Supplement to "Equivalence of Stochastic and Deterministic Mechanisms"

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Appendix B proves the following proposition, which is used in the proof of Theorem 1 in Chen, He, Li, and Sun (2018).

Proposition 2. *Fix a Borel measurable set $D \subseteq V$ with $\lambda(D) > 0$. For any $i \in \mathcal{I}$, let D_i be the projection of D on V_i. For any $v_i \in D_i$, let $D_{-i}(v_i) = \{v_{-i} : (v_i, v_{-i}) \in D\}$. Consider the following system of equations where $\alpha \in L^{\lambda}_{\infty}(D, \mathbb{R})$ are the unknown:*

$$\int_{D_{-i}(v_i)} \alpha(v_i, v_{-i})\, \lambda_{-i}(\mathrm{d}v_{-i}) = 0. \tag{1}$$

for all $i \in \mathcal{I}$ and $v_i \in D_i$. If λ_i is atomless for all $i \in \mathcal{I}$, then the system of equations (1) has a nontrivial bounded solution α.

Appendix C details how to modify the proof of Theorem 1 to prove Theorem 2. Appendix D provides a recipe for the construction of an approximately equivalent mechanism. Appendix E provides examples to illustrate the differences between our approach of mutual purification and the usual purification principle in the literature related to Bayesian games.

B Proof of Proposition 2

We first provide a sketch of the proof. If $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$ are measurable spaces, then a measurable rectangle is a subset $A \times B$ of $X \times Y$, where $A \in \mathcal{X}$ and $B \in \mathcal{Y}$ are measurable

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subsets of X and Y. The sides A, B of the measurable rectangle $A \times B$ can be arbitrary measurable sets. In particular, the sides are not required to be intervals. A measurable rectangle is a discrete rectangle if each of its sides is a finite set. For notational simplicity, we write D_{v_i} rather than $D_{-i}(v_i)$.

Define $\mathscr{E}$ as follows:

$$\mathscr{E} = \Big\{ \sum_{i \in \mathcal{I}} \psi_i(v_i) : \psi_i \in L_\infty^\lambda(D_i, \mathbb{R}), \, \forall i \in \mathcal{I} \Big\}.$$

Then a bounded measurable function $\alpha \in L_\infty^\lambda(D, \mathbb{R})$ is a solution to the system of equations (1) if and only if $\int_D \alpha(v)\varphi(v)\, \lambda(dv) = 0$ for any $\varphi(v) \in \mathscr{E}$. Our objective is to show that $\mathscr{E}$ is not dense in $L_1^\lambda(D, \mathbb{R})$. By Corollary 5.108 in Aliprantis and Border (2006), this implies that the system of equations (1) has a nontrivial bounded solution α.

In what follows, we show that that $\mathscr{E}$ is not dense in $L_1^\lambda(D, \mathbb{R})$ via a series of lemmas. In particular, we construct a measurable function $d(v)$ with a finite set of values, and show that the function cannot be approximated in measure by functions in $\mathscr{E}$. Lemma B.1 and Lemma B.2 are technical preparations for the construction of a discrete rectangle $\tilde{L} = \{(\tilde{v}_1^{i_1}, \tilde{v}_2^{i_2}, \ldots, \tilde{v}_I^{i_I})\}_{1 \le i_j \le \tilde{i}_j, 1 \le j \le I}$ that satisfies certain properties. Lemma B.3 constructs a vector $\bar{w}$ and the measure function $d(v)$ such that it takes a constant value in the neighbourhood of each point in the constructed discrete rectangle. It is then shown that

$$\sum_{1 \le i_j \le \tilde{i}_j, 1 \le j \le I} d(\tilde{v}_1^{i_1}, \tilde{v}_2^{i_2}, \ldots, \tilde{v}_I^{i_I})\, \bar{w}^{\,i_1, i_2, \ldots, i_I} = 1,$$

while

$$\sum_{1 \le i_j \le \tilde{i}_j, 1 \le j \le I} \left(\sum_{1 \le j \le I} \psi_j(\tilde{v}_j^{i_j}) \right) \bar{w}^{\,i_1, i_2, \ldots, i_I} = 0$$

for any $\sum_{1 \le j \le I} \psi_j \in \mathscr{E}$. This further implies that the mapping d cannot be approximated by any function in $\mathscr{E}$. The assumption of atomless distribution ensures that all objects in this proof are well defined.[1]

[1]These lemmas extend the corresponding mathematical results in Arkin and Levin (1972) from the special case with $I = 2$ and λ the uniform distribution on $[0,1] \times [0,1]$ to the general setting in this paper. The corresponding mathematical results in Arkin and Levin (1972) were used to show the following result (see Theorem 2.3 therein): " Suppose that $f_1 \in L_1^\eta(X \times Y, \mathbb{R}^{l_1})$, $f_2 \in L_1^\eta(X \times Y, \mathbb{R}^{l_2})$ and $f_3 \in L_1^\eta(X \times Y, \mathbb{R}^{l_3})$, where $X = Y = [0,1]$ and η is the uniform distribution on $[0,1] \times [0,1]$. Let A be the simplex $\{a = (a_1, \ldots, a_K)\colon \sum_{1 \le k \le K} a_k = 1, a_k \ge 0\}$. Given any measurable function α from $X \times Y$

Lemma B.1. *Let $F \subseteq V$ be a measurable rectangle with sides $Y_i \subseteq V_i$ of measure l_i, $i \in \mathcal{I}$. Assume that*

$$\lambda(D \cap F) \geq (1-\epsilon)\,\lambda(F)$$

for some $0 < \epsilon < 1$. Then, for all i,

$$\lambda_i\{v_i \in V_i:\, \lambda_{-i}(D_{v_i} \cap F_{v_i}) > (1-\sqrt{\epsilon})\,\lambda_{-i}(F_{v_i})\} \geq (1-\sqrt{\epsilon})\,l_i.$$

Proof. Denote

$$\Gamma_i = \{v_i \in V_i:\, \lambda_{-i}(D_{v_i} \cap F_{v_i}) > (1-\sqrt{\epsilon})\,\lambda_{-i}(F_{v_i})\}.$$

Let Γ_i^C be the complement of Γ_i in V_i. We have

$$\begin{aligned}
(1-\epsilon)\,\Pi_{1\leq j\leq I}\, l_j &= (1-\epsilon)\,\lambda(F) \\
&\leq \lambda(D\cap F) \\
&= \int_{V_i} \lambda_{-i}(D_{v_i}\cap F_{v_i})\,\lambda_i(\mathrm{d}v_i) \\
&= \int_{\Gamma_i} \lambda_{-i}(D_{v_i}\cap F_{v_i})\,\lambda_i(\mathrm{d}v_i) + \int_{\Gamma_i^C} \lambda_{-i}(D_{v_i}\cap F_{v_i})\,\lambda_i(\mathrm{d}v_i) \\
&\leq \int_{\Gamma_i} \lambda_{-i}(F_{v_i})\,\lambda_i(\mathrm{d}v_i) + (1-\sqrt{\epsilon}) \int_{\Gamma_i^C} \lambda_{-i}(F_{v_i})\,\lambda_i(\mathrm{d}v_i) \\
&= (\sqrt{\epsilon}+1-\sqrt{\epsilon}) \int_{\Gamma_i} \lambda_{-i}(F_{v_i})\,\lambda_i(\mathrm{d}v_i) + (1-\sqrt{\epsilon}) \int_{\Gamma_i^C} \lambda_{-i}(F_{v_i})\,\lambda_i(\mathrm{d}v_i) \\
&= \sqrt{\epsilon} \int_{\Gamma_i} \lambda_{-i}(F_{v_i})\,\lambda_i(\mathrm{d}v_i) + (1-\sqrt{\epsilon}) \int_{V_i} \lambda_{-i}(F_{v_i})\,\lambda_i(\mathrm{d}v_i) \\
&= \sqrt{\epsilon}\,\lambda_i(\Gamma_i)\,\Pi_{j\neq i}\, l_j + (1-\sqrt{\epsilon})\,\Pi_{1\leq j\leq I}\, l_j,
\end{aligned}$$

where the first line and last line hold because F is a rectangle with sides Y_i of measure l_i, $i \in \mathcal{I}$, the second line follows from the condition that $\lambda(D\cap F) \geq (1-\epsilon)\,\lambda(F)$, the fifth line holds because (1) $D_{v_i} \cap F_{v_i} \subseteq F_{v_i}$; and (2) $\lambda_{-i}(D_{v_i}\cap F_{v_i}) \leq (1-\sqrt{\epsilon})\,\lambda_{-i}(F_{v_i})$ for all $v_i \in \Gamma_i^C$. All other lines are simple algebras. Rearranging the terms, we have $\lambda_i(\Gamma_i) \geq (1-\sqrt{\epsilon})\,l_i$. □

Lemma B.2. *Let $\tilde{i}_1, \tilde{i}_2, \ldots, \tilde{i}_I$ be positive integers, and let $0 < \epsilon < 1$ be sufficiently small*

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to A, there exists another measurable function $\overline{\alpha}$ from $X \times Y$ to the vertices of the simplex A such that $\int_{[0,1]} f_1(x,y)\alpha(x,y)\,\mathrm{d}y = \int_{[0,1]} f_1(x,y)\overline{\alpha}(x,y)\,\mathrm{d}y$, $\int_{[0,1]} f_2(x,y)\alpha(x,y)\,\mathrm{d}x = \int_{[0,1]} f_2(x,y)\overline{\alpha}(x,y)\,\mathrm{d}x$ and $\int_{[0,1]}\int_{[0,1]} f_3(x,y)\alpha(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_{[0,1]}\int_{[0,1]} f_3(x,y)\overline{\alpha}(x,y)\,\mathrm{d}x\,\mathrm{d}y$."

such that

$$\epsilon' = \Pi_{1\le j\le I}\tilde{i}_j\,\epsilon < 1 \ \ \textit{and}\ \ \Pi_{1\le j\le I}\tilde{i}_j\,\epsilon'^{\frac{1}{2^I}} < 1.$$

Consider the system of measurable rectangles $F^{i_1,\dots,i_I} = \prod_{1\le j\le I} Y_j^{i_j}$*, where* $1 \le i_j \le \tilde{i}_j$ *and* $Y_j^1,\dots,Y_j^{\tilde{i}_j}$ *are pairwise disjoint subsets of* V_j *for* $1 \le j \le I$ *such that*

$$\lambda(F^{i_1,i_2,\dots,i_I} \cap D) \ge (1-\epsilon)\lambda(F^{i_1,i_2,\dots,i_I}).$$

Then there exists a discrete rectangle $\{v_1^{i_1}, v_2^{i_1},\dots,v_I^{i_I}\}_{\{1\le i_j\le \tilde{i}_j,\,1\le j\le I\}}$ *such that*

(1) $(v_1^{i_1},\dots,v_I^{i_I}) \in F^{i_1,\dots,i_I}\cap D$ *for* $1 \le i_j \le \tilde{i}_j$ *and* $1 \le j \le I$*;*

(2) for all $1 \le j \le I$*,* $\{v_j^{i_j}\}$ *are different points for* $1 \le i_j \le \tilde{i}_j$*.*

Proof. First, we consider the set

$$\Gamma_1^{i_1,i_2,\dots,i_I} = \{v_1 \in Y_1^{i_i} :\ \lambda_{-1}(D_{v_1} \cap F_{v_1}^{i_1,i_2,\dots,i_I}) > (1-\sqrt{\epsilon'})\,\lambda_{-1}(F_{v_1}^{i_1,i_2,\dots,i_I})\}.$$

It follows from Lemma B.1 that

$$\lambda_1(\Gamma_1^{i_1,i_2,\dots,i_I}) > (1-\sqrt{\epsilon'})\,\lambda_1(Y_1^{i_1}). \tag{2}$$

For all $1 \le i_1 \le \tilde{i}_1$, let $\Gamma_1^{i_1} = \cap_{1\le i_k\le \tilde{i}_k,\,2\le k\le I}\,\Gamma_1^{i_1,i_2,\dots,i_I}$. We have

$$\begin{aligned}
\lambda_1(\Gamma_1^{i_1}) &= \lambda_1(Y_1^{i_1}) - \lambda_1\left(\cup_{1\le i_k\le\tilde{i}_k,\,2\le k\le I}\,(Y_1^{i_1}\setminus \Gamma_1^{i_1,i_2,\dots,i_I})\right)\\
&\ge \lambda_1(Y_1^{i_1}) - \sum_{1\le i_k\le\tilde{i}_k,2\le k\le I}\left(\lambda_1(Y_1^{i_1}) - \lambda_1(\Gamma_1^{i_1,\dots,i_I})\right)\\
&> \lambda_1(Y_1^{i_1}) - \sum_{1\le i_k\le\tilde{i}_k,2\le k\le I}\left(\lambda_1(Y_1^{i_1}) - (1-\sqrt{\epsilon'})\,\lambda_1(Y_1^{i_1})\right)\\
&= \left(1 - \Pi_{2\le k\le I}\tilde{i}_k\cdot\sqrt{\epsilon'}\right)\lambda_1(Y_1^{i_1})\\
&> 0,
\end{aligned}$$

where the first line is due to algebra of sets, the third line follows from (2). Since λ_1 is atomless and $\lambda_1(\Gamma_1^{i_1}) > 0$ for all $1 \le i_1 \le \tilde{i}_1$, we know that the set $\Gamma_1^{i_1}$ is infinite. Thus, we can choose points $y_1^{i_1} \in \Gamma_1^{i_1}$, $1 \le i_1 < \tilde{i}_1$ such that they are all distinct.

Second, let

$$\Gamma_2^{i_1,\ldots,i_I} = \{v_2 \in Y_2^{i_2} : (\bigotimes_{3\le k\le I} \lambda_k)(D_{(y_1^{i_1},v_2)} \cap F^{i_1,\ldots,i_I}_{(y_1^{i_1},v_2)}) > (1-\epsilon'^{\frac{1}{4}})(\bigotimes_{3\le k\le I} \lambda_k)(F^{i_1,\ldots,i_I}_{(y_1^{i_1},v_2)})\}.$$

Since $y_1^{i_1} \in \Gamma_1^{i_1}$ for any i_1, we have $y_1^{i_1} \in \Gamma_1^{i_1,\ldots,i_I}$ and

$$(\bigotimes_{2\le k\le I} \lambda_k)(D_{y_1^{i_1}} \cap F^{i_1,\ldots,i_I}_{y_1^{i_1}}) > (1-\sqrt{\epsilon'})(\bigotimes_{2\le k\le I} \lambda_k)(F^{i_1,\ldots,i_I}_{y_1^{i_1}}).$$

It follows from Lemma B.1 that

$$\lambda_2(\Gamma_2^{i_1,\ldots,i_I}) \ge (1-\epsilon'^{\frac{1}{4}})\lambda_2(Y_2^{i_2}).$$

Denote $\Gamma_2^{i_2} = \cap_{1\le i_j\le \tilde{i}_j, j\ne 2}\Gamma_2^{i_1,\ldots,i_I}$. We have

$$\begin{aligned}
\lambda_2(\Gamma_2^{i_2}) &= \lambda_2(Y_2^{i_2}) - \lambda_2\left(\cup_{1\le i_k\le \tilde{i}_k, k\ne 2}(Y_2^{i_2}\setminus \Gamma_2^{i_1,\ldots,i_I})\right) \\
&\ge \lambda_2(Y_2^{i_2}) - \sum_{1\le i_k\le \tilde{i}_k, k\ne 2}\left(\lambda_2(Y_2^{i_2}) - \lambda_2(\Gamma_2^{i_1,\ldots,i_I})\right) \\
&\ge \lambda_2(Y_2^{i_2}) - \sum_{1\le i_k\le \tilde{i}_k, k\ne 2}\left(\lambda_2(Y_2^{i_2}) - (1-\epsilon'^{\frac{1}{4}})\lambda_2(Y_2^{i_2})\right) \\
&= \left(1 - \Pi_{1\le k\le I, k\ne 2}\tilde{i}_k \cdot \epsilon'^{\frac{1}{4}}\right)\lambda_2(Y_2^{i_2}) \\
&> 0.
\end{aligned}$$

Since λ_2 is atomless and $\lambda_2(\Gamma_2^{i_2}) > 0$, we can fix points $y_2^{i_2} \in \Gamma_2^{i_2}$ arbitrarily, as long as they are all distinct, and are also different from the points $\{y_1^{i_1}\}_{1\le i_1\le \tilde{i}_1}$.

Repeating this procedure until $I-1$, we can find $y_k^{i_k} \in \Gamma_k^{i_k}$ for $1 \le i_k \le \tilde{i}_k$ and $1 \le k \le I-1$, where $\Gamma_k^{i_k} = \cap_{1\le i_j\le \tilde{i}_j, j\ne k}\Gamma_k^{i_1,\ldots,i_I}$ and $\lambda_k(\Gamma_k^{i_k}) > 0$. In particular,

$$\begin{aligned}
\Gamma_{I-1}^{i_1,\ldots,i_I} =& \Big\{v_{I-1} \in Y_{I-1}^{i_{I-1}} : \lambda_I(D_{(y_1^{i_1},\ldots,y_{I-2}^{i_{I-2}},v_{I-1})} \cap F^{i_1,\ldots,i_I}_{(y_1^{i_1},\ldots,y_{I-2}^{i_{I-2}},v_{I-1})}) \\
&> (1-\epsilon'^{\frac{1}{2^{I-1}}})\lambda_I(F^{i_1,\ldots,i_I}_{(y_1^{i_1},\ldots,y_{I-2}^{i_{I-2}},v_{I-1})})\Big\}.
\end{aligned}$$

Finally, consider the set

$$E^{i_I} = \cap_{1\le i_k\le \tilde{i}_k, 1\le k\le I-1}\left(D_{(y_1^{i_1},\ldots,y_{I-1}^{i_{I-1}})} \cap Y_I^{i_I}\right).$$

Notice that $F^{i_1,\cdots,i_I}_{(y_1^{i_1},\ldots,y_{I-1}^{i_{I-1}})} = Y_I^{i_I}$ for any $i_1,\ldots,i_I$. Then

$$\begin{aligned}
\lambda_I(E^{i_I}) &= \lambda_I\left(\cap_{1\le i_k\le \tilde{i}_k, 1\le k\le I-1}(D_{(y_1^{i_1},\ldots,y_{I-1}^{i_{I-1}})}\cap Y_I^{i_I})\right)\\
&= \lambda_I(Y_I^{i_I}) - \lambda_I\left(\cup_{1\le i_k\le \tilde{i}_k, 1\le k\le I-1}(Y_I^{i_I}\setminus D_{(y_1^{i_1},\ldots,y_{I-1}^{i_{I-1}})})\right)\\
&\ge \lambda_I(Y_I^{i_I}) - \sum_{1\le i_k\le \tilde{i}_k, 1\le k\le I-1}\left(\lambda_I(Y_I^{i_I}) - \lambda_I(D_{(y_1^{i_1},\ldots,y_{I-1}^{i_{I-1}})}\cap Y_I^{i_I})\right)\\
&= \lambda_I(Y_I^{i_I}) - \sum_{1\le i_k\le \tilde{i}_k, 1\le k\le I-1}\left(\lambda_I(Y_I^{i_I}) - \lambda_I(D_{(y_1^{i_1},\ldots,y_{I-1}^{i_{I-1}})}\cap F^{i_1,\cdots,i_I}_{(y_1^{i_1},\ldots,y_{I-1}^{i_{I-1}})})\right)\\
&> \lambda_I(Y_I^{i_I}) - \sum_{1\le i_k\le \tilde{i}_k, 1\le k\le I-1}\left(\lambda_I(Y_I^{i_I}) - (1-\epsilon'^{\frac{1}{2^{I-1}}})\lambda_I(F^{i_1,\cdots,i_I}_{(y_1^{i_1},\ldots,y_{I-1}^{i_{I-1}})})\right)\\
&= \lambda_I(Y_I^{i_I}) - \sum_{1\le i_k\le \tilde{i}_k, 1\le k\le I-1}\left(\lambda_I(Y_I^{i_I}) - (1-\epsilon'^{\frac{1}{2^{I-1}}})\lambda_I(Y_I^{i_I})\right)\\
&= \left(1 - \Pi_{1\le k\le I-1}\tilde{i}_k\cdot \epsilon'^{\frac{1}{2^{I-1}}}\right)\lambda_I(Y_I^{i_I})\\
&> 0.
\end{aligned}$$

The second inequality holds since $y_{I-1}^{i_{I-1}} \in \Gamma_{I-1}^{i_{I-1}} \subseteq \Gamma_{I-1}^{i_1,\ldots,i_I}$, and hence

$$\lambda_I(D_{(y_1^{i_1},\ldots,y_{I-1}^{i_{I-1}})}\cap F^{i_1,\ldots,i_I}_{(y_1^{i_1},\ldots,y_{I-1}^{i_{I-1}})}) > (1-\epsilon'^{\frac{1}{2^{I-1}}})\lambda_I(F^{i_1,\ldots,i_I}_{(y_1^{i_1},\ldots,y_{I-1}^{i_{I-1}})}).$$

Since λ_I is atomless and $\lambda_I(E^{i_I}) > 0$, we can fix points $y_I^{i_I} \in E^{i_I}$ arbitrarily, as long as they are all different, and are different from the points $\{y_j^{i_j}\}_{1\le j\le I-1, 1\le i_j\le \tilde{i}_j}$. By the choice of E^{i_I}, $(y_1^{i_1},\ldots,y_I^{i_I}) \in F^{i_1,\ldots,i_I}\cap D$ for any $1\le i_j\le \tilde{i}_j$ and $1\le j\le I$. This completes the proof. □

We are ready to prove that $\mathscr{E}$ is not dense in $L_1^\lambda(D,\mathbb{R})$. Recall that

$$\mathscr{E} = \Big\{\sum_{i\in\mathcal{I}}\psi_i(v_i) :\ \psi_i\in L_\infty^\lambda(D_i,\mathbb{R}),\ \forall i\in\mathcal{I}\Big\}.$$

Lemma B.3. *$\mathscr{E}$ is not dense in $L_1^\lambda(D,\mathbb{R})$.*

Proof of Lemma B.3. We construct a measurable function $d(v)$ with a finite set of values, which cannot be approximated in measure on $(D,\mathcal{B}(D),\lambda)$ by functions in $\mathscr{E}$. Fix positive integers $\tilde{i}_j$, $1\le j\le I$ such that

$$\sum_{1\le j\le I}\tilde{i}_j < \Pi_{1\le j\le I}\tilde{i}_j.$$

Step (1) We construct a linear mapping T from $\mathbb{R}^{\Pi_{1\le j\le I}\tilde{i}_j}$ to $\mathbb{R}^{\sum_{1\le j\le I}\tilde{i}_j}$ as follows:

$$T(w) = \left\{ \sum_{k\neq j,\, 1\le i_k\le \tilde{i}_k} w^{i_1,i_2,\ldots,i_I} \right\}_{1\le i_j\le \tilde{i}_j,\, 1\le j\le I},$$

where w is a $\Pi_{1\le j\le I}\tilde{i}_j \times 1$ column vector with its typical entry denoted by $w^{i_1,i_2,\ldots,i_I}$. Consider the system of $\sum_{1\le j\le I}\tilde{i}_j$ homogeneous linear equations $T(w)=0$ with $\Pi_{1\le j\le I}\tilde{i}_j$ unknowns. By the construction of positive integers $\tilde{i}_j$, $1\le j\le I$, the number of unknowns is more than the number of equations. Therefore, the system of homogeneous linear equations $T(w)=0$ has nontrivial solutions. We denote by $\bar{w}$ an arbitrarily fixed nontrivial solution of $T(w)=0$, and write $\bar{w}^{i_1,i_2,\ldots,i_I}$ for its typical entry. Also pick numbers $\{d^{i_1,i_2,\ldots,i_I}\}_{1\le i_j\le\tilde{i}_j,\,1\le j\le I}$ such that

$$\sum_{1\le i_j\le \tilde{i}_j,\,1\le j\le I} d^{i_1,i_2,\ldots,i_I}\,\bar{w}^{i_1,i_2,\ldots,i_I} = 1. \tag{3}$$

Step (2) Fix a discrete rectangle

$$L = \{(v_1^{i_1}, v_2^{i_2}, \ldots, v_I^{i_I}) \in D:\ 1\le i_j\le \tilde{i}_j, 1\le j\le I\} \subset D.$$

For all $1\le i_j\le\tilde{i}_j$, $1\le j\le I$, we construct the following measurable rectangles:

$$F^{i_1,i_2,\ldots,i_I} = \{v=(v_1,v_2,\ldots,v_I)\in V:\ |v_j - v_j^{i_j}|\le\delta,\ 1\le j\le I\},\ \text{and}$$
$$G^{i_1,i_2,\ldots,i_I} = \{v=(v_1,v_2,\ldots,v_I)\in \mathbb{R}^{Il}:\ |v_j - v_j^{i_j}|\le\delta,\ 1\le j\le I\}.$$

For sufficiently small δ, $\{F^{i_1,i_2,\ldots,i_I}\}_{1\le i_j\le\tilde{i}_j,1\le j\le I}$ are pairwise disjoint, and $\{G^{i_1,i_2,\ldots,i_I}\}_{1\le i_j\le\tilde{i}_j,1\le j\le I}$ are also pairwise disjoint. Furthermore, by construction, for all $1\le i_j\le\tilde{i}_j$, $1\le j\le I$,

$$F^{i_1,i_2,\ldots,i_I} = G^{i_1,i_2,\ldots,i_I}\cap V \subseteq G^{i_1,i_2,\ldots,i_I}. \tag{4}$$

Let $g=\mathbf{1}_D$ be the indicator function on D, and $g_\delta(v) = \frac{1}{\lambda(B(v,\delta))}\int_{B(v,\delta)} g\,\mathrm{d}\lambda$, where $B(v,\delta)$ is a ball with center v and radius δ. By Lemma 4.1.2 in Ledrappier and Young (1985), $g_\delta\to g$ for λ-almost all $v\in\mathbb{R}^{Il}$ as $\delta\to 0$. Therefore, $\frac{1}{\lambda(B(v,\delta))}\int_{B(v,\delta)}\mathbf{1}_D\,\mathrm{d}\lambda \to \mathbf{1}_D(v)$ for each $v\in D$. Since $(v_1^{i_1}, v_2^{i_2},\ldots,v_I^{i_I})\in D$,

$$\lambda(G^{i_1,i_2,\ldots,i_I}\cap D) \ge (1-\epsilon)\lambda(G^{i_1,i_2,\ldots,i_I}) \tag{5}$$

for sufficiently small δ, where ϵ is given in Lemma B.2. Therefore,

$$\begin{aligned}\lambda(F^{i_1,i_2,\ldots,i_I} \cap D) =&\lambda(G^{i_1,i_2,\ldots,i_I} \cap V \cap D)\\ &= \lambda(G^{i_1,i_2,\ldots,i_I} \cap D)\\ &\geq (1-\epsilon)\,\lambda(G^{i_1,i_2,\ldots,i_I})\\ &\geq (1-\epsilon)\,\lambda(F^{i_1,i_2,\ldots,i_I}),\end{aligned}$$

where the first line and the last line follow from (4), and the second line holds because $D \subseteq V$, and the third line is (5).

To summarize our construction above, we pick $\delta > 0$ sufficiently small such that

$$\lambda(F^{i_1,i_2,\ldots,i_I} \cap D) \geq (1-\epsilon)\,\lambda(F^{i_1,i_2,\ldots,i_I})$$

for all $1 \leq i_j \leq \tilde{i}_j$, $1 \leq j \leq I$.

Step (3) Consider the following function $d(v)$ defined on D:

$$d(v) = \begin{cases} d^{i_1,i_2,\ldots,i_I}, & \text{if } v \in F^{i_1,i_2,\ldots,i_I} \cap D;\\ 0, & \text{otherwise.}\end{cases}$$

In what follows, we show that $d(v)$ cannot be approximated by functions in $\mathscr{E}$ on $(D, \mathcal{B}(D), \lambda)$ in measure.

Suppose that $d(v)$ can be approximated by functions in $\mathscr{E}$ on $(D, \mathcal{B}(D), \lambda)$ in measure. By the definition of $\mathscr{E}$, there exists a sequence of functions $d_n(v) = \sum_{i\in\mathcal{I}} \psi_i^n(v_i)$ that converges to d on some Borel measurable subset C of D such that $\lambda(C) = \lambda(D)$.

By the construction in Step (2) and Lemma B.2, there exists a discrete rectangle $\tilde{L} = \{(\tilde{v}_1^{i_1}, \tilde{v}_2^{i_2}, \ldots, \tilde{v}_I^{i_I})\}_{1\leq i_j\leq \tilde{i}_j,\,1\leq j\leq I}$ such that $(\tilde{v}_1^{i_1}, \tilde{v}_2^{i_2}, \ldots, \tilde{v}_I^{i_I}) \in F^{i_1,i_2,\ldots,i_I} \cap C$ for all $1 \leq i_j \leq \tilde{i}_j$, $1 \leq j \leq I$. Since $\bar{w}$ satisfies that $\sum_{1\leq i_k\leq \tilde{i}_k,\, k\neq j} \bar{w}^{i_1,i_2,\ldots,i_I} = 0$ for all $1 \leq i_j \leq \tilde{i}_j$, $1 \leq j \leq I$, we have

$$\sum_{1\leq i_j\leq \tilde{i}_j,\,1\leq j\leq I} d^{i_1,i_2,\ldots,i_I}\,\bar{w}^{i_1,i_2,\ldots,i_I}$$

$$
\begin{aligned}
&= \lim_{n\to\infty} \sum_{1\le i_j\le \tilde{i}_j,\, 1\le j\le I} d_n(\tilde{v}_1^{i_1}, \tilde{v}_2^{i_2}, \dots, \tilde{v}_I^{i_I})\, \bar{w}^{\,i_1,i_2,\dots,i_I} \\
&= \lim_{n\to\infty} \sum_{1\le i_j\le \tilde{i}_j,\, 1\le j\le I} \left(\sum_{1\le j\le I} \psi_j^n(\tilde{v}_j^{i_j}) \right) \bar{w}^{\,i_1,i_2,\dots,i_I} \\
&= \lim_{n\to\infty} \sum_{1\le j\le I} \sum_{1\le i_j\le \tilde{i}_j} \left(\sum_{1\le i_k\le \tilde{i}_k,\, k\ne j} \bar{w}^{\,i_1,i_2,\dots,i_I} \right) \psi_j^n(\tilde{v}_j^{i_j}) \\
&= 0,
\end{aligned}
$$

which contradicts with (7). Therefore, the function d cannot be approximated by functions in $\mathscr{E}$ on $(D, \mathcal{B}(D), \lambda)$ in measure. This completes the proof. $\square$

C Proof of Theorem 2

The proof of Theorem 2 is analogous to the proof of Theorem 1. We shall not repeat all the arguments. Rather, we focus on aspects of the proof that are unique to Theorem 2.

Recall that h is a function taking values in $\mathbb{R}^N_{++}$. For Theorem 2, we work with the following set:

$$
\dot{\Upsilon}_q = \{q' \in \Upsilon : \mathbb{E}(q' h_j | v_i) = \mathbb{E}(q h_j | v_i) \text{ for all } i \in \mathcal{I} \text{ and } \lambda_i\text{-almost all } v_i \in V_i,\ 1 \le j \le N\}.
$$

Following the proof of Theorem 1, it is easy to show $\dot{\Upsilon}_q$ admits extreme points. Then we proceed to show that all extreme points of Υ_q are deterministic at λ-almost all $v \in V$. While the logic is exactly the same to the proof of Theorem 1, the proof of the following proposition requires additional care. In particular, we shall prove the corresponding version of Lemma B.3. We do not need to make any changes to Lemma B.1 and Lemma B.2.

Proposition 3. *Fix a Borel measurable set $D \subseteq V$ with $\lambda(D) > 0$. For any $i \in \mathcal{I}$, let D_i be the projection of D on V_i. For any $v_i \in D_i$, let $D_{-i}(v_i) = \{v_{-i} : (v_i, v_{-i}) \in D\}$. Consider the following system of equations where $\alpha \in L^{\lambda}_{\infty}(D, \mathbb{R})$ are the unknown:*

$$
\int_{D_{-i}(v_i)} \alpha(v_i, v_{-i})\, h(v_i, v_{-i})\, \lambda_{-i}(\mathrm{d}v_{-i}) = 0. \tag{6}
$$

for all $i \in \mathcal{I}$ and $v_i \in D_i$. If λ_i is atomless for all $i \in \mathcal{I}$, then the system of equations (1) has *a nontrivial bounded solution* α*.*

Define the set $\mathscr{E}'$ as

$$\mathscr{E}' = \{h(v) \cdot \sum_{i \in \mathcal{I}} \psi_i(v_i) : \psi_i \in L_\infty^\lambda(D_i, \mathbb{R}^N), \forall i \in \mathcal{I}\}.$$

Then a bounded measurable function α in $L_\infty^\lambda(D, \mathbb{R})$ is a solution to Problem (6) if and only if $\int_D \alpha\varphi d\lambda = 0$ for any $\varphi \in \mathscr{E}'$. Lemma C.1 below shows that $\mathscr{E}'$ is not dense in $L_1^\lambda(D, \mathbb{R})$. By Corollary 5.108 in Aliprantis and Border (2006), the system of equations (6) has a nontrivial bounded solution α.

Lemma C.1. *$\mathscr{E}'$ is not dense in $L_1^\lambda(D, \mathbb{R})$.*

Proof. We construct a measurable function $d(v)$ with a finite set of values, which cannot be approximated in measure on $(D, \mathcal{B}(D), \lambda)$ by functions in $\mathscr{E}'$. Fix positive integers $\tilde{i}_j$, $1 \leq j \leq I$ such that

$$N \sum_{1 \leq j \leq I} \tilde{i}_j < \Pi_{1 \leq j \leq I} \tilde{i}_j,$$

For any discrete rectangle $L = \{(v_1^{i_1}, v_2^{i_2}, \ldots, v_I^{i_I}) \in D : 1 \leq i_j \leq \tilde{i}_j, 1 \leq j \leq I\}$, we associate a linear mapping T_L from $\mathbb{R}^{\Pi_{1 \leq j \leq I} \tilde{i}_j}$ to $\mathbb{R}^{N \sum_{1 \leq j \leq I} \tilde{i}_j}$:

$$T_L(w) = \Big\{ \sum_{k \neq j, 1 \leq i_k \leq \tilde{i}_k} h(v_1^{i_1}, v_2^{i_2}, \ldots, v_I^{i_I})\, w^{i_1, i_2, \ldots, i_I} \Big\}_{1 \leq i_j \leq \tilde{i}_j, 1 \leq j \leq I},$$

where w is a $\Pi_{1 \leq j \leq I} \tilde{i}_j \times 1$ column vector with its typical entry denoted by $w^{i_1, i_2, \ldots, i_I}$.

Fix a discrete rectangle $\bar{L} \subset D$ such that

(1) $\bar{L} = \{(\bar{v}_1^{i_1}, \bar{v}_2^{i_2}, \ldots, \bar{v}_I^{i_I}) \in D : 1 \leq i_j \leq \tilde{i}_j, 1 \leq j \leq I\}$; and

(2) the rank of the mapping $T_{\bar{L}}$ is maximal among all T_L, say r.

Consider the system of $\sum_{1 \leq j \leq I} \tilde{i}_j$ homogeneous linear equations $T_{\bar{L}}(w) = 0$ with $\Pi_{1 \leq j \leq I} \tilde{i}_j$ unknowns. Since the rank of the mapping $T_{\bar{L}}$ is maximal, there exist r equations and r unknowns for which the corresponding determinant is nonzero. Without loss of generality, we focus on this $r \times r$ matrix and denote it by $\bar{L}_r$, then $\det(\bar{L}_r) \neq 0$.

By the construction of positive integers $\tilde{i}_j$, $1 \leq j \leq I$, the number of unknowns is more than the number of equations. Therefore, the system of homogeneous linear equations $T_{\bar{L}}(w) = 0$ has nontrivial solutions. We denote by $w_{\bar{L}}$ an arbitrarily fixed nontrivial solution of

$T_{\bar{L}}(w) = 0$, and write $w_{\bar{L}}^{i_1,i_2,\ldots,i_I}$ for its typical entry. Also pick numbers $\{d^{i_1,i_2,\ldots,i_I}\}_{1\leq i_j\leq \tilde{i}_j,\, 1\leq j\leq I}$ such that

$$\sum_{1\leq i_j\leq \tilde{i}_j,\, 1\leq j\leq I} d^{i_1,i_2,\ldots,i_I}\, w_{\bar{L}}^{i_1,i_2,\ldots,i_I} = 1. \tag{7}$$

For any discrete rectangle L, we denote by L_r the $r\times r$ submatrix with the same r rows and r columns when constructing $\bar{L}_r$ from $T_{\bar{L}}$. For any discrete rectangle L in a small open neighborhood of $\bar{L}$, we have $\det(L_r)\neq 0$.

Let $w_{\bar{L}}$ be a nontrivial solution of the system corresponding to the discrete rectangle $\overline{L}$ in the sense that $T_{\bar{L}}(w_{\bar{L}}) = 0$. For any discrete rectangle $L\subset D$ such that $\det(L_s)\neq 0$, we provide a solution w_L below such that $T_L(w_L) = 0$.

- Since $\det(L_s)\neq 0$, the rank of the system corresponding to the operator T_L is at least r. Due to the choice of $\overline{L}$, the rank of the system corresponding to the operator T_L is at most r, and hence is r. As a result, the equations that do not occur in the determinant $\det(L_s)$ are linear combinations of the r equations that do.
- We focus on the r equations that occur in the determinant $\det(L_s)$, and let $w_L^{i_1,\ldots,i_I} = w_{\bar{L}}^{i_1,i_2,\ldots,i_I}$ if the column corresponding to the unknown $w_L^{i_1,i_2,\ldots,i_I}$ does not occur in the determinant $\det(L_s)$.
- The remaining r unknowns of $w_L^{i_1,\ldots,i_I}$, corresponding to the columns that occur in the determinant $\det(L_s)$, can be obtained by Cramer's rule.

It follows from the construction above that w_L depends continuously on the r nodes of the discrete rectangle L corresponding to the columns of $\det(L_s)$.

For all $1\leq i_j\leq \tilde{i}_j$, $1\leq j\leq I$, we construct the following measurable rectangles:

$$F^{i_1,i_2,\ldots,i_I} = \{v = (v_1,v_2,\ldots,v_I)\in V : |v_j - \bar{v}_j^{i_j}|\leq \delta,\ 1\leq j\leq I\},\ \text{and}$$
$$G^{i_1,i_2,\ldots,i_I} = \{v = (v_1,v_2,\ldots,v_I)\in \mathbb{R}^{Il} : |v_j - \bar{v}_j^{i_j}|\leq \delta,\ 1\leq j\leq I\}.$$

For sufficiently small δ, $\{F^{i_1,i_2,\ldots,i_I}\}_{1\leq i_j\leq \tilde{i}_j,\, 1\leq j\leq I}$ are pairwise disjoint, and $\{G^{i_1,i_2,\ldots,i_I}\}_{1\leq i_j\leq \tilde{i}_j,\, 1\leq j\leq I}$ are also pairwise disjoint. Furthermore, by construction, for all $1\leq i_j\leq \tilde{i}_j$, $1\leq j\leq I$,

$$F^{i_1,i_2,\ldots,i_I} = G^{i_1,i_2,\ldots,i_I}\cap V \subseteq G^{i_1,i_2,\ldots,i_I}. \tag{8}$$

Let $g = \mathbf{1}_D$ be the indicator function on D, and $g_\delta(v) = \frac{1}{\lambda(B(v,\delta))}\int_{B(v,\delta)} g\, \mathrm{d}\lambda$, where $B(v,\delta)$ is a ball with center v and radius δ. By Lemma 4.1.2 in Ledrappier and Young (1985),

$g_\delta \to g$ for λ-almost all $v \in \mathbb{R}^{Il}$ as $\delta \to 0$. Therefore, $\frac{1}{\lambda(B(v,\delta))} \int_{B(v,\delta)} \mathbf{1}_D \, \mathrm{d}\lambda \to \mathbf{1}_D(v)$ for each $v \in D$. Since $\bar{L} \subset D$,

$$\lambda(G^{i_1,i_2,\ldots,i_I} \cap D) \geq (1-\epsilon)\lambda(G^{i_1,i_2,\ldots,i_I}) \tag{9}$$

for sufficiently small δ, where ϵ is given in Lemma B.2. Therefore,

$$\begin{aligned}
\lambda(F^{i_1,i_2,\ldots,i_I} \cap D) =& \lambda(G^{i_1,i_2,\ldots,i_I} \cap V \cap D) \\
&= \lambda(G^{i_1,i_2,\ldots,i_I} \cap D) \\
&\geq (1-\epsilon)\, \lambda(G^{i_1,i_2,\ldots,i_I}) \\
&\geq (1-\epsilon)\, \lambda(F^{i_1,i_2,\ldots,i_I}),
\end{aligned}$$

In addition, since $\sum_{1 \leq i_j \leq \tilde{i}_j, 1 \leq j \leq I} d^{i_1,\ldots,i_I} \cdot w_L^{i_1,\ldots,i_I}$ is continuous in the discrete rectangle, for sufficiently small δ, $\sum_{1 \leq i_j \leq \tilde{i}_j, 1 \leq j \leq I} d^{i_1,\ldots,i_I} \cdot w_L^{i_1,\ldots,i_I} \geq \frac{1}{2}$ for

$$L = \{(v_1^{i_1}, \ldots, v_I^{i_I}) \in F^{i_1,\ldots,i_I} \cap D \colon 1 \leq i_j \leq \tilde{i}_j, 1 \leq j \leq I\}.$$

To summarize our construction above, we pick $\delta > 0$ sufficiently small such that

(1) $\lambda(F^{i_1,i_2,\ldots,i_I} \cap D) \geq (1-\epsilon)\, \lambda(F^{i_1,i_2,\ldots,i_I})$; and

(2) $\sum_{1 \leq i_j \leq \tilde{i}_j,\, 1 \leq j \leq I} d^{i_1,i_2,\ldots,i_I}\, w_L^{i_1,i_2,\ldots,i_I} \geq \frac{1}{2}$ for any discrete rectangle

$$L = \{(v_1^{i_1}, \ldots, v_I^{i_I}) \in F^{i_1,\ldots,i_I} \cap D \colon 1 \leq i_j \leq \tilde{i}_j, 1 \leq j \leq I\}.$$

Step (2). Consider the following function $d(v)$:

$$d(v) = \begin{cases} d^{i_1,i_2,\ldots,i_I}, & \text{if } v \in F^{i_1,i_2,\ldots,i_I} \cap D; \\ 0, & \text{otherwise.} \end{cases}$$

In what follows, we show that the function $d(v)$ cannot be approximated by functions $\mathscr{E}$ on $(D, \mathcal{B}(D), \lambda)$ in measure. Suppose that the function $d(v)$ can be approximated by functions in $\mathscr{E}$ on $(D, \mathcal{B}(D), \lambda)$ in measure. Then there exists a sequence of functions $d_n(v) = h(v) \sum_{i \in \mathcal{I}} \psi_i^n(v_i)$ that converges to d on some Borel measurable subset C such that $\lambda(C) = \lambda(D)$.

By the construction in Step (2) and Lemma B.2, there exists a discrete rectangle $L =$

$\{(v_1^{i_1}, v_2^{i_2}, \ldots, v_I^{i_I})\}_{1\le i_j\le \tilde{i}_j,\,1\le j\le I}$ such that $(v_1^{i_1}, v_2^{i_2}, \ldots, v_I^{i_I}) \in F^{\,i_1,i_2,\ldots,i_I} \cap C$ for all $1 \le i_j \le \tilde{i}_j$, $1 \le j \le I$. Since $\sum_{k\neq j,\,1\le i_k\le \tilde{i}_k} h(v_1^{i_1}, v_2^{i_2}, \ldots, v_I^{i_I})\, w^{\,i_1,i_2,\ldots,i_I} = 0$ for any $j \in \mathcal{I}$, we have

$$
\begin{aligned}
&\sum_{1\le i_j\le \tilde{i}_j,\,1\le j\le I} d^{\,i_1,i_2,\ldots,i_I}\, w_L^{i_1,i_2,\ldots,i_I} \\
&= \lim_{n\to\infty} \sum_{1\le i_j\le \tilde{i}_j,\,1\le j\le I} d_n(v_1^{i_1}, v_2^{i_2}, \ldots, v_I^{i_I})\, w_L^{i_1,i_2,\ldots,i_I} \\
&= \lim_{n\to\infty} \sum_{1\le i_j\le \tilde{i}_j,\,1\le j\le I} \left(h(v_1^{i_1}, v_2^{i_2}, \ldots, v_I^{i_I}) \sum_{j\in\mathcal{I}} \psi_j^n(v_j^{i_j}) \right) w_L^{i_1,i_2,\ldots,i_I} \\
&= \lim_{n\to\infty} \sum_{1\le i_j\le \tilde{i}_j,\,1\le j\le I} \left\{ \left(w_L^{i_1,i_2,\ldots,i_I}\, h(v_1^{i_1}, v_2^{i_2}, \ldots, v_I^{i_I}) \right) \sum_{j\in\mathcal{I}} \psi_j^n(v_j^{i_j}) \right\} \\
&= \lim_{n\to\infty} \sum_{j\in\mathcal{I}} \left\{ \sum_{1\le i_j\le \tilde{i}_j,\,1\le j\le I} w_L^{i_1,i_2,\ldots,i_I}\, h(v_1^{i_1}, v_2^{i_2}, \ldots, v_I^{i_I}) \right\} \psi_j^n(v_j^{i_j}) \\
&= 0.
\end{aligned}
$$

However, $\sum_{1\le i_j\le \tilde{i}_j, 1\le j\le I} d^{\,i_1,i_2,\ldots,i_I}\, w_L^{i_1,i_2,\ldots,i_I} \ge \frac{1}{2}$. We arrive at a contradiction. As a result, the function d cannot be approximated by functions in $\mathscr{E}'$ on $(D, \mathcal{B}(D), \lambda)$ in measure. This completes the proof. $\square$

D Approximately equivalent deterministic mechanisms

Chen, He, Li, and Sun (2018, Section 3) show the existence of equivalent deterministic mechanisms, but do not provide a way of constructing such equivalent deterministic mechanisms. In this section, we explore the construction of an approximately equivalent deterministic mechanism. For simplicity of exposition, we illustrate our approach in the one-dimensional setting in which $V_i = [\underline{v}_i, \bar{v}_i]$ for all $i \in I$, and we focus on approximate equivalence in terms of interim expected allocation probabilities for all agents. For any vector $(z_1, z_2, \ldots, z_K) \in \mathbb{R}^K$, let $\|z\|_1 = \sum_{k\in\mathcal{K}} |z_k|$. For any set S, we write $Card(S)$ for its cardinality.

Fix $\epsilon > 0$. For all $N \ge 1$, we divide V_i into 2^N subintervals $\{V_{i,n}^N\}_{1\le n\le 2^N}$ of equal measure. That is, $\lambda_i(V_{i,n}^N) = \frac{1}{2^N}$ for all $i \in \mathcal{I}$ and $1 \le n \le 2^N$. For each sub-rectangle

$\prod_{i\in\mathcal{I}} V_{i,n_i}^N$, let

$$a_{n_1,n_2,\ldots,n_I}^N = 2^{NI} \int_{\prod_{i\in\mathcal{I}} V_{i,n_i}^N} q(v_1, v_2, \ldots, v_I)\, \lambda(\mathrm{d}v).$$

Note that $a_{n_1,n_2,\ldots,n_I}^N$ is a vector in $\mathbb{R}_+^K$. We write $a_{n_1,n_2,\ldots,n_I}^{N,k}$ to denote the k-th entry of $a_{n_1,n_2,\ldots,n_I}^N$. Since $\sum_{k\in\mathcal{K}} q^k(v) = 1$ for all $v \in V$, we have

$$\sum_{k\in\mathcal{K}} a_{n_1,n_2,\ldots,n_I}^{N,k} = 1.$$

Choose a vector $b_{n_1,n_2,\ldots,n_I}^N \in \mathbb{R}_+^K$ such that

(1) $\|a_{n_1,n_2,\ldots,n_I}^N - b_{n_1,n_2,\ldots,n_I}^N\|_1 < \frac{\epsilon}{4}$;

(2) $\sum_{k\in\mathcal{K}} b_{n_1,n_2,\ldots,n_I}^{N,k} = 1$; and

(3) $b_{n_1,n_2,\ldots,n_I}^{N,k}$ is a nonnegative rational number for all $k \in \mathcal{K}$.

Without loss of generality, we assume that $b_{n_1,n_2,\ldots,n_I}^{N,k} = \frac{1}{\beta_N} c_{n_1,n_2,\ldots,n_I}^{N,k}$, where β_N is a positive integer and $c_{n_1,n_2,\ldots,n_I}^{N,k}$ is a nonnegative integer. Then

$$\sum_{k\in\mathcal{K}} c_{n_1,n_2,\ldots,n_I}^{N,k} = \beta_N \sum_{k\in\mathcal{K}} b_{n_1,n_2,\ldots,n_I}^{N,k} = \beta_N.$$

For all $1 \le i \le I$ and $1 \le n_i \le 2^N$, we further cut V_{i,n_i}^N into β_N subintervals $\{V_{i,n_i}^{N,s_i}\}_{1\le s_i\le\beta_N}$ of equal measure.

We are now ready to construct the allocation rule q_N. Fix N and $(n_1, n_2, \ldots, n_I)$. Any $v = (v_1, v_2, \ldots, v_I) \in V$ necessarily lies in some sub-rectangle $\in V_{1,n_1}^{N,s_1} \times V_{2,n_2}^{N,s_2} \times \ldots \times V_{I,n_I}^{N,s_I}$. Let $\sum_{i\in\mathcal{I}} s_i = \alpha\,\beta_N + \gamma$, where α and γ are nonnegative integers and $0 \le \gamma < \beta_N$. Let

$$q_N(v_1, v_2, \ldots, v_I) = \begin{cases} (1, 0, \ldots, 0) & 0 \le \gamma \le c_{n_1,n_2,\ldots,n_I}^{N,1} - 1, \\ (0, 1, \ldots, 0) & c_{n_1,n_2,\ldots,n_I}^{N,1} \le \gamma \le c_{n_1,n_2,\ldots,n_I}^{N,1} + c_{n_1,n_2,\ldots,n_I}^{N,2} - 1, \\ \cdots & \ldots, \\ (0, \ldots, 0, 1) & \sum_{1\le k\le K-1} c_{n_1,n_2,\ldots,n_I}^{N,k} \le \gamma \le \sum_{k\in\mathcal{K}} c_{n_1,n_2,\ldots,n_I}^{N,k} - 1. \end{cases}$$

Proposition 4. *For any $\epsilon > 0$, there exists a positive integer $\tilde{N}$ such that for all $N \ge \tilde{N}$,*

$$\int_{V_i} \left| \int_{V_{-i}} \left[q^k(v_i, v_{-i}) - q_N^k(v_i, v_{-i}) \right] \lambda_{-i}(\mathrm{d}v_{-i}) \right| \lambda_i(\mathrm{d}v_i) < \epsilon \tag{10}$$

for all $i \in I$ and $k \in \mathcal{K}$.

Proof of Proposition 4. Fix $\epsilon \in (0,1)$. It suffices to show that there exists a positive integer $\tilde{N}$ such that for all $N \geq \tilde{N}$, there exists a subset $D_i^N \subseteq V_i$ with $\lambda_i(D_i^N) < \epsilon$ for all $i \in \mathcal{I}$ such that

$$\left\| \int_{V_{-i}} \left[q(v_i, v_{-i}) - q_N(v_i, v_{-i}) \right] \lambda_{-i}(\mathrm{d}v_{-i}) \right\|_1 < \epsilon \tag{11}$$

for all $i \in \mathcal{I}$ and $v_i \in V_i \setminus D_i^N$.

It follows from Lusin's Theorem that there exists a continuous function $\tilde{q}$ that is an approximation for q. By the continuity of $\tilde{q}$, Step (2) constructs $\tilde{q}_N$ from $\tilde{q}$ such that $\tilde{q}_N$ is an approximation for $\tilde{q}$. We then show in Step (3) and Step (4) that q_N is an approximation for $\tilde{q}_N$. Combining the arguments above, we show that q_N is an approximation for q.

Step (1) By Lusin's Theorem (see Royden and Fitzpatrick (2010, p. 66)), there exists a continuous function $\tilde{q} \colon V \to \triangle(\{1, 2, \ldots, K\})$ such that

$$\lambda(\{v \in V \colon q(v) \neq \tilde{q}(v)\}) < \frac{\epsilon^3}{128K^2}. \tag{12}$$

Let $D = \{v \in V : q(v) \neq \tilde{q}(v)\}$. For each $i \in \mathcal{I}$, let $D(v_i) = \{v_{-i} : (v_i, v_{-i}) \in D\}$ for $v_i \in V_i$, and let $D_i = \{v_i : \lambda_{-i}(D(v_i)) \geq \frac{\epsilon}{8K}\}$. It follows from (12) that $\lambda_i(D_i) < \frac{\epsilon^2}{16K}$ for all $i \in \mathcal{I}$. By the definition of D_i, for all $v_i \in V_i \setminus D_i$, $\lambda_{-i}(D(v_i)) < \frac{\epsilon}{8K}$ and

$$\left\| \int_{V_{-i}} \left[q(v_i, v_{-i}) - \tilde{q}(v_i, v_{-i}) \right] \lambda_{-i}(\mathrm{d}v_{-i}) \right\|_1 \leq K\, \lambda_{-i}(D(v_i)) < K\, \frac{\epsilon}{8K} = \frac{\epsilon}{8}. \tag{13}$$

Step (2) Parallel to the construction of $a^N_{n_1, n_2, \ldots, n_I}$ from q, for each sub-rectangle $\prod_{i \in \mathcal{I}} V^N_{i,n_i}$, let

$$\tilde{a}^N_{n_1, n_2, \ldots, n_I} = 2^{NI} \int_{\prod_{i \in \mathcal{I}} V^N_{in_i}} \tilde{q}(v_1, \ldots, v_I)\, \lambda(\mathrm{d}v).$$

Let $\tilde{q}_N(v) = \tilde{a}^N_{n_1, n_2, \ldots, n_I}$ for $v \in \prod_{i \in \mathcal{I}} V^N_{i,n_i}$. Then for $i \in \mathcal{I}$, $1 \leq n_i \leq 2^N$ and $v_i \in V^N_{i,n_i}$,

$$\int_{V_{-i}} \tilde{q}_N(v_i, v_{-i})\, \lambda_{-i}(\mathrm{d}v_{-i}) = \frac{1}{2^{N(I-1)}} \sum_{1 \leq n_j \leq 2^N,\, j \neq i} \tilde{a}^N_{(n_1, n_2, \ldots, n_I)}.$$

Since $\tilde{q}$ is continuous, $\tilde{q}_N(v)$ converges to $\tilde{q}(v)$ as $N \to \infty$ for all $v \in V$. By Lebesgue's dominated convergence theorem (see Royden and Fitzpatrick (2010, p. 88)),

$$\left\| \int_{V_{-i}} \left[\tilde{q}(v_i, v_{-i}) - \tilde{q}_N(v_i, v_{-i}) \right] \lambda_{-i}(\mathrm{d}v_{-i}) \right\|_1 \to 0$$

for all $v_i \in V_i$. By Egoroff's Theorem (see Royden and Fitzpatrick (2010, p. 64)), there exists a subset $\tilde{D}_i \subseteq V_i$ with $\lambda_i(\tilde{D}_i) < \frac{\epsilon}{4}$ such that $\int_{V_{-i}} \tilde{q}_N(v_i, v_{-i})\, \lambda_{-i}(\mathrm{d}v_{-i})$ uniformly converges to $\int_{V_{-i}} \tilde{q}(v_i, v_{-i})\, \lambda_{-i}(\mathrm{d}v_{-i})$ on $V_i \setminus \tilde{D}_i$. Then there exists $\tilde{N}$ such that for $N \geq \tilde{N}$ and $v_i \in V_i \setminus \tilde{D}_i$,

$$\left\| \int_{V_{-i}} \left[\tilde{q}(v_i, v_{-i}) - \tilde{q}_N(v_i, v_{-i}) \right] \lambda_{-i}(\mathrm{d}v_{-i}) \right\|_1 < \frac{\epsilon}{4}. \tag{14}$$

Step (3) Recall from Step (1) that $D_i = \{v_i : \lambda_{-i}(D(v_i)) \geq \frac{\epsilon}{8K}\}$ and $\lambda_i(D_i) < \frac{\epsilon^2}{16K}$. For all $i \in \mathcal{I}$ and $N \geq 1$, let $E_i^N = \{n_i : \lambda_i(D_i \cap V_{i,n_i}^N) \geq \frac{1}{2^N} \frac{\epsilon}{8K},\ 1 \leq n_i \leq 2^N\}$. Since

$$\frac{\epsilon^2}{16K} > \lambda_i(D_i) \geq \sum_{n_i \in E_i^N} \lambda_i(D_i \cap V_{i,n_i}^N) \geq Card(E_i^N)\, \frac{1}{2^N}\, \frac{\epsilon}{8K},$$

we have $\frac{Card(E_i^N)}{2^N} < \frac{\epsilon}{2}$. Let $\hat{D}_i^N = \cup_{n_i \in E_i^N} V_{i,n_i}^N$. Then $\lambda_i(\hat{D}_i^N) = \frac{Card(E_i^N)}{2^N} < \frac{\epsilon}{2}$. For $v_i \in V_i \setminus \hat{D}_i^N$ (i.e., $v_i \in V_{i,n_i}^N$ with $n_i \notin E_i^N$),

$$\begin{aligned}
&\left\| \frac{1}{2^{N(I-1)}} \sum_{1 \leq n_j \leq 2^N\, j \neq i} a^N_{n_1, n_2, \ldots, n_I} - \int_{V_{-i}} \tilde{q}_N(v_i, v_{-i})\, \lambda_{-i}(\mathrm{d}v_{-i}) \right\|_1 \\
&= \left\| \frac{1}{2^{N(I-1)}} \sum_{1 \leq n_j \leq 2^N\, j \neq i} \left(a^N_{n_1, n_2, \ldots, n_I} - \tilde{a}^N_{n_1, n_2, \ldots, n_I} \right) \right\|_1 \\
&\leq 2^N \int_{V_{i,n_i}^N} \int_{V_{-i}} \left\| q(v_1, v_2 \ldots, v_I) - \tilde{q}(v_1, v_2, \ldots, v_I) \right\|_1 \lambda_{-i}(\mathrm{d}v_{-i})\, \lambda_i(\mathrm{d}v_i) \\
&= 2^N \int_{V_{i,n_i}^N \setminus D_i} \int_{V_{-i}} \left\| q(v_1, v_2, \ldots, v_I) - \tilde{q}(v_1, v_2, \ldots, v_I) \right\|_1 \lambda_{-i}(\mathrm{d}v_{-i})\, \lambda_i(\mathrm{d}v_i) \\
&+ 2^N \int_{V_{i,n_i}^N \cap D_i} \int_{V_{-i}} \left\| q(v_1, v_2, \ldots, v_I) - \tilde{q}(v_1, v_2, \ldots, v_I) \right\|_1 \lambda_{-i}(\mathrm{d}v_{-i})\, \lambda_i(\mathrm{d}v_i) \\
&\leq 2^N \lambda_i(V_{i,n_i}^N)\, K\, \frac{\epsilon}{8K} + 2^N \lambda_i(V_{i,n_i}^N \cap D_i)\, K \\
&\leq \frac{\epsilon}{4}.
\end{aligned} \tag{15}$$

Step (4) By the construction of q_N, for all $i \in I$, $1 \leq n_i \leq 2^N$, and $v_i \in V_{i,n_i}^N$,

$$\int_{V_{-i}} q_N(v_i, v_{-i})\, \lambda_{-i}(\mathrm{d}v_{-i}) = \frac{1}{2^{N(I-1)}} \sum_{1 \leq n_j \leq 2^N, j \neq i} b^N_{n_1, n_2, \ldots, n_I}.$$

Therefore, for all N and $v_i \in V_i$,

$$\begin{aligned}
&\left\| \int_{V_{-i}} q_N(v_i, v_{-i})\, \lambda_{-i}(\mathrm{d}v_{-i}) - \frac{1}{2^{N(I-1)}} \sum_{1 \le n_j \le 2^N,\, j \ne i} a^N_{n_1,n_2,\ldots,n_I} \right\|_1 \\
&= \left\| \frac{1}{2^{N(I-1)}} \sum_{1 \le n_j \le 2^N,\, j \ne i} b^N_{n_1,n_2,\ldots,n_I} - \frac{1}{2^{N(I-1)}} \sum_{1 \le n_j \le 2^N,\, j \ne i} a^N_{n_1,n_2,\ldots,n_I} \right\|_1 \\
&\le \frac{1}{2^{N(I-1)}} \sum_{1 \le n_j \le 2^N\, j \ne i} \left\| b^N_{(n_1,n_2,\ldots,n_I)} - a^N_{(n_1,n_2,\ldots,n_I)} \right\|_1 \\
&< \frac{\epsilon}{4}.
\end{aligned} \tag{16}$$

Finally, let $D_i^N = D_i \cup \tilde{D}_i \cap \hat{D}_i^N$. Then $\lambda_i(D_i^N) \le \epsilon$. Recall that $\tilde{N}$ has been defined in Step (2). For all $N \ge \tilde{N}$ and $v_i \in V_i \setminus D_i^N$, (11) follows from (13-16). □

E Self purification and mutual purification

Our mechanism equivalence result builds on the methodology of mutual purification. We emphasize that the notion of mutual purification is both conceptually and technically different from the usual purification principle in the literature related to Bayesian games, as illustrated by the following two examples.

Example 6 studies a generalized matching pennies game, and Example 7 studies a single-unit auction. The two games share the following features:

1. There are two agents.
2. Agent 1's type is uniformly distributed on $(0, 1]$ with total probability $1 - \lambda_1(0)$, and the distribution has an atom at the point 0 with $\lambda_1(0) > 0$.
3. Agent 2's type is uniformly distributed on $[0, 1]$.
4. Agents' types are independently distributed.

Example 6 below illustrates the idea of self purification. The behavioral strategy of agent 2 can be purified since the distribution of agent 2's type is atomless, whereas the behavioral strategy of agent 1 cannot be purified since agent 1's type has an atom.

Example 6 (Generalized matching pennies)**.** Consider the following $m \times m$ zero-sum game with incomplete information, where m is sufficiently large ($\frac{1}{m} < \lambda_1(0)$). The action space for both agents is $A_1 = A_2 = \{a_1, a_2, \dots, a_m\}$. The payoff matrix for agent 1 is given in Figure 1. In words, agent 1 would like to match the action of agent 2 and avoid the action that is one step below the action of agent 2 (including the case that she takes the action a_m and agent 2 takes the action a_1). The payoffs of the agents do not depend on the type profile.

		Agent 2				
		a_1	a_2	a_3	$\cdots$	a_m
Agent 1	a_1	1	-1	0	$\cdots$	0
	a_2	0	1	-1	$\cdots$	0
	a_3	0	0	1	$\cdots$	0
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	a_m	-1	0	$\cdots$	0	1

Figure 1: Agent 1's payoff matrix

Consider the behavioral strategy that each agent mixes over all actions with equal probability. Formally,

$$f_1(v) = f_2(v) = \frac{1}{m} \sum_{1 \le s \le m} \delta_{a_s}$$

for all $v \in [0,1]$, where δ_{a_s} is the Dirac measure at a_s. It is easy to verify that (f_1, f_2) is a Bayesian Nash equilibrium and the expected payoffs of both agents are 0.

Claim 1. *Agent 2 has a pure strategy f_2' such that (f_1, f_2') is a Bayesian Nash equilibrium and provides the same expected payoffs for both agents, whereas agent 1 does not have such a pure strategy.*

Proof of Claim 1. Consider the following pure strategy f_2' of agent 2:

$$f_2'(v) = \begin{cases} a_s, & v \in [\frac{s-1}{m}, \frac{s}{m}),\ 1 \le s \le m-1; \\ a_m, & v \in [\frac{m-1}{m}, 1]. \end{cases}$$

It is easy to see that (f_1, f_2') is a Bayesian Nash equilibrium and provides the same expected payoffs for both agents.

Next, we show that there does not exist a pure strategy g_1 of agent 1 such that g_1 is a component of a Bayesian Nash equilibrium with both agents' expected payoffs being 0. Suppose that (g_1, g_2) is a Bayesian Nash equilibrium such that g_1 is a pure strategy of agent 1. For each $1 \leq s \leq m$, let $D_s = \{v_1 \in V_1 : g_1(v_1) = a_s\}$ denote the collection of types of agent 1 that play a_s. Without loss of generality, we assume that $0 \in D_1$. Let $S = \arg\max_{1 \leq s \leq m} \lambda_1(D_s)$. Since $\lambda_1(D_s) \geq \lambda_1(D_1) \geq \lambda_1(0) > \frac{1}{m}$ for all $s \in S$, it must be that S is a strict subset of $\{1, 2, \ldots, m\}$. Therefore, at least one of the following is true: (1) there exists $1 \leq s^* < m$ such that $s^* \in S$ and $s^* + 1 \notin S$; and (2) $m \in S$ and $1 \notin S$. In the former case, playing a_{s^*+1} for all her types gives agent 2 a strictly positive expected payoff $\lambda_1(D_{s^*}) - \lambda_1(D_{s^*+1}) > 0$. In the latter case, playing a_1 for all her types gives agent 2 a strictly positive expected payoff $\lambda_1(D_m) - \lambda_1(D_1) > 0$. Since in either case, agent 2 has a strategy that gives her a strictly positive expected payoff, the expected payoff of agent 2 when playing g_2 must be strictly positive in the equilibrium (g_1, g_2). We arrive at a contradiction. □

Example 7 below shows how a purification for an agent relies on the dispersed information of the other agent, which partially illustrates the idea of mutual purification. In particular, for some given stochastic mechanism in the 2-agent setting with independent types as specified above, agent 1 who has an atom in her type space can achieve the same interim expected payoff by some deterministic mechanism,[2] whereas there does not exist such a deterministic mechanism for agent 2 who has dispersed information.

Example 7. *Consider a single-unit auction with two bidders. The payoff function of agent i is $\epsilon v_i + (1 - v_j)^m$ for $i, j = 1, 2$ and $i \neq j$, where m is sufficiently large and ϵ is sufficiently small such that*

$$\frac{\lambda_1(0)}{2} > \epsilon + \frac{1}{m+1}.$$

Consider the stochastic allocation rule $q = (q_1, q_2)$ with $q_1(v) = q_2(v) = 1/2$ for all $v \in V$, where q_i is the probability of agent i getting the object for $i \in \{1, 2\}$. The interim expected utility of agent 1 with type v_1 is

$$\int_{V_2} (\epsilon v_1 + (1 - v_2)^m)\, q_1(v_1, v_2)\, \lambda_2(\mathrm{d}v_2) = \frac{\epsilon v_1}{2} + \frac{1}{2(m+1)},$$

and the interim expected utility of agent 2 with type v_2 is

$$\int_{V_1} (\epsilon v_2 + (1 - v_1)^m)\, q_2(v_1, v_2)\, \lambda_1(\mathrm{d}v_1) = \frac{\epsilon v_2}{2} + \frac{\lambda_1(0)}{2} + (1 - \lambda_1(0))\, \frac{1}{2(m+1)}.$$

[2] For simplicity, we only consider such an equivalence in terms of interim expected payoffs.

Claim 2. *There exists a deterministic mechanism which gives agent* 1 *the same interim expected utility, whereas there does not exist such a deterministic mechanism for agent* 2*.*

Proof of Claim 2. We first construct a deterministic mechanism which gives agent 1 the same interim expected payoff. Define a function G on $V_1 \times V_2 = [0,1]^2$ by letting

$$G(v_1, v_2) = \int_0^{v_2} \Big[\epsilon v_1 + (1 - v_2')^m\Big] \lambda_2(\mathrm{d}v_2') - \Big[\frac{\epsilon v_1}{2} + \frac{1}{2(m+1)}\Big],$$

for any $(v_1, v_2) \in V_1 \times V_2$. It is easy to see that for any $v_1 \in [0,1]$, $G(v_1, 0) < 0 < G(v_1, 1) = \frac{\epsilon v_1}{2} + \frac{1}{2(m+1)}$. Furthermore, $\frac{\partial G}{\partial v_2} = \epsilon v_1 + (1 - v_2)^m > 0$ for any $v_1 \in [0,1]$ and $v_2 \in [0,1)$. Therefore, for each $v_1 \in [0,1]$, there exists a unique $g(v_1) \in (0,1)$ such that $G(v_1, g(v_1)) = 0$. By the implicit function theorem, g is differentiable, and hence measurable. Let $\hat{q}_1(v_1, v_2) = 1$ if $0 \le v_2 \le g(v_1)$ and 0 otherwise, and $\hat{q}_2(v_1, v_2) = 1 - \hat{q}_1(v_1, v_2)$. Then the mechanism $\hat{q}$ gives agent 1 the same interim expected utility.

Next, we show that there does not exist any deterministic mechanism that gives agent 2 the same interim expected utility. Suppose that there exists a deterministic mechanism $\tilde{q} = (\tilde{q}_1, \tilde{q}_2)$ that gives agent 2 the same interim expected utility. Fix $v_2 \in V_2 = [0,1]$.

Suppose that $\tilde{q}_2(0, v_2) = 1$. Then the interim expected utility of agent 2 with type v_2 is

$$\int_{V_1} (\epsilon v_2 + (1 - v_1)^m)\, \tilde{q}_2(v_1, v_2)\, \lambda_1(\mathrm{d}v_1) \ge (\epsilon v_2 + 1)\, \lambda_1(0).$$

Recall that $\frac{\lambda_1(0)}{2} > \epsilon + \frac{1}{m+1}$. Hence we have

$$(\epsilon v_2 + 1)\lambda_1(0) \ge \lambda_1(0) > \frac{\lambda_1(0)}{2} + \epsilon + \frac{1}{m+1} > \frac{\epsilon v_2}{2} + \frac{\lambda_1(0)}{2} + (1 - \lambda_1(0))\frac{1}{2(m+1)}.$$

Thus, the interim expected payoff of agent 2 under the mechanism $\tilde{q}$ is strictly greater than the interim expected payoff of agent 2 under the mechanism q. This is a contradiction. Therefore, it must be that $\tilde{q}_2(0, v_2) = 0$ since $\tilde{q}$ is a deterministic mechanism.

Next, since $\tilde{q}^2(0, v_2) = 0$, the interim expected payoff of agent 2 is

$$\int_{V_1} (\epsilon v_2 + (1 - v_1)^m)\, \tilde{q}_2(v_1, v_2)\, \lambda_1(\mathrm{d}v_1) = \int_{(0,1]} (\epsilon v_2 + (1 - v_1)^m)\, \tilde{q}_2(v_1, v_2)\, \lambda_1(\mathrm{d}v_1)$$
$$\le (1 - \lambda_1(0)) \int_0^1 (\epsilon v_2 + (1 - v_1)^m)\, \mathrm{d}v_1 = (1 - \lambda_1(0))\, \epsilon v_2 + \frac{1 - \lambda_1(0)}{m+1}$$

$$< \epsilon + \frac{1}{m+1} < \frac{\lambda_1(0)}{2} < \frac{\epsilon v_2}{2} + \frac{\lambda_1(0)}{2} + (1 - \lambda_1(0))\frac{1}{2(m+1)}.$$

That is, the interim expected payoff of agent 2 under the mechanism $\tilde{q}$ is strictly less than the interim expected payoff of agent 2 under the mechanism q. This is also a contradiction. Therefore, there does not exist any deterministic mechanism that gives agent 2 the same interim expected payoff. □

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