t_i(v^1, v_{-i}) \\
> 0 \quad \text{for} \enspace j = 2, 3.
\]

Therefore, the mechanism designer could increase \( t_i(v^j, v_{-i}), j = 1, 2, 3 \) by the same amount slightly and achieve a higher expected revenue.

Second, since

\[
v^1 \cdot p(v^1, v_{-i}) - t_i(v^1, v_{-i}) \geq v^1 \cdot p(v^3, v_{-i}) - t_i(v^3, v_{-i}) \quad \text{and} \\
v^3 \cdot p(v^3, v_{-i}) - t_i(v^3, v_{-i}) \geq v^3 \cdot p(v^1, v_{-i}) - t_i(v^1, v_{-i}),
\]

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we have
\[(v^3 - v^1) \cdot p(v^3, v_{-i}) \geq (v^3 - v^1) \cdot p(v^1, v_{-i}),\]
or equivalently
\[(v^2 - v^1) \cdot p(v^1, v_{-i}) \geq (v^2 - v^1) \cdot p(v^3, v_{-i}).\]

Therefore,
\[
v^2 \cdot p(v^1, v_{-i}) - t_i(v^1, v_{-i}) \geq v^1 \cdot p(v^1, v_{-i}) - t_i(v^1, v_{-i}) = 0 \text{ and}
\]
\[
v^2 \cdot p(v^1, v_{-i}) - t_i(v^1, v_{-i}) = (v^2 - v^1) \cdot p(v^1, v_{-i}) + v^1 \cdot p(v^1, v_{-i}) - t_i(v^1, v_{-i}) \geq (v^2 - v^1) \cdot p(v^3, v_{-i}) + v^1 \cdot p(v^3, v_{-i}) - t_i(v^3, v_{-i}) \geq (v^2 - v^1) \cdot p(v^3, v_{-i}) + v^1 \cdot p(v^3, v_{-i}) - t_i(v^3, v_{-i}) = v^2 \cdot p(v^3, v_{-i}) - t_i(v^3, v_{-i}).
\]

It must be the case that \(v^2 \cdot p(v^2, v_{-i}) - t_i(v^2, v_{-i}) = v^2 \cdot p(v^1, v_{-i}) - t_i(v^1, v_{-i}).\) Suppose not, \(v^2 \cdot p(v^2, v_{-i}) - t_i(v^2, v_{-i}) > v^2 \cdot p(v^1, v_{-i}) - t_i(v^1, v_{-i}).\) The mechanism designer could increase \(t_i(v^2, v_{-i})\) slightly and achieve a higher expected revenue.

Similarly,
\[
v^3 \cdot p(v^3, v_{-i}) - t_i(v^3, v_{-i}) = v^3 \cdot p(v^1, v_{-i}) - t_i(v^1, v_{-i}).
\]

A.8 Proof of Theorem 3

The theorem follows from the following two lemmas.

**Lemma 8.** If \(p\) is dsIC-BB, then \(p\) is BIC-BB on all type spaces.

The proof of this lemma is standard and hence omitted.

**Lemma 9.** If \(p\) is BIC-BB on all type spaces, then \(p\) is dsIC-BB.

**Proof.** Suppose to the contrary, \(p\) is BIC-BB on all type spaces and yet, \(p\) is not dsIC-BB.

Case i) \(p\) does not satisfy the cyclical monotonicity constraints. If we first ignore the budget balance constraints, we have a separable environment. Since \(p\) does not satisfy the cyclical monotonicity constraints, \(p\) is not dominant strategy implementable. We can apply the main result of Bergemann and Morris (2005), \(p\) is not interim implementable on all type spaces. Since the choice set of \(t\) is even smaller by further imposing the weak budget balance constraints, \(p\) is not BIC-BB on all type spaces. We have a contradiction.
Case ii) $p$ satisfies the cyclical monotonicity constraints. If we first ignore the budget balance constraints, $p$ is dominant strategy implementable. There exists transfer scheme $t$ such that the mechanism $(p, t)$ satisfies the constraints:

$$\forall i \in I, \forall m, l = 1, 2, ..., M, \forall v_{-i} \in V_{-i},$$

$$p(v^m, v_{-i}) \cdot A_i v^m - t_i(v^m, v_{-i}) \geq 0,$$

$$p(v^m, v_{-i}) \cdot A_i v^m - t_i(v^m, v_{-i}) \geq p(v^l, v_{-i}) \cdot A_i v^m - t_i(v^l, v_{-i})$$

and

Therefore, the mechanism $(p, t')$ also satisfies these constraints, where

$$t'_i(v^m, v_{-i}) = p(v^m, v_{-i}) \cdot A_i v^m - \gamma \sum_{m'=1}^{m-1} p(v^{m'}, v_{-i}) \cdot A_i$$

Since $p$ is not dsIC-BB, it cannot be the case

$$\sum_{i \in I} t'_i(v) \geq 0 \text{ for all } v.$$ 

In other words, there exists $v^* \in V$ such that

$$\sum_{i \in I} t'_i(v^*) < 0.$$ 

Next, consider the following type space: $\Omega_i = V_i, f_i(v_i) = v_i$ and $g_i(v_i)[v^*_{-i}] = 1$. For this type space, Bayesian incentive constraints reduces to the (a subset of) constraints in the dominant-strategy implementation problem. By the definition of uniform graph, for the optimal $t^*$,

$$t^*_i(v^*) = t'_i(v^*).$$

Since $\sum_{i \in I} t^*_i(v^*) < 0$, the decision rule $p$ is not BIC-BB on the constructed type space. Again, we have a contraction. 

References


