Maskin Meets Abreu and Matsushima*

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Abstract

We study the classical Nash implementation problem due to Maskin (1999), but allow for the use of lottery and monetary transfer as in Abreu and Matsushima (1992, 1994). We therefore unify two well-established but somewhat orthogonal approaches of implementation theory. We first show that Maskin monotonicity is a necessary and sufficient condition for pure-strategy Nash implementation by a *direct* mechanism. Second, taking mixed strategies into consideration, we show that Maskin monotonicity is a necessary and sufficient condition for mixed-strategy Nash implementation by a *finite* (albeit indirect) mechanism. Third, we extend our analysis to implementation in rationalizable strategies. In contrast to previous papers, our approach possesses many appealing features simultaneously, e.g., finite mechanisms (with no integer or modulo game) are used; mixed strategies are handled explicitly; neither transfer nor bad outcomes are used on the equilibrium path; our mechanism is robust to information perturbations; and the size of off-equilibrium transfers can be made arbitrarily small. Finally, our result can be extended to continuous settings and ordinal settings.

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Although the theory of implementation has been quite successful in identifying the social choice functions which can be implemented in different informational settings, a nagging criticism of the theory is that the mechanisms used in the general constructive proofs have "unnatural" features. A natural response to this criticism is that the mechanisms in the constructive proofs are designed to apply to a broad range of environments and social choice functions. Given this versatility, it is not surprising that the mechanisms possess questionable features. With this in mind, we would hope that for particular settings and social choice functions we could find "natural" mechanisms with desirable properties. To the extent that there are social choice functions which we can only implement using questionable mechanisms, the existing theory of implementation is inadequate.

—Jackson (1992, pp. 757-758)

1 Introduction

Mechanism design can be seen as reverse engineering of game theory. Suppose that a society has decided on a social choice rule – a recipe for choosing the socially optimal alternatives on the basis of individuals' preferences over alternatives. To implement the social choice rule, a mechanism designer chooses a game/mechanism so that the equilibrium outcomes of the mechanism coincide with the social outcomes designated by the choice rule.

There are two prominent paradigms in the theory of implementation—partial implementation and full implementation. One critical difference between the two paradigms is that the former requires that *one* equilibrium outcome achieve the social choice rule, while the latter requires that *all* equilibrium outcomes be socially desirable. Conventional wisdom suggests that many fewer social choice rules can be fully implemented; even when they can be, this is often accomplished by invoking more complicated indirect mechanisms. The historical development of these two paradigms, however, also leads to another important, and perhaps unexpected, difference: full implementation usually focuses on general social choice environments, while partial implementation/mechanism design is explored mainly in economic environments in which both lotteries and monetary transfer are available. Indeed, while economic theory has gone a long way around the Gibbard-Satterthwaite impossibility theorem by exploiting dominant-strategy mechanisms in quasilinear environments, we do not observe a similar development in full Nash implementation due to Maskin (1977, 1999).

In this paper, we study the full Nash implementation problem but allow for lotteries and monetary transfer. We focus on the monotonicity condition (hereafter, Maskin monotonicity) which Maskin shows is necessary and "almost sufficient" for Nash implementation. We aim to implement Maskin-monotonic *social choice functions* (henceforth, SCFs) in pure or mixed strategy Nash equilibrium by mechanisms with no questionable feature. Specifically, we restrict attention to *finite mechanisms* that make use of neither the *integer game device* nor the *modulo game* device which prevails in the full implementation literature.[1]

In the integer game, each player announces some integer and the person who announces the highest integer becomes a dictator. In the absence of a common best outcome, an integer game has no pure-strategy Nash equilibria. This questionable feature is also shared by a modulo game. The modulo game is considered a finite version of the integer game in which agents announce integers from a finite set. The agent whose identification matches the modulo of the sum of the integers gets to name the allocation. In order to "knock out" undesirable equilibria in general environments, most constructive proofs in the literature have taken advantage of the fact that the integer/modulo game has no solution. In particular, without imposing any domain restriction on the (even finite) environment, Jackson (1992, Example 4) shows that it is generally impossible to achieve the mixed-strategy Nash implementation of a Maskin-monotonic SCF by a finite mechanism.[2]

Abreu and Matsushima (henceforth, AM, 1992, 1994) also studied full implementation problem in environments with lottery and transfer. AM obtain permissive implementation results using finite mechanisms without the aforementioned questionable features.[3] However, AM do not investigate Nash implementation but rather appeal to a different notion of implementation: virtual implementation in Abreu and Matsushima (1992) or exact implementation under iterated weak dominance in Abreu and Matsushima (1994).[4] Virtual

[1]More precisely, the implementing mechanism which we construct is finite as long as each agent has only finitely many possible preferences. We consider infinite environments in Section 7.3 where we construct infinite yet well-behaved implementing mechanisms.

[2]Nevertheless, our Theorem 7.2 shows that the SCF which Jackson (1992) constructs can be implemented in mixed-strategy Nash equilibrium in a finite mechanism with arbitrarily small off-equilibrium transfers.

[3]To be precise, Abreu and Matsushima (1992) do not need transfers but rather assume existence of lotteries only. "Reducing the probability of a favorable social choice outcome" in their setup plays the same role of "penalizing players by decreasing transfer" in our setup.

[4]Iterated weak dominance in Abreu and Matsushima (1994) also yields the unique undominated Nash

implementation means that the planner contents herself with implementing the SCF with arbitrarily high probability.[5] In contrast, by studying *exact* Nash implementation in the specific setting, we unify the two well-established but somewhat orthogonal approaches to implementation theory due to Maskin (1999) and AM. Our exercise is directly comparable to Maskin (1999) and highlights the pivotal trade-off between the class of environments and the feature of implementing mechanisms. We consider this as a major step in advancing Jackson's (1992) research program, cited in the beginning of this section.

Our first result (Theorem 1) establishes a revelation principle for pure-strategy Nash implementation by making use of monetary transfer alone. Specifically, we show that with three or more agents, every Maskin-monotonic SCF is fully implementable in pure-strategy Nash equilibrium by a direct mechanism. In the direct mechanism, each agent announces only a state (which consists of the agents' preference profile) and hence, the mechanism employs neither integer games nor modulo games. The result also involves no randomization and contrasts with the folk understanding that direct mechanisms are generally inadequate for full implementation.

While the direct mechanism in Theorem 1 is finite and possesses nice properties, it might admit mixed-strategy equilibria whose outcomes are not socially desirable. Our second result (Theorem 2) shows that as long as there are two agents, Maskin monotonicity is a necessary and sufficient condition for mixed-strategy Nash implementation by a finite mechanism.[6] In the finite mechanism, each agent is asked to announce his type twice in addition to reporting the state. Once again, the result makes use of neither integer games

equilibrium outcome. For undominated Nash implementation by finite mechanisms, see also Jackson et al. (1994) and Sjostrom (1994).

[5] Virtual implementation allows for the possibility that an outcome not in the SCF is selected with positive probability even on the equilibrium path. This feature is problematic in situations where the planner is free to renege. Specifically, if agents believe that the planner will not adopt a questionable outcome x when (s)he knows (according to the equilibrium) that a different outcome y is an element of the SCF, the equilibrium falls apart. The random mechanism which we adopt do not share this issue, since no randomization takes place in equilibrium, and out of equilibrium the planner will not know if any particular outcome belongs to the SCF. See (Benoît and Ok, 2008, Section 3.3) for more discussion.

[6] In addition, if the SCF satisfies Maskin monotonicity in the restricted domain without transfer, we show that it is implementable in mixed-strategy Nash equilibrium by a finite mechanism such that the size of transfers remains zero on the equilibrium path and can be made arbitrarily small off the equilibrium path (Theorem 5).

nor refinements which are by far the standard way to handle mixed-strategy equilibria.

Indeed, with refinements of Nash equilibria such as undominated Nash equilibrium or subgame-perfection equilibrium, essentially any (monotonic or non-monotonic) SCF is implementable in a complete-information environment.[7] However, Chung and Ely (2003) and Aghion et al. (2012) have pointed out that if we were to achieve exact implementation in refinements which is robust to a "small amount of incomplete information," Maskin monotonicity would come back as a necessary condition. Indeed, we invoke Theorem 2 to verify that Maskin monotonicity is not only a necessary but also a sufficient condition for exact and *robust* implementation in Nash equilibrium and hence also in any refinement (Proposition 1).[8]

For our third result, we study the notion of rationalizable implementation due to Bergemann et al. (2011). Specifically, in Theorem 3 we show that as long as there are two agents, an SCF is implementable in rationalizable strategies by a finite mechanism if and only if it satisfies Maskin monotonicity*, a condition proposed by Bergemann et al. (2011).[9] Any Maskin-monotonic SCF that is responsive/injective satisfies Maskin monotonicity*. The implementing mechanism is obtained by modifying the implementing mechanism of Theorem 2. The result implies AM's result in our environment, i.e., any SCF is virtually implementable in rationalizable strategies by a finite mechanism. Moreover, it also follows that every Maskin-monotonic* SCF is continuously implementable – another notion of implementation proposed by Oury and Tercieux (2012) which is robust to a broader class of information perturbations than those considered in Chung and Ely (2003) and Aghion et al. (2012).

[7]See, for instance, Moore and Repullo (1988), Abreu and Sen (1991), Palfrey and Srivastava (1991), and Abreu and Matsushima (1994). In an economic environment similar to ours, Moore and Repullo (1988) construct a simple mechanism with no mixed strategy "subgame-perfect" equilibria, while Abreu and Matsushima (1994), Jackson et al. (1994), and Sjostrom (1994) construct a finite mechanism with no mixed-strategy "undominated" Nash equilibria.

[8]Harsanyi (1973) shows that a mixed Nash equilibrium outcome may occur as the limit of a sequence of pure-strategy Bayesian Nash equilibria for "nearby games" in which players are uncertain about the exact profile of preferences. Hence, ignoring mixed strategy equilibria would be particularly problematic if we were to achieve implementation which is robust to information perturbations.

[9]By making use of integer games, Bergemann et al. (2011) show that Maskin monotonicity* is a necessary and almost sufficient condition for rationalizable implementation by a mechanism satisfying the best response peorpty (e.g., a finite mechanism). See Bergemann et al. (2011) for the definition of the best response property.

To sum up, we list our main results as follows:

- **Pure-Strategy Nash Implementation** (Theorem 1): When there are at least three agents, Maskin monotonicity is a necessary and sufficient condition for pure-strategy Nash implementation by a direct mechanism.

- **Mixed-Strategy Nash Implementation** (Theorem 2): When there are two or more agents, Maskin monotonicity is a necessary and sufficient condition for mixed-strategy Nash implementation by a finite mechanism.

- **Rationalizable Implementation** (Theorem 3): When there are two or more agents, Maskin monotonicity* is a necessary and sufficient condition for rationalizable implementation by a finite mechanism.

We also provide several extensions of our main results. First, we extend our mixed-strategy Nash implementation result to cover social choice *correspondences* (i.e., multivalued social choice rules) which Maskin (1999) as well as many other papers have studied. Formally, we show that when there are at least three agents, any Maskin-monotonic social choice correspondence (henceforth, SCC) is mixed-strategy Nash implementable (Theorem 4). Furthermore, if the range of the SCC is a finite set, we guarantee that the implementing mechanism is still finite. Second, we show that if the SCF satisfies Maskin monotonicity in the restricted domain without transfer, then it is implementable in mixed-strategy Nash equilibrium by a finite mechanism in which the size of transfers remains zero on the equilibria and can be made arbitrarily small off the equilibria (Theorem 5).

Third, we consider an infinite setting which the state space is a compact set, and the utility functions and the SCF are continuous. In this setting, we show that Maskin monotonicity is a necessary and sufficient condition for mixed-strategy Nash implementation by a mechanism with a compact message space, a continuous outcome function, and a continuous transfer rule (Theorem 6). Such an extension to an infinite setting is by far not available in the literature, even for virtual implementation.[10] This extension covers many applications and verifies that our finite setting is indeed a good approximation of settings with a continuum of states.

[10]This was a prominent open question that was raised in Section 5 of Abreu and Matsushima (1992), and which remains open to the best of our knowledge.

The extension to an infinite setting yields a further interesting extension. Specifically, in proving Theorem 2, we have assumed that each agent is an expected utility maximizer with a fixed cardinal utility function over pure outcomes. This raises the question as to whether our result is an artifact of the fixed finite set of cardinalizations. To answer the question, we adopt the concept of *ordinal* Nash implementation proposed by Mezzetti and Renou (2012). The notion requires that a *single* mechanism achieve mixed-strategy Nash implementation for *any* cardinal representation of preferences over lotteries. We say that an SCF satisfies *ordinal* Maskin monotonicity if it is Maskin-monotonic for any cardinal representation. By making use of our implementing mechanism in the infinite setting, we show that ordinal Maskin monotonicity is a necessary and sufficient condition for ordinal Nash implementation (Theorem 7).

The rest of the paper is organized as follows. Section 2 uses two examples to illustrate the main idea of the paper. In Section 3, we present the basic setup and definitions. Section 4 studies pure-strategy Nash implementation in a direct mechanism. Section 5 studies mixed-strategy Nash implementation in a finite mechanism and its robustness to information perturbations. Section 6 studies rationalizable implementation. We discuss the extensions of our Nash implementation result in Section 7. Section 8 concludes.

2 An Illustration of Our Mechanism and Results

We provide two examples to illustrate our main result on implementation in mixed-strategy Nash equilibrium. In the first example, we illustrate the essential features of the implementing mechanism which we propose and how it works. In the second example, we argue that Maskin monotonicity is a mild requirement in a bilateral trading environment, a prominent applied setting of implementation theory.

2.1 Example 1: King Solomon's Dilemma

Two women came to King Solomon with a baby and both claimed to be the baby's true mother. King Solomon faced the problem of finding out which of them was the true mother of the baby. Denote the two mothers by Anna (A) and Bess (B). Let a be the alternative of giving the baby to Anna and b the alternative of giving the baby to Bess. In the original setup, King Solomon introduces another alternative c, which is to cut the baby in half. We write α (β) as the state where A (B) is the true mother. King Solomon's goal is summarized by an SCF f mapping from the set of states $\{\alpha, \beta\}$ to outcomes $\{a, b, c\}$ such that $f(\alpha) = a$ and $f(\beta) = b$, i.e., his goal is to give the baby to the true mother. Both mothers know which of them is the true mother, but King Solomon does not.

In the original setup, Anna has the preference order $a \succ_A^\alpha b \succ_A^\alpha c$ at state α and $a \succ_A^\beta c \succ_A^\beta b$ at state β, whereas Bess has the preference order $b \succ_B^\alpha c \succ_B^\alpha a$ at state α, and $b \succ_B^\beta a \succ_B^\beta c$ in state β. Maskin monotonicity of f requires that if the desired outcome $f(\theta)$ (where $\theta = \alpha$ or β) never moves down any agent's rankings in switching from state θ to state θ', then $f(\theta)$ must continue to be the desired outcome in state θ' (see Definition 2). Here, for instance, when the state switches from α to β, $f(\alpha) = a$ stays the same in mother A's ranking and goes up mother B's ranking among the three alternatives. Yet, $f(\beta)$ is different from $f(\alpha)$. Hence, f is not Maskin-monotonic and not implementable in Nash equilibrium.

The situation is different once we introduce transfers. To be precise, suppose that we now denote the true mother's valuation of getting the baby by $\bar{v}$ and the false mother's valuation by $\underline{v}$ where $\bar{v}$ is higher than $\underline{v}$ and both positive. A mother who values the baby at v and receives transfer t derives utility $v + t$ if she gets the baby and transfer t.

We use a triplet to denote an outcome or allocation $(l, -t_A, -t_B)$, where l denotes a lottery over a and b which determines who gets the baby and t_i is the payment of mother i. In the setup, we write the SCF as $f(\alpha) = (a, 0, 0)$ and $f(\beta) = (b, 0, 0)$. Then, when the state switches from α to β, the social outcome $f(\alpha)$ moves down mother A's ranking against the allocation

$$y_A \equiv (b, -v_m, 0) \tag{1}$$

where $v_m = \frac{\underline{v}+\bar{v}}{2}$. In other words, at state α, mother A would strictly prefer keeping the baby to selling the baby to mother B at price v_m while such preference order is reversed when the state is β. Symmetrically, in switching from state β to state α, the social outcome $f(\beta)$ moves down mother B's ranking against the allocation

$$y_B \equiv (a, , 0, -v_m). \tag{2}$$

Hence, f satisfies Maskin monotonicity.

The usual interpretation is that mother A (B) can serve as the "whistle blower" to knock out a bad equilibrium resulted from a (mis-)reported state α (β) by proposing allocation y_A (y_B) when the state is actually β (α). Such whistle-blowing is credible, since the allocation y_A (y_B) would be worse than the social outcome under state α (β). That is, the "right mother" has incentive to blow the whistle if and only if she is supposed to do so.

2.1.1 The Mechanism

We now provide a finite mechanism which implements King Solomon's SCF in mixed-strategy Nash equilibrium. In the mechanism, each mother i is asked to submit three separate envelopes. Each of the envelopes contains a state written in a card. We denote the three envelopes submitted by mother i as m_i^1, m_i^2, and m_i^3, respectively. This mechanism consists of two parts such that each agent's final utility is the sum of the utility from the two parts. The first part defines the allocation rule as follows:

Rule 1: If $m_A^2 = m_A^3 = m_B^2 = m_B^3 = \tilde{\theta}$ (i.e., the two mothers' second and third envelopes all match), then implement $(f(\tilde{\theta}), 0, 0)$.

Rule 2: Otherwise, we trigger Rule 2-1 with small probability ε and Rule 2-2 with probability $1 - \varepsilon$:

Rule 2-1 (dictator lottery): the outcome is determined by the mothers' first envelopes which is described as follows:

	$m_B^1 = \alpha$	$m_B^1 = \beta$
$m_A^1 = \alpha$	y_A	$\frac{1}{2}y_A + \frac{1}{2}y_B$
$m_A^1 = \beta$	$\frac{1}{2}y_A + \frac{1}{2}y_B$	y_B

where y_A and y_B are defined as in (1) and (2), and $\frac{1}{2}y_A + \frac{1}{2}y_B$ denote the lottery in which both y_A and y_B have probability $\frac{1}{2}$.

Rule 2-2 (whistle blower): the outcome is determined by the mothers' third envelopes which is described as follows:

	$m_B^3 = \alpha$	$m_B^3 = \beta$
$m_A^3 = \alpha$	y_A	$\frac{1}{2}y_A + \frac{1}{2}y_B$
$m_A^3 = \beta$	$\frac{1}{2}y_A + \frac{1}{2}y_B$	y_B

The second part defines a tansfer rule which we use to discipline the two mothers' announcements. For each mother i and mother $j \neq i$, the payment is defined as follows (where $\eta > \bar{v}$):

	$m_A^2 = m_B^2$	$m_A^2 \neq m_B^2$	
		$m_i^2 = m_j^1$	$m_i^2 \neq m_j^1$
The payments of agent i :	0	$-\eta$	η

In words, if both mothers report the same state in their second envelope, then mother i makes no payment. Otherwise, we distinguish two cases: (i) if mother i's second report matches mother j's first report, mother i receives a reward η; (ii) if mother i's second report does not match mother j's first report, then mother i pays a penalty η.

Observation (★): For each mother i, the first envelope affects her payoff only when Rule 2-1 is triggered; moreover, once Rule 2-1 is triggered with positive probability, bother mother must strictly prefer reporting the true state in the first envelope.

2.1.2 The Implementation in Mixed-Strategy Nash Equilibrium

We now show that the mechanism implements King Solomon's SCF in mixed-strategy Nash equilibrium. Suppose that θ is the true state. We first argue that truth-telling (i.e., $m_i = (\theta, \theta, \theta)$ for each mother $i = A, B$) constitutes a pure-strategy Nash equilibrium. That is, we claim that any unilateral deviation from truth-telling is not profitable. First, a false report in the second envelope leads to the penalty of η and hence cannot be a profitable deviation. Second, a truthful report in the second envelope together with a false report in the third envelope also leads to a worse outcome: either the true mother pays money to get the baby instead of getting it for free, or the false mother buys the baby at the price v_m which exceeds her willingness to pay. Finally, together with a truthful second and third report, any false report in the first envelope affects neither the allocation nor mother i's payment.

We next show that under any mixed-strategy Nash equilibrium σ, both mothers will announce the same state $\tilde{\theta}$ in both their second and third envelopes with probability one. This implies that the social outcome $f(\tilde{\theta})$ is implemented and no transfer is induced. Furthermore, by Maskin monotonicity, $f(\tilde{\theta})$ must be the desirable social outcome in state θ. Indeed, if $f(\tilde{\theta})$ is not socially desirable, the supposed whistle-blower would find it profitable to choose a report that is different from $\tilde{\theta}$ in her third envelope.

The proof is divided into three steps. Step 1 is "contagion of truth": if mother j tells the truth in her first envelope, then mother i must tell the truth in her second envelope.

Step 2 is "consistency": both mothers report the same state $\tilde{\theta}$ in their second envelope. Step 3 is "no challenge": both mothers report the same state $\tilde{\theta}$ in their third envelope.

Step 1. Contagion of Truth: This step follows from the transfer table above. Indeed, we can summarize the payment of mother i as the following table:

mother i's payment	$m_j^2 = \theta$	$m_j^2 \neq \theta$
$m_i = \theta$	0	$-\eta$
$m_i \neq \theta$	η	0

Hence, by announcing the true state θ in her second envelope, mother i will save the payment by η which exceeds $\bar{v}$, the maximal gain from misreporting in the second envelope.

Step 2. Consistency: We argue that with probability one, both mothers report the same state in their second envelope. We complete this argument by considering three cases. For the first case, we assume that both mothers tell the truth with certainty in their first envelope. Then, the state in the second envelope is the true state θ by Step 1. For the second case, we suppose that both mothers tell a lie in their first envelope with positive probability. Observation ($\bigstar$) implies that each mother must believe that with probability one, the other mother has the same report in the second envelope, say, $\tilde{\theta}$. Hence, bother mothers report the same $\tilde{\theta}$ in the second envelope with probability one. Finally, if mother i tells the truth in her first envelope with certainty, while mother $j \neq i$ lies in her first envelope with positive probability. First, Step 1 implies that mother j's report in the second envelope must be truthful with probability one. Furthermore, as in the second case, mother j who lies in her first envelope must believe that with probability one mother i has the same report in the second envelope as j reports. Hence, both mothers announce the same state (truth) in the second envelope with probability one.

Step 3. No Challenge: By Step 2, both mothers report the same state $\tilde{\theta}$ in their second envelope. First, we claim that $\tilde{\theta}$ is the true state. Suppose not. Then, the true mother should tell the truth in the third envelope, and by doing this with probability $1 - \varepsilon$ she gets the baby by paying v_m. This is better than selling the baby to the false mother at price $\bar{v}$, with a small enough ε. However, this triggers Rule 2-1 with positive probability. Hence, both mothers should tell the truth. By Step 1, $\tilde{\theta}$ should be the true state. This is a contradiction. Finally, it is never a best response for anyone to report a lie in the third envelope against the truth in the second envelope and hence there is no challenge.

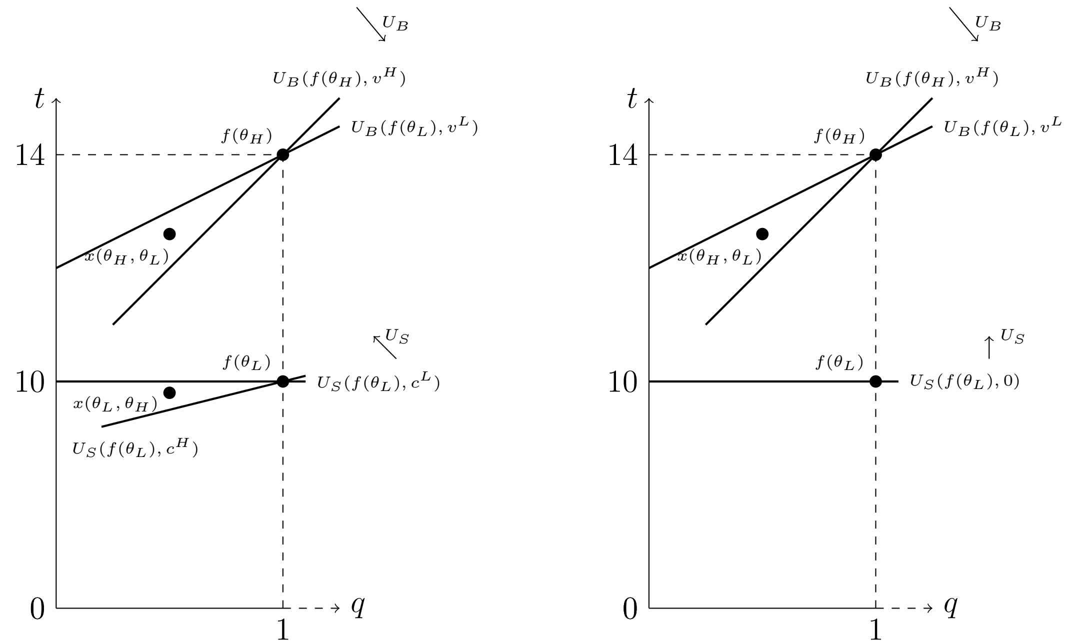
2.2 Example 2: Bilateral Trade

A seller S has an object for sale to a buyer B. The quality θ of the good is either θ^L or θ^H. The designer can impose transfers and hence the set of outcomes A is the set of triplets (q, t_B, t_S) with $q \in [0,1]$ representing the amount of the good being traded, t_B is the price paid by B and t_S is the payment received by S. For any outcome (q, t_B, t_S), B's utility is $u_B = qv + t_B$ when the good quality is v, and the seller's utility is $u_S = t_S - qc$. We identify θ^H with the pair $\left(v^H, c^H\right)$ and θ^L with the pair $\left(v^L, c^L\right)$.

We want to implement an efficient allocation rule in which the good is always traded with a surplus division rule $t^L \in [0,10]$ when the state is θ^L and $t^H \in [0,14]$ when the state is θ^H. That is, the social choice function we want to implement is $f\left(\theta^L\right) = \left(1, -t^L, t^L\right)$ and $f\left(\theta^H\right) = \left(1, -t^H, t^H\right)$. The leading example of Hart and Moore (2003) and Aghion et al. (2012) sets $v^H > v^L$ and $c^H = c^L = 0$; moreover, $t^H = v^H$ and $t^L = v^L$. That is, the buyer pays his value and all the surplus goes to the seller.

As we show in the following figure, however, the specification in Hart and Moore (2003) is a knife-edge case which results in the non-Maskin-monotonicity of f. To see this, we draw the indifference curves for both buyer and seller for the case with $v^H > v^L$ and $c^H > c^L = 0$. When the state switches from θ^L to θ^H, the buyer can serve as a whistle-blower with the allocation $x\left(\theta^L, \theta^H\right)$. Indeed, from state θ^L to state θ^H, $f\left(\theta^L\right)$ moves down the buyer's ranking against $x\left(\theta^L, \theta^H\right)$. Likewise, when the state switches from θ^H to θ^L, the seller can serve as a whistle-blower with the allocation $x\left(\theta^H, \theta^L\right)$. As a result, f is Maskin-monotonic and in fact it remains so for *any* surplus division rule. It is also clear from the figure that if $c^H = c^L = 0$ instead, then we can no longer find a room for the test allocation $x\left(\theta^H, \theta^L\right)$ and hence f is no longer Maskin-monotonic.

A typical solution to this problem with a non-Maskin-monotonic SCF is to appeal to implementation with some refinements. For instance, the well known Irrelevance Theorem of nonverifiable/indescribable information due to Maskin and Tirole (1999) is based on the implementation in subgame-perfect Nash equilibrium via the Moore-Repullo mechanism. However, (Aghion et al., 2012, Theorem 3) shows that no finite mechanism, whether it is static or dynamic, can implement the SCF in subgame-perfect Nash equilibrium in a manner that is robust to small information perturbations. In contrast, once we move from the knife-edge case to have $c^H > c^L \geq 0$, our Theorem 2 implies that the Maskin-monotonic SCF f



(a) Maskin-monotonic (b) non-Maskin-monotonic

can be implemented in mixed-strategy Nash equilibrium in a finite mechanism. Moreover, Proposition 1 shows that the implementation is robust to any small information perturbations in the sense of Aghion et al. (2012).

3 Preliminaries

3.1 Environment

There are a finite set of agents $\mathcal{I} = \{1, 2, ..., I\}$ with $I \geq 2$; a finite set of possible states Θ; and a set of pure alternatives A. We consider an environment with lotteries and transfers. Specifically, we work with the space of allocations $X \equiv \Delta(A) \times \mathbb{R}^I$ where $\Delta(A)$ denotes the set of lotteries on A that have a countable support and $\mathbb{R}^I$ denotes the set of transfers to the agents. Each $\theta \in \Theta$ induces a preference relation θ_i over A for each agent $i \in \mathcal{I}$. Thus, the vector $(\theta_i)_{i \in \mathcal{I}}$ specifies every agent's preference under θ. In what follows, we call θ_i agent i's *type*. Assume that Θ has no redundancy, i.e., $\theta \neq \theta' \implies \theta_i \neq \theta'_i$ for some i. Hence, we can identify a state θ with its induced type profile $(\theta_i)_{i \in \mathcal{I}}$; moreover, we say that a type profile $(\theta_i)_{i \in \mathcal{I}}$ identifies a state θ' if $\theta_i = \theta'_i$ for every $i \in \mathcal{I}$. We focus on *complete information* environments in which the state θ is common knowledge among the agents but unknown to the designer.[11]

Let Θ_i be the set of types of agent i. We also assume that each $\theta_i \in \Theta_i$ induces a utility function $u_i(\cdot, \theta_i) : X \to \mathbb{R}$ which is quasilinear in transfers and has a bounded expected utility representation on $\Delta(A)$. That is, for each $x = \left(l, (t_i)_{i\in\mathcal{I}}\right) \in X$, we have $u_i(x, \theta_i) = v_i(l, \theta_i) + t_i$ for some bounded expected utility function $v_i(l, \theta_i)$ over $\Delta(A)$. The designer's objective is specified by a *social choice function* (henceforth, SCF) $f : \Theta \to \Delta(A)$. Finally, we define

$$D \equiv \sup_{i\in\mathcal{I}, \theta_i\in\Theta_i, a, a'\in A} 2 \times |u_i(a, \theta_i) - u_i(a', \theta_i)|. \tag{3}$$

3.2 Mechanism and Solution

A mechanism $\mathcal{M}$ is a triplet $((M_i), g, (\tau_i))_{i\in\mathcal{I}}$ where M_i is the nonempty finite *set of messages* available to agent i; $g : M \to \Delta(A)$ ($M \equiv \times_{i=1}^{I} M_i$) is the *outcome function*; and $\tau_i : M \to \mathbb{R}$ is the *transfer rule* which specifies the payment or subsidy to agent i. The environment and the mechanism together constitute a *game with complete information* at each state $\theta \in \Theta$ which we denote by $\Gamma(\mathcal{M}, \theta)$. A direct mechanism is a mechanism $((M_i), g, (\tau_i))_{i\in\mathcal{I}}$ where $M_i = \Theta$ for every agent i and $g(\theta, ..., \theta) = f(\theta)$ for each θ.

Let $\sigma_i \in \Delta(M_i)$ be a (possibly mixed) *strategy* of agent i in the game $\Gamma(\mathcal{M}, \theta)$. A strategy profile $\sigma = (\sigma_1, \ldots, \sigma_I) \in \times_{i\in\mathcal{I}}\Delta(M_i)$ is said to be a (mixed-strategy) *Nash equilibrium* of the game $\Gamma(\mathcal{M}, \theta)$ if, for any player $i \in \mathcal{I}$, any messages $m_i \in$supp(σ_i) and $m_i' \in M_i$, we have

$$\begin{aligned} &\sum_{m_{-i}\in M_{-i}} \sigma_{-i}(m_{-i}) \left\{u_i(g(m_i, m_{-i}); \theta_i) + \tau_i(m_i, m_{-i})\right\} \\ \geq &\sum_{m_{-i}\in M_{-i}} \sigma_{-i}(m_{-i}) \left\{u_i(g(m_i', m_{-i}); \theta_i) + \tau_i(m_i', m_{-i})\right\}. \end{aligned}$$

A pure-strategy Nash equilibrium is a Nash equilibrium σ such that for each agent i, $\sigma_i(m_i) = 1$ for some $m_i \in M_i$.

Let $NE(\Gamma(\mathcal{M}, \theta))$ denote the set of Nash equilibria of the game $\Gamma(\mathcal{M}, \theta)$. We also denote by supp $(NE(\Gamma(\mathcal{M}, \theta)))$ as the set of message profiles that can be played with

[11] Thanks to the complete-information assumption, it is without loss of generality to assume that agents' values are private. In Section 5.3, we also explain why it still entails no loss of generality even with information perturbations around complete information.

positive probability under some Nash equilibrium $\sigma \in NE(\Gamma(\mathcal{M}, \theta)$, i.e.,

$$\text{supp }(NE(\Gamma(\mathcal{M}, \theta))) = \{m \in M : \text{there exists } \sigma \in NE(\Gamma(\mathcal{M}, \theta)) \text{ such that } \sigma(m) > 0\}$$

We propose our concept of Nash implementation.

Definition 1 *An SCF f is* ***implementable in Nash equilibria*** *if there exists a mechanism $\mathcal{M} = ((M_i), g, (\tau_i))_{i \in \mathcal{I}}$ such that for any state $\theta \in \Theta$ and $m \in M$, (i) there exists a pure-strategy Nash equilibrium in the game $\Gamma(\mathcal{M}, \theta)$; and (ii) $m \in$ supp $(NE(\Gamma(\mathcal{M}, \theta))) \Rightarrow g(m) = f(\theta)$ and $\tau_i(m) = 0$ for every $i \in \mathcal{I}$.*

Remark: We adopt Maskin's (1999) definition of mixed-strategy Nash implementation. Mezzetti and Renou (2012) propose another definition of Nash implementation that keeps requirement (ii) but weaken requirement (i) in requiring only the existence of mixed-strategy Nash equilibrium. See Section 7.4 for more details.

3.3 Dictator Lottery

Recall that $v_i(\cdot, \theta_i)$ is the bounded expected utility function of agent i of type θ_i. We maintain the following weak assumption throughout the paper:

Assumption 1 *$\theta_i \neq \theta_i' \Rightarrow u_i(\cdot, \theta_i)$ and $u_i(\cdot, \theta_i')$ induce different preference orders on X.*

Given the assumption, we have the following result borrowed from Abreu and Matsushima (1992).

Lemma 1 *Suppose that Assumption 1 holds. Then, for each $i \in \mathcal{I}$, there exists a function $y_i : \Theta_i \to X$ such that for any $\theta_i, \theta_i' \in \Theta_i$ with $\theta_i \neq \theta_i'$,*

$$v_i(y_i(\theta_i), \theta_i) > v_i(y_i(\theta_i'), \theta_i). \tag{4}$$

3.4 Maskin Monotonicity

For $(\theta_i, x) \in \Theta_i \times X$, we use $\mathcal{L}_i(x, \theta_i)$ to denote the lower-contour set at x in X for type θ_i, i.e.,

$$\mathcal{L}_i(x, \theta_i) = \{x' \in X : u_i(x, \theta_i) \geq u_i(x', \theta_i)\},$$

We use $\mathcal{SU}_i(x,\theta_i)$ to denote the strict upper-contour set of $x \in X$ for type θ_i, i.e.,

$$\mathcal{SU}_i(x,\theta_i) = \{x' \in X : u_i(x',\theta_i) > u_i(x,\theta_i)\}.$$

We now definition Maskin monotonicity which Maskin (1999) proposes for Nash implementation:

Definition 2 *Say an SCF f satisfies* ***Maskin monotonicity*** *if, for any pair of states θ and θ' with $f(\theta) \neq f(\theta')$, there is some agent $i \in \mathcal{I}$ such that*

$$\mathcal{L}_i(f(\theta),\theta_i) \cap \mathcal{SU}_i(f(\theta),\theta'_i) \neq \varnothing. \tag{5}$$

The agent i in Definition 2 is called a "whistle-blower" or a "test agent" for the ordered pair of states (θ,θ').

To see the idea of Maskin monotonicity, suppose that f is implemented in Nash equilibrium by a mechanism. When θ is the true state, there exists a Nash equilibrium m in $\Gamma(\mathcal{M},\theta)$ which induces $f(\theta)$. If $f(\theta) \neq f(\theta')$, when θ' becomes the true state, the strategy profile m cannot be a Nash equilibrium, i.e., there exists some agent i who has a profitable deviation. Suppose this deviation induces outcome x, i.e., agent i strictly prefers x to $f(\theta)$ at state θ'. However, since m is a Nash equilibrium at state θ, such a deviation cannot be profitable, and hence, agent i weakly prefers $f(\theta)$ to x at state θ. Therefore, Maskin monotonicity is a necessary condition for Nash implementation.

Next, we introduce the notion of strict Maskin monotonicity defined in Bergemann et al. (2011). For $(\theta_i, x) \in \Theta_i \times X$, we use $\mathcal{SL}_i(x,\theta_i)$ to denote the strict lower-contour set at allocation x for type θ_i, i.e.,

$$\mathcal{SL}_i(x,\theta_i) = \{x' \in X : u_i(x,\theta_i) > u_i(x',\theta_i)\},$$

Definition 3 *Say an SCF f satisfies* ***strict Maskin monotonicity*** *if, for any pair of states θ and θ' with $f(\theta) \neq f(\theta')$, there is some agent $i \in \mathcal{I}$ such that*

$$\mathcal{SL}_i(f(\theta),\theta_i) \cap \mathcal{SU}_i(f(\theta),\theta'_i) \neq \varnothing.$$

Observe that strict Maskin monotonicity is equivalent to Maskin monotonicity in our transferable utility setup.

3.5 Best Challenge Scheme

We now define a notion called *the best challenge scheme*, which plays a crucial role in proving our main results. Fix agent i of type θ_i. For each state $\tilde{\theta} \in \Theta$, if $\mathcal{SL}_i(f(\tilde{\theta}), \tilde{\theta}_i) \cap \mathcal{SU}_i(f(\tilde{\theta}), \theta_i) \neq \varnothing$, we select some $x(\tilde{\theta}, \theta_i) \in \mathcal{SL}_i(f(\tilde{\theta}), \tilde{\theta}_i) \cap \mathcal{SU}_i(f(\tilde{\theta}), \theta_i)$. Then, the best challenge scheme for agent i of type θ_i is a function $B_{\theta_i} : \Theta \to X$ such that for any $\tilde{\theta} \in \Theta$,

$$B_{\theta_i}(\tilde{\theta}) = \begin{cases} f(\tilde{\theta}), & \text{if } \mathcal{SL}_i(f(\tilde{\theta}), \tilde{\theta}_i) \cap \mathcal{SU}_i(f(\tilde{\theta}), \theta_i) = \varnothing; \\ x(\tilde{\theta}, \theta_i), & \text{if } \mathcal{SL}_i(f(\tilde{\theta}), \tilde{\theta}_i) \cap \mathcal{SU}_i(f(\tilde{\theta}), \theta_i) \neq \varnothing. \end{cases}$$

We may understand the notion of the best challenge scheme in conjuction with the (strict) monotonicity of f. Indeed, if $f(\theta) \neq f\left(\tilde{\theta}\right)$, monotonicity of f requires that there be a whistle-blower i together with a test allocation

$$x(\tilde{\theta}, \theta_i) \in \mathcal{SL}_i(f(\tilde{\theta}), \tilde{\theta}_i) \cap \mathcal{SU}_i(f(\tilde{\theta}), \theta_i).$$

When the true state is θ and the designer is about to implement a misported social outcome $f\left(\tilde{\theta}\right)$, the whistle-blower can make use of the test allocation $x(\tilde{\theta}, \theta_i)$ to convince the designer that $\tilde{\theta}$ is false and gain from blowing the whistle. The best challenge scheme saves the whistle-blower from reporting the test allocation. As the scheme pre-selects the test allocations for each state-type pair, the whistle-blower can just report the true state θ to challenge a bad equilibrium misreport $\tilde{\theta}$ to obtain the allocation $x(\tilde{\theta}, \theta_i)$.

4 Pure-Strategy Nash Implementation

Recall that a mechanism is direct if every agent announces the preference *profile* of all agents (i.e., a state) but nothing else. We prove our first result which shows that with three or more agents, every Maskin-monotonic SCF can be implemented in pure-strategy Nash equilibrium by a *direct* mechanism. This is clearly at odds with the conventional wisdom that revelation principle does not hold in the full Nash implementation problem, which is why all previous papers resort to *indirect* mechanisms. Hence, Theorem 1 illustrates how monetary transfers can be used to simplify the implementing mechanism to the largest extent and to dispense with devices such as integer games or modulo games.

Reall that a direct mechanism is a mechanism $((M_i), g, (\tau_i))_{i \in \mathcal{I}}$ where $M_i = \Theta$ for every agent i and $g(\theta, ..., \theta) = f(\theta)$ for each θ. We now state and prove the main result of the section.

Theorem 1 *Suppose that there are at least three agents, i.e., $I \geq 3$. Then, an SCF f is implementable in pure-strategy Nash equilibria by a direct mechanism if and only if it satisfies Maskin monotonicity.*

Proof. Suppose that f satisfies Maskin-monotonicity. We define the implementing mechanism according to three rules:

Rule 1. If there exists $\tilde{\theta}$ such that $m_i = \tilde{\theta}$ for each $i \in I$, then $g(m) = f(\tilde{\theta})$;

Rule 2. If there exist $\tilde{\theta}$ and agent $i \in I$ such that $m_j = \tilde{\theta}$ for all $j \neq i$ and $m_i \neq \tilde{\theta}$, then $g(m) = B_{\theta_i}(\tilde{\theta})$. Moreover, charge player $i+1 \pmod{I}$ a large penalty 2η, where $\eta > D$ and D is as defined in (3).

Rule 3. Otherwise, $g(m) = f(m_1)$. Moreover, every agent i pays a penalty of η if and only if $\arg\max_{\tilde{\theta}} |\{j \in I : m_j = \tilde{\theta}\}| \neq \{m_i\}$ (i.e., agent i does not report a state reported by the *unique majority*).[12]

It follows from Rule 2 that if $\tilde{\theta}$ is the true state, then $B_{\theta_i}(\tilde{\theta}) \neq f\left(\tilde{\theta}\right)$ implies that $B_{\theta_i}(\tilde{\theta}) \in \mathcal{SL}_i(f(\tilde{\theta}), \tilde{\theta}_i)$. Hence, everyone reporting the true state constitutes a pure-strategy Nash equilibrium.

Now fix an arbitrary pure-strategy Nash equilibrium m. First, we claim that m cannot trigger Rule 2. To see this, suppose that Rule 2 is triggered, and let agent i be the odd man out. Then, agent $i+1$ finds it strictly profitable to deviate to announce m_i: After such a deviation, since $I \geq 3$, either Rule 3 is triggered or it remains in Rule 2, but agent i is no longer the odd man out. Thus, agent $i+1$ saves at least η (from paying 2η to paying η or 0). Such a deviation may also change the allocation selected by the outcome function $g(\cdot)$, which induces utility change bounded by D. Since $\eta > D$, agent $i+1$ strictly prefers deviating to announce m_i, which contradicts the hypothesis that m is a Nash equilibrium.

Second, we claim that m cannot trigger Rule 3 either. To see this, suppose that Rule 3 is triggered. Pick any state reported by some (not necessarily unique) majority of agents, i.e., $\hat{\theta} \in \arg\max_{\tilde{\theta}} |\{j \in I : m_j = \tilde{\theta}\}|$. Let $I_{\hat{\theta}}$ be the set of agents who report $\hat{\theta}$. Clearly, $I_{\hat{\theta}} \subsetneq I$, because Rule 3 (rather than Rule 1) is triggered. Then, we can find an agent $i^* \in I_{\hat{\theta}}$ such that agent $i^* + 1 \pmod{I}$ is not in $I_{\hat{\theta}}$. Since agent $i^* + 1$ does not belong to the unique majority, he must pay η under m. Then, agent $i^* + 1$ will strictly prefer deviating to announce $m_{i^*} = \hat{\theta}$. Indeed, after such a deviation, either Rule 3 is triggered, and agent

[12] Hence, when there are multiple groups of majority, everyone has to pay η.

$i^* + 1$ falls in the unique majority who reports $\hat{\theta}$; or Rule 2 is triggered, but agent i^* cannot be the odd man out. Thus, agent $i^* + 1$ saves at least η (from paying η to paying 0) and $\eta > D$, the maximal utility change induced by different allocations in $g(\cdot)$. The existence of profitable deviation of agent $i^* + 1$ contradicts the hypothesis that m is a Nash equilibrium.

Hence, we conclude that m must trigger Rule 1. It follows that $f(\tilde{\theta}) = f(\theta)$. Otherwise, by Maskin monotonicity, a whistle blower can deviate to trigger Rule 2. ■

Remark: We adopt the notion of direct (revelation) mechanism from Dutta and Sen (1991) and Osborne and Rubinstein (1994, Definition 179.2) but our notion differs from the one adopted by Dasgupta et al. (1979) in which agents report only their own types/preferences. In particular, Dasgupta et al. (1979, Theorem 7.1.1) shows that only strategy-proof SCFs are "partially" implemented in Nash equilibrium by the notion of direct mechanism in Theorem 7.1.1 of Dasgupta et al. (1979).

Remark: Theorem 1 does not hold when there are only two agents. We provide a counterexample in Appendix A.1.

We note that Benoît and Ok (2008) also studies the exact Nash implementation problem in economic environments. There are three major differences between Benoît and Ok (2008) and this paper, however. First, Benoît and Ok (2008) consider a more general environment in which either lotteries alone *or* transfers alone are allowed, while we need transfers in proving Theorem 1[13] and both lotteries and transfers in proving Theorems 2 and 3. Second, Benoît and Ok (2008) prove that Maskin monotonicity is a necessary and sufficient condition for pure-strategy Nash implementation using the integer game device, whereas we prove the result by using a direct mechanism. Third, Benoît and Ok (2008) focus on pure-strategy Nash implementation with three or more agents, while we fully characterize mixed-strategy Nash implementation and rationalizable implementation in the following two sections.

5 Mixed-Strategy Nash Implementation

Theorem 1 establishes a revelation principle for Nash implementation. That is, in environments with transfers, we need only direct mechanisms to fully implement any Maskin-monotonic SCF in Nash equilibrium. The direct mechanism is a deterministic mechanism

[13] It is possible that lotteries exist in the direct mechanism when the definition of Maskin monotonicity involves lotteries, otherwise we do not employ any randomization device in constructing the direct mechanism.

and we completely ignores the existence of mixed-strategy equilibria. We now show that by invoking both lotteries and transfers, we can implement any Maskin-monotonic SCF in mixed-strategy Nash equilibrium in a finite mechanism. Its proof will be provided after we propose a canonical mechanism to be used in the theorem below.

Theorem 2 *An SCF f is implementable in Nash equilibria by a finite mechanism if and only if it satisfies Maskin monotonicity.*

Jackson (1992, Example 4) constructs a monotonic SCF such that any finite mechanism which implements the SCF in pure-strategy Nash equilibrium must also possess a mixed-strategy equilibrium which Pareto-dominates the outcome associated with the pure-strategy equilibria. The example shows that it is generally impossible to implement a monotonic SCF in mixed-strategy equilibria in a finite mechanism without making use of lotteries and transfers.

5.1 The Mechanism

We propose a mechanism $\mathcal{M} = (M, g, \tau)$ that is used to prove Theorem 2. We define the message space, allocation rule, and transfer rule as follows.

5.1.1 Message Space

A generic message of agent i is described as follows:

$$m_i = \left(m_i^1, m_i^2, m_i^3\right) \in M_i = M_i^1 \times M_i^2 \times M_i^3 = \Theta_i \times \Theta \times \Theta_i.$$

That is, agent i is asked to make (1) two announcements about his own type (i.e., m_i^1, m_i^3); and (2) an announcement about the state (i.e., m_i^2). To simplify the notation, we write $m_{i,j}^2 = \tilde{\theta}_j$ if agent i reports in m_i^2 that agent j is of type $\tilde{\theta}_j$.

5.1.2 Allocation Rule

For each message profile $m \in M$, the allocation is defined as follows:

$$g(m) = \frac{1}{I(I-1)} \sum_{i \in \mathcal{I}} \sum_{j \neq i} \left[e_{i,j}(m_i, m_j) \frac{1}{I} \sum_{k \in \mathcal{I}} y_k\left(m_k^1\right) + \left(1 - e_{i,j}(m_i, m_j)\right) B_{m_j^3}\left(m_i^2\right) \right]$$

where $y_k : \Theta \to X$ is the dictator lottery for agent k defined in Lemma 1 and

$$e_{i,j}(m_i, m_j) = \begin{cases} 0, & \text{if } m_i^2 = m_j^2 \text{ and } B_{m_j^3}(m_i^2) = f(m_i^2); \\ \varepsilon, & \text{otherwise.} \end{cases}$$

That is, the designer first chooses each pair of agents (i, j) with equal probability. A pair of (i, j) will be treated differently from a pair of (j, i), i.e., the order of the pair matters in determining the allocation. In what follows, say the second reports of agent i and agent j are *consistent* if $m_i^2 = m_j^2$; moreover, say agent j does not challenge agent i if $B_{m_j^3}(m_i^2) = f(m_i^2)$. We distinguish two cases: (1) if the second reports of agent i and agent j are consistent and agent j does not challenge agent i, then we implement $f(m_i^2)$; (2) if either the second reports of agent i and agent j are inconsistent or agent j challenges agent i, then we implement the compound lottery:

$$C_{i,j}^{\varepsilon}(m) \equiv \varepsilon \times \frac{1}{I} \sum_{k \in \mathcal{I}} y_k\left(m_k^1\right) + (1 - \varepsilon) \times B_{m_j^3}\left(m_i^2\right)$$

where $y_k(\cdot)$ are the dictator lotteries defined in Lemma 1.

By strict Maskin monotonicity of the SCF f, for every $m \in M$, $\theta \in \Theta$ and $j \in \mathcal{I}$, we can choose $\varepsilon > 0$ sufficiently small such that it does not disturb the monotonicity property, i.e.,

$$u_j(C_{i,j}^{\varepsilon}(m), \theta_j) < u_j(f(m_i^2), \theta_j) \text{ if } B_{m_j^3}(m_i^2) \neq f(m_i^2) \text{ where } m_i^2 = \theta; \tag{6}$$

$$u_j(C_{i,j}^{\varepsilon}(m), \theta_j) > u_j(f(m_i^2), \theta_j) \text{ if } B_{m_j^3}(m_i^2) \neq f(m_i^2) \text{ where } m_j^3 = \theta_j, \tag{7}$$

That is, $C_{i,j}^{\varepsilon}(m) \in \mathcal{SL}_j\left(f\left(m_i^2\right), m_j\right) \cap \mathcal{SU}_j\left(f\left(m_i^2\right), m_j^3\right)$ whenever $B_{m_j^3}(m_i^2) \neq f(m_i^2)$, i.e., whenever it is an "effective" challenge, $C_{i,j}^{\varepsilon}(m)$ is worse than $f(m_i^2)$ for agent j when agent i tells the truth; $C_{i,j}^{\varepsilon}(m)$ is better than $f(m_i^2)$ for agent j when agent i tells a lie.

5.1.3 Transfer Rule

We now define the transfer rule. For any message profile $m \in M$ and agent $i \in \mathcal{I}$, we specify the transfer to agent i as follows:

$$\tau_i(m) = \sum_{j \neq i} \left[\tau_{i,j}^1(m) + \tau_{i,j}^2(m)\right]$$

where

$$\tau_{i,j}^{1}(m) = \begin{cases} 0, & \text{if } m_{i,j}^2 = m_{j,j}^2; \\ -\eta & \text{if } m_{i,j}^2 \neq m_{j,j}^2 \text{ and } m_{i,j}^2 \neq m_j^1; \\ \eta & \text{if } m_{i,j}^2 \neq m_{j,j}^2 \text{ and } m_{i,j}^2 = m_j^1. \end{cases} \tag{8}$$

$$\tau_{i,j}^{2}(m) = \begin{cases} 0, & \text{if } m_{i,i}^2 = m_{j,i}^2; \\ -\eta, & \text{if } m_{i,i}^2 \neq m_{j,i}^2, \end{cases} \tag{9}$$

and

$$\eta > D. \tag{10}$$

Recall that D is the maximal utility difference defined in (3). The transfer rule can be summarized in the table below:

Transfer to agents	$m_{i,j}^2 = m_{j,j}^2$	$m_{i,j}^2 \neq m_{j,j}^2$	
	$m_{i,j}^2 = m_j^1$ or $m_{i,j}^2 \neq m_j^1$	$m_{i,j}^2 = m_j^1$	$m_{i,j}^2 \neq m_j^1$
$\left(\tau_{i,j}^1(m), \tau_{j,i}^2(m)\right)$	$(0,0)$	$(\eta, -\eta)$	$(-\eta, -\eta)$

In words, for each pair of agents (i,j) to be chosen, they may incur the following transfers: If their second reports on agent j's type match, then they incur no transfer; if their second reports on j's type differ, then distinguish two subcases: (a) if agent i's report matches agent j's first report, then agent j pays η to agent i; (b) if agent i's report does not match agent j's first report either, then both agents pay η to the designer.

The mechanism has the following crucial feature which will be used in proving Theorem 2.

Claim 1 *Let σ be a Nash equilibrium of the game $\Gamma(\mathcal{M},\theta)$. If $m_i^1 \neq \theta_i$ for some $m_i \in$supp(σ_i), then we must have $e_{k,j}(m_k, m_j) = e_{j,k}(m_j, m_k) = 0$ for every $m_j \in$supp(σ_j) and every agent $j \neq k$.*

Proof. Observe that m_i^1 only affects agent i's own payoff through controlling the dictator lottery y_i. Hence, if $e_{i,j}(m_i, m_j) = \varepsilon$ or $e_{j,i}(m_j, m_i) = \varepsilon$, then $m_i^1 = \theta_i$ by (4). ■

Note $e_{i,j}(m_i, m_j) = \varepsilon$ (and similarly for $e_{j,i}(m_j, m_i) = \varepsilon$) when $m_i^2 \neq m_j^2$ or $B_{m_j^3}(m_i^2) \neq f(m_i^2)$. Hence, the claim says that agent i must report his true type in m_i^1 in any of his equilibrium message(s) whenever he believes that (m_i, m_{-i}) will be inconsistent or result in challenge for some agents j, k.

5.2 Proof of Theorem 2

We first note that Maskin monotonicity is already shown to be a necessary condition. Thus, we focus on the "if" part of the proof. Consider an arbitrary true state $\theta = (\theta_i)_{i \in \mathcal{I}}$.

First, we argue that truth-telling m where $m_i = (\theta_i, \theta, \theta_i)$ for each $i \in \mathcal{I}$ constitutes a pure-strategy equilibrium. Under the message profile m, for any $i, j \in \mathcal{I}$, we have $B_{m_j^3}(m_i^2) = f(\theta)$ and $e_{i,j}(m_i, m_j) = 0$. Firstly, reporting $\tilde{m}_i$ with either $\tilde{m}_{i,i}^2 \neq \theta_i$ or $\tilde{m}_{i,j}^2 \neq \theta_j$ suffers the penalty of $\eta > D$ and hence cannot be a profitable deviation. Secondly, reporting $\tilde{m}_i$ with $\tilde{m}_i^2 = \theta$ and $\tilde{m}_i^3 \neq \theta_i$ either leads to $B_{\tilde{m}_i^3}(\theta) = f(\theta)$ and results in no change in payoff or $B_{\tilde{m}_i^3}(\theta) \neq f(\theta)$ which is strictly worse than $f(\theta)$ by (6). Finally, reporting $\tilde{m}_i$ with $\tilde{m}_i^2 = \theta$, $\tilde{m}_i^3 = \theta_i$, and $\tilde{m}_i^1 \neq \theta_i$ does not affect the allocation or transfer, since we still have $\tau_i(m) = 0$ and $e_{j,k}(m_j, m_k) = 0$ for every j and k.

Second, we show that for any Nash equilibrium σ of the game $\Gamma(\mathcal{M}, \theta)$ and any $m \in$supp(σ), $g(m) = f(\theta)$ and $\tau_i(m) = 0$ for any $i \in \mathcal{I}$. The proof is divided into three steps: (Step 1) *contagion of truth*: if agent j announces his type truthfully in his first report, then every agent must also report agent j's type truthfully in their second report; (Step 2) *consistency*: every agent reports the same state $\tilde{\theta}$ in the second report; and (Step 3) *no challenge*: no agent challenges the common reported state $\tilde{\theta}$, i.e., $B_{m_j^3}(\tilde{\theta}) = f(\tilde{\theta})$ for any $j \in \mathcal{I}$. Consistency implies that $\tau_i(m) = 0$ for any $i \in \mathcal{I}$, whereas no challenge together with monotonicity of f implies that $g(m) = f(\tilde{\theta}) = f(\theta)$. We now proceed to establish the three steps.

5.2.1 Contagion of Truth

Claim 2 *We establish two results:*
(a) If agent j reports the truth in his first report with probability one (i.e., $m_j^1 = \theta_j$ for any $m_j \in$supp(σ_j)), then every agent $i \neq j$ must report agent j's type truthfully in his second report with probability one (i.e., $m_{i,j}^2 = \theta_j$ for any $m_i \in$supp(σ_i)).
(b) If every agent i reports a fixed type $\tilde{\theta}_j$ of agent j in his second report with probability one (i.e., $m_{i,j}^2 = \tilde{\theta}_j$ for any $m_i \in$supp(σ_i)), then agent j must report $\tilde{\theta}_j$ in his second report with probability one (i.e., $m_{j,j}^2 = \tilde{\theta}_j$ for any $m_j \in$supp(σ_j)).

Proof. We first prove (a). Suppose instead that there exists some $m_i \in$supp(σ_i) such that $m_{i,j}^2 \neq \theta_j$. Let $\tilde{m}_i$ be a message that is identical to m_i except that $\tilde{m}_{i,j}^2 = \theta_j$. Consider any m_{-i} $\in$supp(σ_{-i}) and distinguish two cases: If $m^2_{j,j} = \theta_j$, due to the construction of $\tau^1_{i,j}(\cdot)$, we have $\tau^1_{i,j}(m) = -\eta$ whereas $\tau^1_{i,j}(\tilde{m}_i, m_{-i}) = 0$. If $m^2_{j,j} \neq \theta_j$, then according to the construction of $\tau^1_{i,j}(\cdot)$, we have $\tau^1_{i,j}(m_i, m_{-i})$ is either 0 or $-\eta$ whereas $\tau^1_{i,j}(\tilde{m}_i, m_{-i}) = \eta$. Thus, in terms of transfers, the gain from reporting $\tilde{m}_i$ rather than m_i is at least η; while, in terms of allocation, the potential loss is at most D. Since $\eta > D$ by (10), $\tilde{m}_i$ is a better response than m_i against any $m_{-i} \in$ supp(σ_{-i}). This is a contradiction to the hypothesis that $m_i \in$ supp(σ_i). This concludes (a).

We then prove (b). Suppose on the contrary that there exists $m_j \in$ supp(σ_j) such that $m^2_{j,j} \neq \tilde{\theta}_j$. We then construct $\tilde{m}_j$ as a message that is identical to m_j except that $\tilde{m}^2_{j,j} = \tilde{\theta}_j$. According to the construction of $\tau^2_{j,i}(\cdot)$ and because $\eta > D$ by (10), we conclude that $\tilde{m}_j$ is a better response than m_j against any $m_{-j} \in$ supp(σ_{-j}). This contradicts the hypothesis that $m_j \in$ supp(σ_j). This concludes (b). ■

5.2.2 Consistency

Claim 3 *Everyone announces the same state in their second report. That is, there exists a state $\tilde{\theta}$ such that, for any agent $i \in \mathcal{I}$ and $m_i \in$ supp(σ_i), we have $m^2_i = \tilde{\theta}$.*

Proof. We prove consistency in the following three cases:

Case 1: *Everyone tells the truth in the first report with probability one, i.e., $m^1_i = \theta_i$ for every m_i $\in$supp(σ_i) and every agent $i \in \mathcal{I}$.*

It follows directly from Claim 2 that $m^2_i = \theta$ for every m_i $\in$supp(σ_i) and every agent $i \in \mathcal{I}$.

Case 2: *Two or more agents tell a lie in their first report with positive probability, i.e., $m^1_i \neq \theta_i$ and $m^1_j \neq \theta_j$ for some m_i $\in$supp(σ_i) and m_j $\in$supp(σ_j).*

Since $m^1_i \neq \theta_i$, it follows from Claim 1 that $e_{i,k}(m_i, m_k) = 0$ for every m_k $\in$supp(σ_k) and every agent k. Hence,(m_i, m_{-i}) is consistent for every m_{-i} $\in$supp(σ_{-i}). Similarly, (m_j, m_{-j}) is consistent for every m_{-j} $\in$supp(σ_{-j}). Hence, everyone reports the same state in the second report.

Case 3: *Only one agent, say agent i, tells a lie in the first report with positive probability (i.e., $m^1_i \neq \theta_i$ for some m_i $\in$supp(σ_i)) and for every agent $j \neq i$, we have $m^1_j = \theta_j$ for every m_j $\in$supp(σ_j).*

First, since $m^1_j = \theta_j$ for every m_j $\in$supp(σ_j), it follows from Claim 2(a) that for every agent $j \neq i$, we must have $m^2_{k,j} = \theta_j$ for every $k \neq j$ and m_k $\in$supp(σ_k). Second, since

$m_i^1 \neq \theta_i$, by Claim 1, (m_i, m_{-i}) is consistent for every m_{-i} $\in$supp(σ_{-i}). In particular, every agent $j \neq i$ must report a common $\tilde{\theta}_i = m_{i,i}^2$ with probability one. It thus follows from Claim 2(b) that $\tilde{m}_i^2 = \tilde{\theta}_i$ for every $\tilde{m}_i$ $\in$supp(σ_i). Hence, $\tilde{m}_k^2 = (\tilde{\theta}_i, \theta_{-i})$ for every agent m_k $\in$supp(σ_k) and every agent $k \in \mathcal{I}$. ■

5.2.3 No challenge

Claim 4 *No one challenges the common state $\tilde{\theta}$ announced in the second report, i.e., $B_{m_j^3}(\tilde{\theta}) = f(\tilde{\theta})$ for every m_j $\in$suppσ_j and for every agent $j \in \mathcal{I}$.*

Proof. By Claim 3, denote by $\tilde{\theta}$ the common state announced in the second report. Suppose that $\mathcal{SL}_j(f(\tilde{\theta}), \tilde{\theta}_j) \cap \mathcal{SU}_j(f(\tilde{\theta}), \theta_j) = \varnothing$ for every agent j. Then, if $B_{m_j^3}(\tilde{\theta}) \neq f(\tilde{\theta})$, then $B_{m_j^3}(\tilde{\theta}) \in \mathcal{SL}_j(f(\tilde{\theta}), \tilde{\theta}_j)$ must be strictly worse than $f(\tilde{\theta})$ under the true type of agent j.[14] Then, if m_j^3 triggers a challenge and hence the allocation $B_{m_j^3}(\tilde{\theta})$, agent j can profitably deviate from announcing m_j to annoucing $\tilde{m}_j = \left(m_j^1, m_j^2, \tilde{\theta}_i\right)$. Hence, $B_{m_j^3}(\tilde{\theta}) = f(\tilde{\theta})$ and hence the claim holds.

It remains to show that $\mathcal{SL}_j(f(\tilde{\theta}), \tilde{\theta}_j) \cap \mathcal{SU}_j(f(\tilde{\theta}), \theta_j) = \varnothing$. Suppose to the contrary that $\mathcal{SL}_j(f(\tilde{\theta}), \tilde{\theta}_j) \cap \mathcal{SU}_j(f(\tilde{\theta}), \theta_j) \neq \varnothing$. Then, we must have $B_{m_j^3}(\tilde{\theta}) \neq f(\tilde{\theta})$ for every m_j $\in$supp(σ_j). Indeed, if $B_{m_j^3}(\tilde{\theta}) = f(\tilde{\theta})$, agent j can profitably deviate from announcing m_j to announcing $\tilde{m}_j = \left(m_j^1, m_j^2, \theta_i\right)$. This deviation results in the better allocation $B_{\tilde{m}_j^3}(\tilde{\theta}) \in \mathcal{SU}_j(f(\tilde{\theta}), \theta_j)$. Finally, since $B_{m_j^3}(\tilde{\theta}) \neq f(\tilde{\theta})$ for every m_j $\in$supp(σ_j), it follows that the dictator lottery is triggered with positive probability. Thus, by (4), each agent i has strict incentive to announce the true type in his first report, i.e., $m_i^1 = \theta_i$ for each m_i $\in$suppσ_i and agent $i \in \mathcal{I}$. By Claim 2, we conclude that $\tilde{\theta} = \theta$ and hence $\mathcal{SL}_j(f(\theta), \theta_j) \cap \mathcal{U}_j(f(\theta), \theta_j) \neq \varnothing$, which is impossible. ■

5.3 Robustness to Information Perturbations

Chung and Ely (2003) and Aghion et al. (2012) consider a designer who not only wants all equilibria of her mechanism to yield a desirable outcome under complete information, but is also concerned about the possibility that agents may entertain small doubts about the true

[14] Since $\mathcal{SL}_j(f(\tilde{\theta}), \tilde{\theta}_j) \cap \mathcal{SU}_j(f(\tilde{\theta}), \theta_j) = \varnothing$, we must have $u_j\left(B_{m_j^3}(\tilde{\theta}), \theta_j\right) \leq u_j\left(f(\tilde{\theta}), \theta_j\right)$. If $u_j\left(B_{m_j^3}(\tilde{\theta}), \theta_j\right) = u_j\left(f(\tilde{\theta}), \theta_j\right)$, then adding small transfer to agent j in $B_{m_j^3}(\tilde{\theta})$ will make $B_{m_j^3}(\tilde{\theta}) \in \mathcal{SL}_j(f(\tilde{\theta}), \tilde{\theta}_j) \cap \mathcal{SU}_j(f(\tilde{\theta}), \theta_j)$. This is a contradiction to $\mathcal{SL}_j(f(\tilde{\theta}), \tilde{\theta}_j) \cap \mathcal{SU}_j(f(\tilde{\theta}), \theta_j) = \varnothing$.

state. They argue that such a designer should insist on implementing the SCF in the closure of a solution concept as incomplete information about the state vanishes. Chung and Ely (2003) adopt undominated Nash equilibrium and Aghion et al. (2012) adopt subgame-perfect equilibrium as a solution concept in studying the robustness issue.

To allow for information perturbations, we now dispense with the private-value assumption. Suppose that each state θ associates for each agent i a utility function $u_i(\cdot,\theta) : X \to \mathbb{R}$ which is quasilinear in transfers and has a bounded expected utility representation on $\Delta(A)$. Indeed, while the private-value assumption entails no loss of generality under complete information, it need not be the case once we relax the complete-information assumption.

Formally, suppose that the agents do not observe the state directly but are informed of the state via signals. The set of agent i's signals is denoted as S_i which is identified with Θ, i.e., $S_i \equiv \Theta$.[15] A signal profile is an element $s = (s_1, ..., s_I) \in S \equiv \times_{i \in I} S_i$. When the realized signal profile is s, agent i observes only his own signal s_i. Let s_i^θ be the signal in which agent i's signal is θ and write $s^\theta = \left(s_i^\theta\right)_{i \in I}$. State and signals are drawn from some prior distribution over $\Theta \times S$. In particular, complete information can be modeled as a prior μ such that $\mu(\theta, s) = 0$ whenever $s \neq s^\theta$. Such μ will be called a *complete-information prior*. We assume for each $i \in \mathcal{I}$, the marginal distribution on i's signals places strictly positive weight on each of i's signals in every state, that is, $\text{marg}_{S_i}\mu(s_i) > 0$ for every $s_i \in S_i$, so that Bayes's rule is well defined. For any prior ν, we also write $\nu(\cdot|s_i)$ for the conditional distribution of ν on signal s_i.

The distance between two priors is measured by the supremum metric. That is, for any two priors μ and ν, $d(\mu,\nu) \equiv \max_{(\theta,s)\in\Theta\times S} |\mu(\theta,s) - \nu(\theta,s)|$. Write $\nu^\varepsilon \to \mu$ if $d(\nu^\varepsilon, \mu) \to 0$ as $\varepsilon \to 0$. A prior ν together with a mechanism $\mathcal{M}$ induces an incomplete-information game which we denote as $\Gamma(\mathcal{M}, \nu)$. A (mixed-)strategy of agent i is now a mapping $\sigma_i : S_i \to \Delta(M_i)$. Note that here $NE(\mathcal{M}, \nu)$ is a Bayesian Nash equilibrium. We recap the standard notion of Bayesian Nash equilibrium (BNE) in the current setup:

Definition 4 *A strategy profile σ constitute a (mixed-strategy)* ***Bayesian Nash equilibrium (BNE)*** *in $\Gamma(\mathcal{M}, \nu)$ if and only if for any agent i with signal s_i and for any messages*

[15] We adopt this formulation from Chung and Ely (2003) and Aghion et al. (2012). Our result holds for any alternative formulation so long as the (Bayesian) Nash equilibrium correspondence has closed graph.

$m_i \in supp(\sigma_i(s_i))$ and $m_i' \in M_i$, we have

$$\begin{aligned} & \sum_{\theta, s_{-i}} \nu(\theta, s_{-i}|s_i) \sum_{m_{-i}} \sigma_{-i}(s_{-i})[m_{-i}][u_i(g(m_i, m_{-i}), \theta) + \tau_i(m_i', m_{-i})] \\ \geq & \sum_{\theta, s_{-i}} \nu(\theta, s_{-i}|s_i) \sum_{m_{-i}} \sigma_{-i}(s_{-i})[m_{-i}][u_i(g(m_i', m_{-i}), \theta) + \tau_i(m_i', m_{-i})]. \end{aligned}$$

More generally, the designer may subscribe a solution concept $\mathcal{E}$ for the game $\Gamma(\mathcal{M}, \nu)$ (such as BNE) which induces a set of mappings (which we call *acts* following Chung and Ely (2003)) from $\Theta \times S$ to X. Denote the set of acts induced by the solution concept $\mathcal{E}$ as $\mathcal{E}(\mathcal{M}, \nu)$. We now define $\overline{\mathcal{E}}$-implementable as follows.

Definition 5 *An SCF f is $\overline{\mathcal{E}}$-implementable under the complete-information prior μ if there exists a mechanism $\mathcal{M} = ((M_i), g, (\tau_i))_{i \in \mathcal{I}}$ such that for any $\theta \in \Theta$ and any sequence of priors $\{\nu^n\}$ converging to μ, the following two requirements hold: (i) there is a sequence of acts $\{\alpha_n\}$ with $\alpha_n \in \mathcal{E}(\mathcal{M}, \nu_n)$ such that $\alpha_n(\theta, s) \to f(\theta)$; (ii) for every sequence of acts $\{\alpha_n\}$ with $\alpha_n \in \mathcal{E}(\mathcal{M}, \nu_n)$, we have $\alpha_n(\theta, s) \to f(\theta)$.*

Chung and Ely (2003) and Aghion et al. (2012) show that Maskin monotonicity is a necessary condition for $\overline{UNE}$-implementation and $\overline{SPE}$-implementation, respectively.[16] The result of Chung and Ely (2003) implies that implementation of a non-monotonic SCF in undominated Nash equilibrium such as the result in Abreu and Matsushima (1994) is necessarily vulnerable to information perturbations. Moreover, both Chung and Ely (2003, Theorem 2) and Aghion et al. (2012) establish the sufficiency result by restricting attention to pure-strategy equilibrium and by using infinite mechanisms. This raises the question as to whether their robustness test may be too demanding when it is applied to finite mechanisms such as the implementing mechanism of Jackson et al. (1994), that of Abreu and Matsushima (1994), or a simple mechanism in Section 5 of Moore and Repullo (1988) where mixed-strategy equilibria have to be taken seriously.

The canonical mechanism which we propose in the proof of Theorem 2 is indeed finite, and we show that the finite mechanism implements any Maskin-monotonic SCF in mixed-strategy Nash equilibrium. Since $NE(\mathcal{M}, \nu)$ viewed as a correspondence on priors, has a closed graph, it follows that the mechanism achieves $\overline{NE}$-implementation. Note that this

[16] Aghion et al. (2012) adopt sequential equilibrium as the solution concept for the incomplete-information game $\Gamma(\mathcal{M}, \nu)$.

closed-graph property holds even when we allow for interdependent values. Hence, if a solution concept $\mathcal{E}$ refines Nash equilibrium, then (ii) implies (i) in Definition 5. We now obtain the following result as a corollary of Theorem 2.

Proposition 1 *Let $\mathcal{E}$ be a solution concept such that $\varnothing \neq \mathcal{E}(\mathcal{M}, \nu) \subset NE(\mathcal{M}, \nu)$ for each finite mechanism $\mathcal{M}$ and prior ν. Then, every Maskin-monotonic SCF f is $\overline{\mathcal{E}}$-implementable.*

The condition $\varnothing \neq \mathcal{E}(\mathcal{M}, \nu) \subset NE(\mathcal{M}, \nu)$ is satisfied for virtually any refinement of Nash equilibrium, because we allow for mixed-strategy equilibrium and the requirement is imposed only for finite mechanisms.

6 Rationalizable Implementation

In this section, we adopt the solution concept of *correlated rationalizability* of Brandenburger and Dekel (1987), allowing the agents' beliefs to be correlated, and investigate the implications of implementation in rationalizable strategies. Our goal is to show that by modifying the finite implementing mechanism used in our Theorem 2, we can also implement the largest possible class of SCFs in rationalizable strategies.

First, we define rationalizability for the finite game $\Gamma(\mathcal{M}, \theta)$ as follows. Let $S_i^0(\mathcal{M}, \theta) = M_i$, and we define $S_i^k(\mathcal{M}, \theta)$ inductively: for any $k > 0$, we set

$$S_i^k(\mathcal{M}, \theta) = \left\{ m_i \in M_i \;\middle|\; \begin{array}{l} \text{there exists } \lambda_i \in \Delta(M_{-i}) \text{ such that} \\ (1)\ \lambda_i(m_{-i}) > 0 \Rightarrow m_j \in S_j^{k-1}(\mathcal{M}, \theta) \text{ for each } j \neq i, \\ (2)\ m_i \in \arg\max_{m_i} \lambda_i(m_{-i})\, u_i(g(m_i, m_{-i}), \theta_i). \end{array} \right\}.$$

Then, $S_i^\infty(\mathcal{M}, \theta) = \bigcap_{k=0}^{\infty} S_i^k(\mathcal{M}, \theta)$ is the set of rationalizable messages of agent i and $S^\infty(\mathcal{M}, \theta) = \prod_{i \in \mathcal{I}} S_i^\infty(\mathcal{M}, \theta)$ is the set of rationalizable message profiles in $\Gamma(\mathcal{M}, \theta)$.

Throughout this section, we impose a technical assumption that Θ has a product structure, i.e., $\Theta = \prod_{i \in \mathcal{I}} \Theta_i$, which is due to the fact that rationalizable strategy profiles have a product structure, i.e., $S^\infty(\mathcal{M}, \theta) = \prod_{i \in \mathcal{I}} S_i^\infty(\mathcal{M}, \theta)$.

Definition 6 *An SCF f is **implementable in rationalizable strategies** if there exists a mechanism $\mathcal{M} = ((M_i), g, (\tau_i))_{i \in \mathcal{I}}$ such that for any $\theta \in \Theta$, (i) $S^\infty(\mathcal{M}, \theta) \neq \varnothing$; and (ii) for any $m \in S^\infty(\mathcal{M}, \theta)$, we have $g(m) = f(\theta)$ and $\tau_i(m) = 0$.*

Remark: Since we propose a finite implementing mechanism below, $S^{\infty}(\mathcal{M},\theta)$ is always nonempty. That is, requirement (i) of rationalizable implementation is automatically satisfied.

Second, we introduce a central condition to our rationalizable implementation result, which is called *Maskin monotonicity**. The condition is proposed by Bergemann et al. (2011) as a necessary condition for rationalizable implementation using "well-behaved" mechanisms (such as finite one). However, Bergemann et al. (2011) has left open the question as to when an SCF is implementable in rationalizable strategies in a "well-behaved" mechanism. Theorem 3 fills the gap in the environment with lotteries and transfers.

For $(\theta_i, x) \in \Theta_i \times X$, recall that $\mathcal{SL}_i(x,\theta_i)$ denotes the strict lower-contour set at allocation x for type θ_i. Given an SCF f, we let $\mathcal{P}_f = \{\Theta_z\}_{z\in f(\Theta)}$ be the partition on Θ induced by f where $\Theta_z = \{\theta \in \Theta |\ f(\theta) = z\}$. For each partition $\mathcal{P}$ on Θ, we denote by $\mathcal{P}(\theta)$ the atom in $\mathcal{P}$ which contains state θ. Define

$$\mathcal{L}_i(x,\mathcal{P}(\theta)) \equiv \bigcap_{\hat{\theta}\in\mathcal{P}(\theta)} \mathcal{L}_i(x,\hat{\theta}_i) \text{ and } \mathcal{SL}_i(x,\mathcal{P}(\theta)) \equiv \bigcap_{\hat{\theta}\in\mathcal{P}(\theta)} \mathcal{SL}_i(x,\hat{\theta}_i)$$

The following definition is obtained by adapting Definition 5 of Bergemann et al. (2011) to our current setup.

Definition 7 *Say an SCF f satisfies* ***Maskin monotonicity**** *if there exists a partition $\mathcal{P}$ of Θ such that (i) $\mathcal{P}$ is weakly finer than $\mathcal{P}_f$; (ii) for any $\theta, \theta' \in \Theta$, whenever $\theta' \notin \mathcal{P}(\theta)$, there exists $i \in \mathcal{I}$ for whom*

$$\mathcal{L}_i(f(\theta),\mathcal{P}(\theta)) \cap \mathcal{SU}_i(f(\theta),\theta'_i) \neq \varnothing. \tag{11}$$

As we introduce strict Maskin monotonicity, we say that an SCF f satisfies strict Maskin monotonicity* if we replace $\mathcal{L}_i(f(\theta),\mathcal{P}(\theta))$ in (11) with $\mathcal{SL}_i(f(\theta),\mathcal{P}(\theta))$. Again, in the environment with transfers, strict Maskin monotonicity* and Maskin monotonicity* are equivalent.

Remark: Clearly, Maskin monotonicity* implies Maskin monotonicity. Moreover, Jain (2017, Appendix 2) constructs an example to show that strict Maskin monotonicity* is strictly stronger than strict Maskin monotonicity. In Appendix A.2, we modify Jain's example to make the same point in our setup, which accommodates the case with two agents and lotteries and transfers. Since (strict) Maskin monotonicity* is a necessary condition for rationalizable implementation by a finite mechanism, we conclude that rationalizable implementation is generally more restrictive than Nash implementation.

Let $\mathcal{P}$ be the partition in the definition of strict Maskin monotonicity*. As the case of Nash implementation, we also make use of the best challenge scheme with respect to $\mathcal{P}$. Fix agent i of type θ_i. For each state $\tilde{\theta} \in \Theta$, if $\mathcal{SL}_i(f(\tilde{\theta}), \mathcal{P}(\theta)) \cap \mathcal{SU}_i(f(\tilde{\theta}), \theta_i) \neq \varnothing$, we select some $x(\tilde{\theta}, \theta_i) \in \mathcal{SL}_i(f(\tilde{\theta}), \mathcal{P}(\theta)) \cap \mathcal{SU}_i(f(\tilde{\theta}), \theta_i)$. The best challenge scheme for agent i of type θ_i with respect to $\mathcal{P}$ is defined as a function $B_{\theta_i} : \Theta \to X$ such that for any $\tilde{\theta} \in \Theta$,

$$B_{\theta_i}(\tilde{\theta}) = \begin{cases} f(\tilde{\theta}), & \text{if } \mathcal{SL}_i(f(\tilde{\theta}), \mathcal{P}(\tilde{\theta})) \cap \mathcal{SU}_i(f(\tilde{\theta}), \theta_i) = \varnothing; \\ x(\tilde{\theta}, \theta_i), & \text{if } \mathcal{SL}_i(f(\tilde{\theta}), \mathcal{P}(\tilde{\theta})) \cap \mathcal{SU}_i(f(\tilde{\theta}), \theta_i) \neq \varnothing \end{cases}$$

where we omit the reference to $\mathcal{P}$ in B_{θ_i} for simplicity.

We now state our main result on rationalizable implementation as follows.

Theorem 3 *An SCF f is implementable in rationalizable strategies by a finite mechanism if and only if it satisfies Maskin monotonicity**.*

Since a finite mechanism satisfies the *best response property* defined in Bergemann et al. (2011, Definition 6), the "only if" part of Theorem 3 follows from Proposition 3 of Bergemann et al. (2011). In the following subsections, we will construct a mechanism to prove the "if" part of Theorem 3.

6.1 The Mechanism

Let Γ_i denote the set of functions from Θ to Θ_i. Observe that Γ_i is a finite set because both Θ and Θ_i are finite. Call each γ_i a *challenge function* of agent i which is viewed as a plan to challenge contingent on the state realization.

6.1.1 Message Space

A generic message of agent i is:

$$m_i = \left(m_i^1, m_i^2, m_i^3, m_i^4\right) \in M_i^1 \times M_i^2 \times M_i^3 \times M_i^4 = M_i = \Theta_i \times \Theta \times \Theta \times \Gamma_i.$$

That is, in this mechanism, agent i is asked to make (1) an announcement of his own type (i.e., m_i^1); (2) two announcements of the state (i.e., m_i^2, and m_i^3); (3) an announcement of a challenge function (i.e., m_i^4).

6.1.2 Allocation Rule

Say two states θ and θ' are equivalent (denoted as $\theta \sim \theta'$) if they belong to the same atom of $\mathcal{P}$. Given a message profile m, we say that m is *consistent* if there exists $\tilde{\theta} \in \Theta$ such that

$$m^1 \text{ identifies } \tilde{\theta} \text{ and } m_i^2 \sim m_i^3 \sim \tilde{\theta} \text{ for every } i \in \mathcal{I}.$$

That is, consistency requires that the type profile m^1 identify a state $\tilde{\theta}$ that is equivalent to each of the two states (i.e., m_i^2 and m_i^3) reported by every agent. Alternatively, we also say that m *is consistent on* $\tilde{\theta}$.

For each message profile $m \in M$, the allocation is defined as follows:

$$g\left(m\right) = \frac{1}{I^2} \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{I}} \left[e_{i,j}\left(m_i, m_j\right) \frac{1}{I} \sum_{k \in \mathcal{I}} y_k \left(m_k^1\right) + \left(1 - e_{i,j}\left(m_i, m_j\right)\right) B_{m_j^4\left(m_i^3\right)} \left(m_i^3\right) \right]$$

where $y_k : \Theta \to X$ is the dictator lottery for agent k defined in Lemma 1 and

$$e_{i,j}\left(m_i, m_j\right) = \begin{cases} 0, & \text{if } m_i \text{ is consistent with } m_j \text{ and } B_{m_j^4\left(m_i^3\right)}\left(m_i^3\right) = f(m_i^3); \\ \varepsilon, & \text{otherwise.} \end{cases}$$

Note that there are two differences from the allocation rule from Nash implementation: (1) here we allow any pair of (i, j), even including the case of $i = j$, to be chosen with positive probability; (2) the best challenge scheme we use is a challenge scheme contingent on every possible state.

6.1.3 Transfer Rule

We now define the *transfer rule*. For any message profile $m \in M$ and agent $i \in \mathcal{I}$, we specify the transfer from agent i as follows:

$$\tau_i(m) = \tau_i^2(m) + \tau_i^3(m),$$

where

$$\tau_i^2(m) = \begin{cases} 0, & \text{if } m^1 \text{ identifies } \tilde{\theta} \text{ and } m_i^2 \sim \tilde{\theta}; \\ \eta'', & \text{otherwise.} \end{cases}$$

$$\tau_i^3(m) = \begin{cases} 0 & \text{if } m_i^3 \sim m_{i+1}^2; \\ \eta & \text{otherwise.} \end{cases}$$

In words, $\tau_i^2(m)$ and $\tau_i^3(m)$ are the cross-checking penalties which ensure that once the type profile m^1 identifies a unique state $\tilde{\theta}$, each agent i will only want to announce states which are equivalent to $\tilde{\theta}$ when reporting m_i^2 and m_i^3. Specifically, $\tau_i^2(m)$ requires that agent i pay η'' if his announcement m_i^2 is "not" equivalent to the state identified by m^1; $\tau_i^3(m)$ requires that agent i pay η if his announcement m_i^3 is *not* equivalent to agent $(i+1)$'s announcement m_{i+1}^2 where $I+1 \equiv 1$.

By strict Maskin monotonicity* of the SCF f, for every $m \in M$, $\theta \in \Theta$ and $j \in \mathcal{I}$, we can choose $\varepsilon > 0$ sufficiently small such that

$$u_j(C_{i,j}^{\varepsilon}(m), \theta_j) > u_j(f(m_i^3), \theta_j), \text{ if } B_{m_j^4(m_i^3)}(m_i^3) \neq f(m_i^3) \text{ and } m_j^4(m_i^3) = \theta_j; \tag{12}$$

$$u_j(C_{i,j}^{\varepsilon}(m), \theta_j) < u_j(f(m_i^3), \theta_j), \text{ if } B_{m_j^4(m_i^3)}(m_i^3) \neq f(m_i^3) \text{ and } m_i^3 = \theta \tag{13}$$

where

$$C_{i,j}^{\varepsilon}(m) \equiv \varepsilon \times \frac{1}{I} \sum_{k \in \mathcal{I}} y_k\left(m_k^1\right) + (1-\varepsilon) \times B_{m_j^4(m_i^3)}\left(m_i^3\right)$$

That is, $C_{i,j}^{\varepsilon}(m) \in \mathcal{SL}_j\left(f\left(m_i^3\right), \theta_j\right) \cap \mathcal{SU}_j\left(f\left(m_i^3\right), m_j^4\left(m_i^3\right)\right)$ whenever $B_{m_j^4(m_i^3)}(m_i^3) \neq f(m_i^3)$ and $m_i^3 = \theta$.

Once again, since Θ is finite, we can find $d > 0$ such that for any $i \in \mathcal{I}$ and $\theta_i, \theta_i' \in \Theta_i$ with $\theta_i \neq \theta_i'$, the dictator lotteries satisfy

$$u_i\left(y_i\left(\theta_i\right), \theta_i\right) > u_i\left(y_i\left(\theta_i'\right), \theta_i\right) + d. \tag{14}$$

Finally, we choose $\eta'' > 0$ small enough and $\eta > 0$ large enough such that

$$\frac{\varepsilon}{I^3} d > \eta''; \tag{15}$$

$$\eta > D. \tag{16}$$

We relegate the proof of Theorem 3 to Appendix A.3. Here we give only an outline of the proof and highlight its difference from the proof of Theorem 2. In this proof, we start by arguing that if m is rationalizable, then m^1 identifies a state which is equivalent to the true state. Next, the cross-checking transfers τ_i^2 and τ_i^3 ensure that each m_i^2 and m_i^3 also identify a state equivalent to the true state.

Unlike the mechanism we constructed for Nash implementation, we check the coherence between the first and second reports of the same agent. Specifically, whether agent i needs to pay η'' depends on both his own first and his own second reports. Recall that agent i's first report does not affect his own transfer in our proof of Nash implementation. In contrast, such self-checking is crucial for the mechanism to achieve rationalizable implementation. Indeed, here we can argue only that m^1 identifies a state which is equivalent to the true state. If we only cross-check each agent's announcement of the other agents' types (instead of all agents') in the second report (as $\tau_{i,j}^1$ does for Nash implementation), then the state identified by the second report may no longer be equivalent to the true state.[17] Clearly, the self-checking may interfere with the truth-telling incentive in m_i^1, and hence we add condition (15) to make sure that η'' is not too large. As a result, we cannot have m_i^2 control the allocation but only use m_i^2 to "preserve" the truth identified by m^1. This explains why we need two state announcements instead of one, as in Nash implementation.

The lack of correct belief in rationalizability necessitates that agent i has an opportunity to challenge his own state announcement. Otherwise, agent i may report a state that is outside the equivalence class of the true state if he believes that the lie will not be challenged by any other agent.

6.2 Continuous Implementation

Oury and Tercieux (2012) consider the following notion of robustness for partial implementation: the designer wants not only that there be an equilibrium that implements the SCF but also that the same equilibrium continue to implement the SCF in all the models *close* to

[17]To see this, consider an example with two agents, each of whom has two types. Consider an SCF: $f(\theta_1, \theta_2) = f(\theta_1', \theta_2') = a$ and $f(\theta_1', \theta_2) = f(\theta_1, \theta_2') = b$. Let $\mathcal{P} = \{\{(\theta_1, \theta_2), (\theta_1', \theta_2')\}, \{(\theta_1', \theta_2), (\theta_1, \theta_2')\}\}$ be a partition over Θ. Suppose that m^1 identifies the atom which contains the true state (θ_1, θ_2). If agent 1 is asked to announce only agent 2's types in his second report, agent 1 may well announce θ_2 or θ_2' since the "true atom" contains both (θ_1, θ_2) and (θ_1', θ_2'). However, the same situation occurs when the true state is (θ_1', θ_2'). In other words, we cannot preserve the "true atom" identified by m^1 without cross-checking the entire type profile.

her initial model. Hence, the SCF is *continuously* implementable. Oury and Tercieux (2012) obtain the following characterization of continuous implementation in their Theorem 4: an SCF is continuously implementable by a finite mechanism if it is exactly implementable in rationalizable strategies by a finite mechanism.[18] Since this result says nothing about the class of SCFs that are exactly implementable in rationalizable strategies by finite mechanisms, we view this as an important open question in the literature.[19] We establish the following continuous implementation result, which is a direct consequence of our Theorem 3 and Theorem 4 of Oury and Tercieux (2012).

Proposition 2 *If an SCF satisfies Maskin monotonicity*, it is continuously implementable by a finite mechanism.*

To the best of our knowledge, Proposition 2 is the first to delineate the class of SCFs which are continuously implementable. The condition identified is Maskin monotonicity*, which is stronger than Maskin monotonicity itself. Recall the example in Appendix A.2 which shows that strict Maskin monotonicity* is strictly stronger than strict Maskin monotonicity. There are two caveats in relating Proposition 2 to Theorem 4 of Oury and Tercieux (2012). The first caveat is that we focus on complete-information environments, whereas Oury and Tercieux (2012) deal with incomplete-information environments in which the baseline model can be an arbitrary finite type space. The second caveat is that we specialize in environments with lottery and transfer, whereas Oury and Tercieux (2012) impose no restriction on the environment. Oury and Tercieux (2012) also allow for any degree of interdependence of preferences.

6.3 Responsive SCFs

Bergemann et al. (2011) introduce a condition on SCFs.

[18] In fact, assuming that sending messages is slightly costly, Oury and Tercieux (2012) also prove the converse: an SCF is continuously implementable by a finite mechanism only if it is rationalizably implementable by a finite mechanism.

[19] In particular, in their study of exact rationalizable implementation, Bergemann et al. (2011) invoke an infinite mechanism with integer games to implement strict Maskin-monotonic* SCFs. Hence, it is not possible to combine Theorem 4 of Oury and Tercieux (2012) with the result of Bergemann et al. (2011) to get a possibility result for continuous implementation.

Definition 8 *An SCF f is **responsive** if, for any pair of states $\theta, \theta' \in \Theta$, $f(\theta) = f(\theta') \Rightarrow \theta = \theta'$.*

Responsiveness requires that the SCF "responds" to a change in the state with a change in the social choice outcome. Observe that a responsive SCF that satisfies Maskin monotonicity must satisfy Maskin monotonicity*. Indeed, since $\mathcal{P}_f$ is the finest partition on Θ, $\theta' \notin \mathcal{P}(\theta)$ if and only if $\theta' \neq \theta$. We thus obtain the following corollary for the case of responsive SCFs.

Corollary 1 *Let f be a responsive SCF. Then, the SCF f is implementable in rationalizable strategies by a finite mechanism if and only if it satisfies Maskin monotonicity.*

Remark: Bergemann et al. (2011) prove that under the no-worst alternative condition (see Definition 4 of Bergemann et al. (2011)), if there are at least three agents, and f is responsive and satisfies strict Maskin monotonicity, then f is implementable in rationalizable strategies by an infinite mechanism. In contrast, with use of lotteries and transfer, we achieve rationalizable implementation by a finite mechanism and we can handle the case of two agents.

In what follows, we argue that the responsiveness of SCFs is tightly connected to the permissive result of virtual implementation by Abreu and Matsushima (1992), who show that when there are at least three agents, any SCF is *virtually* implementable in rationalizable strategies by a finite mechanism. An SCF f is said to be virtually implementable if, for any $\varepsilon \in (0, 1)$, the SCF f is exactly implementable with probability $1 - \varepsilon$. Fix an SCF f. For each $\varepsilon \in (0, 1)$, define $f^\varepsilon : \Theta \to \Delta(A)$ as follows: for any $\theta \in \Theta$,

$$f^\varepsilon(\theta) = \frac{\varepsilon}{I} \sum_{i \in \mathcal{I}} y_i(\theta_i) + (1 - \varepsilon) f(\theta),$$

where $y_i(\theta_i)$ is the dictator lottery for type θ_i, as constructed in Lemma 1. Moreover, by adding small transfers to the dictator lotteries, we can make $\sum_{i \in \mathcal{I}} y_i(\theta_i) \neq \sum_{i \in \mathcal{I}} y_i(\theta_i')$ whenever $\theta \neq \theta'$, without affecting the conclusion of Lemma 1 (i.e., (17) below). Therefore, $f^\varepsilon(\theta) \neq f^\varepsilon(\theta')$ whenever $\theta \neq \theta'$. In other words, we can make f^ε responsive. We now argue that f^ε is also Maskin-monotonic.[20] Fix two states θ and θ' with $\theta \neq \theta'$ (and hence

[20] One additional property Abreu and Matsushima (1992) obtain in their mechanism is that they can make the size of transfers arbitrarily small. We discuss this below.

$f^{\varepsilon}(\theta) \neq f^{\varepsilon}(\theta')$). Since $\theta \neq \theta'$ and due to the construction of dictator lotteries, there must exist agent i for whom

$$u_i(y_i(\theta_i), \theta_i) > u_i(y_i(\theta_i'), \theta_i) \text{ and } u_i(y_i(\theta_i'), \theta_i') > u_i(y_i(\theta_i), \theta_i'). \quad (17)$$

We construct the following lottery $x(\theta, \theta_i') \in X$:

$$x(\theta, \theta_i') \equiv \frac{\varepsilon}{I}\left(y_i(\theta_i') + \sum_{j \neq i} y_j(\theta_j)\right) + (1-\varepsilon) f(\theta).$$

That is, $x(\theta, \theta_i')$ is constructed by replacing $y_i(\theta_i)$ in $f(\theta)$ with $y_i(\theta_i')$. By (17), we have

$$x(\theta, \theta_i') \in \mathcal{SL}_i(f(\theta), \theta_i) \cap \mathcal{SU}_i(f(\theta), \theta_i').$$

This shows that f^{ε} satisfies strict Maskin monotonicity. By Theorem 3, we provide the following result without proof.

Corollary 2 *Any SCF f is virtually implementable in rationalizable strategies by a finite mechanism.*

Recall that our mechanism is different from that of Abreu and Matsushima (1992), who do not use transfers but rather introduce a domain restriction in the lottery space. AM's (1992) domain restriction requires that for every player i and state θ, there exist a pair of lotteries which are strictly ranked for player i and for which other players have the (weakly) opposite ranking. Since we can choose the size of transfers to be as small as possible using the technique developed by AM (see also Section 7.2), we obtain the following result:

Corollary 3 *Any SCF f is virtually implementable in rationalizable strategies with arbitrarily small transfer by a finite mechanism.*

7 Extensions

We now establish several extensions of our Nash implementation results. In Section 7.1, we extend our result to the case of social choice correspondences. Section 7.2 shows that one can make the size of transfers arbitrarily small in our implementation results. In Section 7.3, we extend our results to an infinite state space model. Finally, by making use of the infinite state space extension in Section 7.3, we handle the ordinal approach to Nash implementation in Section 7.4. The proofs of Sections 7.1, 7.2, 7.3, and 7.4 are relegated to the Appendix.

7.1 Social Choice Correspondences

Many papers in the literature on Nash implementation deal with social choice *correspondences*, i.e., multivalued social choice rules. In this section, we extend our Nash implementation result to the case of social choice correspondences (henceforth, SCCs). The designer's objective is now specified by an SCC $F : \Theta \rightrightarrows \Delta(A)$. We first extend Maskin monotonicity to the case of SCCs.

Definition 9 *Say an SCC F satisfies **Maskin monotonicity** if, for any pair of states θ and θ' and $l \in F(\theta) \setminus F(\theta')$, there is some agent $i \in \mathcal{I}$ such that*

$$\mathcal{L}_i(l, \theta_i) \cap \mathcal{SU}_i(l, \theta'_i) \neq \varnothing.$$

Similarly, we extend *strict* Maskin monotonicity to the case of SCCs. We say that an SCC F satisfies strict Maskin monotonicity, if for any pair of states θ and θ' and $l \in F(\theta) \setminus F(\theta')$, there is some agent $i \in \mathcal{I}$ such that $\mathcal{SL}_i(l, \theta_i) \cap \mathcal{SU}_i(l, \theta'_i) \neq \varnothing$. As in the case of SCF, in our transferable utility setup, strict Maskin monotonicity is equivalent to Maskin monotonicity for the case of SCCs.

We extend *the best challenge scheme* to the case of SCCs. Fix agent i of type θ_i. For each state $\tilde{\theta} \in \Theta$, and $x \in F(\tilde{\theta})$, if $\mathcal{SL}_i(l, \tilde{\theta}_i) \cap \mathcal{SU}_i(l, \theta_i) \neq \varnothing$, we select some $x(l, \tilde{\theta}, \theta_i) \in \mathcal{SL}_i(l, \tilde{\theta}_i) \cap \mathcal{SU}_i(l, \theta_i)$. Then, the best challenge scheme for agent i of type θ_i is defined as a function B_{θ_i} such that for any $\tilde{\theta} \in \Theta$ and $l \in F(\tilde{\theta})$,

$$B_{\theta_i}(\tilde{\theta}, l) = \begin{cases} l, & \text{if } \mathcal{SL}_i(l, \tilde{\theta}_i) \cap \mathcal{SU}_i(l, \theta_i) = \varnothing; \\ x(l, \tilde{\theta}, \theta_i), & \text{if } \mathcal{SL}_i(l, \tilde{\theta}_i) \cap \mathcal{SU}_i(l, \theta_i) \neq \varnothing. \end{cases}$$

We propose the concept of Nash implementation for the case of SCCs.

Definition 10 *An SCC F is **implementable in Nash equilibria** if there exists a mechanism $\mathcal{M} = ((M_i), g, (\tau_i))_{i \in \mathcal{I}}$ such that for any state $\theta \in \Theta$, the following two conditions are satisfied: (i) for any $l \in F(\theta)$, there exists a pure-strategy Nash equilibrium $m \in \Gamma(\mathcal{M}, \theta)$ such that $g(m) = l$ and $\tau_i(m) = 0$ for every $i \in \mathcal{I}$; and (ii) for every $m \in supp\ (NE(\Gamma(\mathcal{M}, \theta)))$, we have $supp(g(m)) \subset F(\theta)$ and $\tau_i(m) = 0$ for every $i \in \mathcal{I}$.*

Remark: This definition is the same as that of Maskin (1999). The definition of Nash implementation proposed by Mezzetti and Renou (2012) keeps requirement (ii) but weakens requirement (i) so that any outcome in the range of the SCC is possibly supported by a mixed-strategy Nash equilibrium.[21] We discuss their notion in Section 7.4.

We are now ready to state our Nash implementation result for the case of SCCs.

Theorem 4 *Assume that there are at least three agents. An SCC F is implementable in Nash equilibria if and only if it satisfies Maskin monotonicity.*

Remark: The implementing mechanism that will be constructed for this result may be infinite. However, if $F(\Theta)$ is a finite set, Theorem 4 establishes Nash implementation by a finite mechanism even for the case of SCCs.

Remark: When there are only two agents, we can still show that every Maskin-monotonic SCC F is **weakly** implementable in Nash equilibrium. That is, there exists a mechanism $\mathcal{M} = ((M_i), g, (\tau_i))_{i \in \mathcal{I}}$ such that for any state $\theta \in \Theta$ and $m \in \text{supp}\,(NE(\Gamma(\mathcal{M}, \theta)))$, it follows that $\text{supp}(g(m)) \in F(\theta)$ and $\tau_i\,(m) = 0$ for every $i \in \mathcal{I}$. The difficulty of strengthening weak implementation so that it becomes the standard full implementation lies in how to specify an outcome when the two agents disagree on which outcome to be chosen under the true state.

7.2 Small Transfer

One potential deficiency of the mechanisms we propose for Theorems 2 is that the size of transfers may be large. In this section, we use the technique introduced by Abreu and Matsushima (1994) to show that if the SCF satisfies Maskin monotonicity in the restricted domain without transfer, then it is Nash-implementable with arbitrarily small transfers.

We first propose a notion of Nash implementation with zero transfer on the equilibrium and bounded transfer off the equilibrium.

Definition 11 *An SCF f is implementable in Nash equilibria* ***with transfer bounded by*** $\bar{\tau}$ *if there exists a mechanism $\mathcal{M} = ((M_i), g, (\tau_i))_{i \in \mathcal{I}}$ such that for any state $\theta \in \Theta$*

[21] We have another way of accommodating SCCs. Let $F : \Theta \rightrightarrows A$ be a deterministic SCC and for simplicity suppose that A is a fintie set. We construct a stochastic SCF f such that $\text{supp}(f(\theta)) = F(\theta)$ for each $\theta \in \Theta$. Due to the linearity of expected utility, it is easy to see that if F is a Maskin-monotonic SCC or set-monotonic SCC (a weaker monotonicity condition than Maskin-monotonicity defined in Mezzetti and Renou (2012)), then the translated SCF f is a Maskin-monotonic. Hence, f is Nash implementable in our sense.

and $m \in M$, *(i) there exists a pure-strategy Nash equilibrium in the game* $\Gamma(\mathcal{M}, \theta)$*; (ii)* $m \in$ *supp* $(NE(\Gamma(\mathcal{M}, \theta))) \Rightarrow g(m) = f(\theta)$ *and* $\tau_i(m) = 0$ *for every* $i \in \mathcal{I}$*; and (iii)* $|\tau_i(m)| \leq \bar{\tau}$ *for any* $m \in M$.

We propose a notion of Nash implementation in which there are no transfers on the equilibrium and only arbitrarily small transfers off the equilibrium:

Definition 12 *An SCF* f *is implementable in Nash equilibria* ***with arbitrarily small transfer*** *if for any* $\bar{\tau} > 0$, *the SCF* f *is implementable in Nash equilibria with transfer bounded by* $\bar{\tau}$.

We say that an SCF f satisfies Maskin monotonicity in the restricted domain $\Delta(A)$ if $f(\theta) \neq f(\theta')$ implies that there is an agent i and some $l(\theta, \theta_i') \in \Delta(A)$ such that $l(\theta, \theta_i') \in \mathcal{SL}_i(f(\theta), \theta_i) \cap \mathcal{SU}_i(f(\theta), \theta_i')$.

Remark: Maskin monotonicity in the restricted domain $\Delta(A)$ indeed a strictly stronger condition than Maskin monotonicity in the domain X as shown in Section 2.1, King Solomon's dilemma. Here, by imposing a stronger monotonicity condition, we obtain a stronger result, Theorem 5.

Finally, we strengthen Assumption 1 throughout the paper:

Assumption 2 $\theta_i \neq \theta_i' \Rightarrow u_i(\cdot, \theta_i)$ *and* $u_i(\cdot, \theta_i')$ *induce different preference orders on* $\Delta(A)$.

Again, given the assumption, we have the following result:

Lemma 2 *Suppose that Assumption 2 holds. Then, for each* $i \in \mathcal{I}$, *there exists a function* $l_i : \Theta_i \to \Delta(A)$ *such that for any* $\theta_i, \theta_i' \in \Theta_i$ *with* $\theta_i \neq \theta_i'$,

$$v_i(l_i(\theta_i), \theta_i) > v_i(l_i(\theta_i'), \theta_i).$$

Theorem 5 *Under Assumption 2, an SCF* $f_A : \Theta \to \Delta(A)$ *is* ***implementable in Nash equilibria*** *with arbitrarily small transfer if* f_A *satisfies Maskin monotonicity in the restricted domain.*

7.3 Infinite State Space

One significant assumption we have made in this paper is that the state space is finite. In Appendix A.6, we extend Theorem 2 to a compact state space in which the agents' utility functions are continuous. A similar extension was raised as a question left open in Abreu and Matsushima (1992) (see their Section 5), and it remains unknown to the best of our knowledge. In appendix A.6, we construct an infinite extension of the implementing mechanism for mixed-strategy Nash implementation. A notable feature of this extension is that as long as the setting is compact and continuous, the resulting implementing mechanism will also be compact and continuous. Keeping this feature is important in differentiating our construction from the traditional way of using integer games. We state the result as follows:

Theorem 6 *Suppose that A is a finite set of pure alternatives and Θ is a Polish space. Then, an SCF f satisfies Maskin monotonicity if and only if there exists a mechanism which implements f in mixed-strategy Nash equilibrium. Moreover, if Θ is compact and both the cardinalization $v_i(a, \cdot)$ and the SCF are continuous functions on Θ, then the mechanism has a compact set of messages, a continuous outcome function, and a continuous transfer rule.*

This extension overcomes two main difficulties. First, in a finite state space, the transfer rules $\tau_{i,j}^1$ and $\tau_{i,j}^2$ which we define in (8) and (9) impose either a large penalty or a large reward as long as the designer sees a discrepancy in the agents' announcements. With a continuum of states/types, however, such a drastic change in transfer will necessarily result in discontinuity. Hence, our first challenge is to suitably define $\tau_{i,j}^1$ and $\tau_{i,j}^2$ so that they vary continuously in incentivizing truth-telling. Second, in a finite setting, we can choose just the contingent weight (i.e., the ε in the function $e_{i,j}(\cdot)$) so that a test agent challenges only when he is supposed to (in the sense that conditions (6) and (7) are satisfied). This is because in a finite world, there is a uniform lower bound for the loss from a false challenge and for the gain from making a correct challenge. Without a uniform lower bound, the weighting function $e_{i,j}(\cdot)$ can no longer take a binary value and needs to vary continuously, depending on the gain or loss from the challenge associated with a message profile. In particular, we will establish a counterpart of the conditions (6) and (7) as Lemmas 4 and 5 in Appendix A.6.

7.4 The Ordinal Approach

We have assumed that the agents are expected utility maximizers and have used lotteries to elicit their cardinal preferences. Therefore, it is natural to ask whether our implementation results critically depend on the cardinalization of the preferences over lotteries. To answer this question, we now introduce the notion of *ordinal* Nash implementation, which requires that the mixed-strategy Nash implementation holds for *any* cardinal representation of the ordinal preferences over a finite set of pure alternatives A. This is the approach proposed by Mezzetti and Renou (2012).

Suppose that at state $\theta \in \Theta$, agents only have common knowledge about their ordinal rankings over the set of pure alternatives A. We write the induced ordinal preference profile at state θ by $(\succeq_i^{\theta_i})_{i\in\mathcal{I}}$. We denote $(v_i)_{i\in\mathcal{I}}$ as a *cardinal representation* of $(\succeq_i^{\theta_i})_{i\in\mathcal{I},\theta_i\in\Theta_i}$, i.e., for each $a, a^{'} \in A,\ i \in \mathcal{I}$, and $\theta \in \Theta$, we have $v_i(a;\theta_i) \geq v_i(a^{'};\theta_i) \Leftrightarrow a \succeq_i^{\theta_i} a^{'}$. We assume that all cardinal representations are bounded and normalize have the range $[0, 1]$. Again, each cardinal representation v_i induces an expected utility function on $\Delta(A)$ which we abuse the notation to also denote as v_i. We denote V_i^{θ} the set of all cardinal representations $v_i(\cdot, \theta_i)$ of $\succeq_i^{\theta_i}$. Following Mezzetti and Renou (2012), we focus our discussion on the case of an SCF $f : \Theta \to A$.

An SCF f is said to be *ordinally* Nash implementable if it is implementable in Nash equilibria "independently of the cardinal representation," i.e., there exists a mechanism $\mathcal{M}$ such that, for any profile of cardinal representations $v = (v_i)_{i\in\mathcal{I}}$ of $(\succeq_i^{\theta_i})_{i\in\mathcal{I},\theta_i\in\Theta_i}$ and $\theta \in \Theta$, the following two conditions are satisfied: (i) there exists a pure-strategy Nash equilibrium $m \in \Gamma(\mathcal{M}, \theta, v)$ such that $g(m) = f(\theta)$ and $\tau_i(m) = 0$ for every $i \in \mathcal{I}$; and (ii) for every $m \in \text{supp}(NE(\Gamma(\mathcal{M}, \theta, v)))$, we have $\text{supp}(g(m)) = f(\theta)$ and $\tau_i(m) = 0$ for every $i \in \mathcal{I}$. To prove our main theorem in this section, we strengthen Assumption 1 into the following:

Assumption 3 *$\theta_i \neq \theta_i^{'} \Rightarrow \succeq_i^{\theta_i}$ and $\succeq_i^{\theta_i'}$ induce different preference orders on A.*

With this assumption, we obtain a stronger version of Lemma 1, namely, there is a set of dictator lotteries that work regardless of the cardinal representation. By Assumption 3, the dictator lottery constructed remains valid as long as the preferences exhibit monotonicity with respect to first-order stochastic dominance. (See the proof of Lemma in Abreu and Matsushima (1992).)

Lemma 3 *Suppose that Assumption 3 holds. For each $i \in \mathcal{I}$, there exists a function $y_i : \Theta_i \to \Delta(A)$ such that for any $\theta_i, \theta_i^{'} \in \Theta_i$ with $\theta_i \neq \theta_i^{'}$ and any cardinal representation $v_i(\cdot)$ of $(\succeq_i^{\theta_i})_{\theta_i \in \Theta_i}$,*

$$u_i(y_i(\theta_i), \theta_i) > u_i(y_i(\theta_i^{'}), \theta_i).$$

We first introduce the following definitions of contour set under ordinal preferences.

For $(a, \theta_i) \in A \times \Theta_i$, under ordinal preference $\succeq_i^{\theta_i}$, we denote the upper-contour set, the lower-contour set, the strict upper-contour set, and the strict lower-contour set as follows:

$$\begin{aligned} U_i(a, \theta_i) &= \left\{a' \in A : a' \succeq_i^{\theta_i} a\right\}; \\ L_i(a, \theta_i) &= \left\{a' \in A : a \succeq_i^{\theta_i} a'\right\}; \\ SU_i(a, \theta_i) &= \left\{a' \in A : a' \succ_i^{\theta_i} a\right\}; \\ SL_i(a, \theta_i) &= \left\{a' \in A : a \succ_i^{\theta_i} a'\right\}; \end{aligned}$$

where $\succ_i^{\theta_i}$ denotes the strict preference induced by $\succeq_i^{\theta_i}$. We now introduce the notion of ordinal almost monotonicity proposed by Sanver (2006) as the key condition in this section.

Definition 13 *An SCF f satisfies **ordinal almost monotonicity** if for any pair of states θ and θ', with $f(\theta) \neq f(\theta')$, there is some agent $i \in \mathcal{I}$ such that*

$$L_i(f(\theta), \theta_i) \cap SU_i(f(\theta), \theta_i') \neq \varnothing,$$

or

$$SL_i(f(\theta), \theta_i) \cap U_i(f(\theta), \theta_i') \neq \varnothing.$$

Thus, we obtain the following result:

Theorem 7 *Suppose that Assumption 3 holds. Then, an SCF f is ordinally Nash implementable if and only if it satisfies ordinal almost monotonicity.*

We prove Theorem 7 as a straightforward application of Theorem 6. While Mezzetti and Renou (2012) and Theorem 7 both study ordinal implementation in mixed-strategy Nash equilibrium, there are three essential differences. First, Mezzetti and Renou (2012) requires only the existence of mixed-strategy equilibria but we require the existence of pure-strategy equilibria following Maskin (1999). Second, we use monetary transfers, while Mezzetti and Renou (2012) do not. The first difference makes our ordinal approach more demanding than that of Mezzetti and Renou (2012). Third, Mezzetti and Renou (2012) studies the case of SCC which we omit here. Specifically, Mezzetti and Renou (2012) propose a notion called ordinal set-monotonicity for SCC. They show that the notion of ordinal set-monotonicity is weaker than Maskin monotonicity and is necessary and almost sufficient in their notion of implementation in mixed-strategy Nash equilibrium. Ordinal almost monotonicity is slightly weaker than set-monotonicity and yet characterizes the stronger notion of ordinal mixed-strategy Nash implementation *à la* Maskin (1999) for the case of SCF in the environment with transfer.

8 Concluding Remarks

Despite its tremendous success, implementation theory has also been criticized on various fronts. In particular, the major criticism is that the mechanisms used to achieve full implementation are not "natural," as reflected in the quote from Jackson (1992) at the beginning of the paper. To address such criticism, Jackson (1992) proposes that we may restrict our attention to "natural mechanisms" and study which SCFs can be fully implemented, even at the cost of restricting attention to more specific environments.

We consider the results in this paper as a first yet important step in advancing the Jackson program. Specifically, we focus on environments with lotteries and transfer and provide well-behaved implementing mechanisms for pure-strategy Nash implementation, mixed-strategy Nash implementation, and rationalizable implementation. We also show that our result and the idea of our implementing mechanism are amenable to prominent extensions to the case of SCC, infinite settings, and ordinal settings.

As a first benchmark, we follow Maskin and AM in focusing our study of full Nash implementation on the complete-information setup. In addition, our results also invite possible extensions to a Bayesian setup (Jackson (1991)) and a robust setup (Bergemann and Morris (2009)), which we would like to explore in future research. Indeed, our approach is intimately related to the burgeoning literature on repeated implementation in particular, such as Lee and Sabourian (2011) and Mezzetti and Renou (2017), where continuation values can serve as transfers in our construction.[22] We leave the study on repreated implementation for future research.

[22]We thank Hamid Sabourian for drawing our attention to this point.

A Appendix

In this Appendix, we provide the proofs omitted from the main body of the paper.

A.1 Theorem 1 with Two Agents

We now provide an example to show that we cannot prove Theorem 1 with two agents. The example demonstrates that when there are only two agents, a direct mechanism cannot implement some Maskin-monotonic SCF in pure-strategy Nash equilibrium. This suggests that, even with the help of lottery and transfer, Nash implementation with two agents generally requires indirect mechanisms as in the proof of Theorem 2.

Suppose that there are two agents: A and B; two states: α and β; and four pure alternatives: a, b, c, and d. Define an SCF such that $f(\alpha) = a$ and $f(\beta) = b$. Agents' utilities across different states are described in the following table where $\overline{v} > \underline{v} > 0$ and $0 < \varepsilon < (\overline{v} - \underline{v})/2$:

v_A	α	β
a	$\underline{v}$	$\overline{v}$
b	0	0
c	$-\varepsilon$	ε
d	ε	$-\varepsilon$

v_B	α	β
a	0	0
b	$\overline{v}$	$\underline{v}$
c	$-\varepsilon$	ε
d	ε	$-\varepsilon$

.

Specifically, since $d \in \mathcal{SL}_A(f(\beta), \beta) \cap \mathcal{SU}_A(f(\beta), \alpha)$ and $c \in \mathcal{SL}_B(f(\alpha), \alpha) \cap \mathcal{SU}_B(f(\alpha), \beta)$, it follows that f is Maskin-monotonic. Hence, by Theorem 2, f is implementable in mixed-strategy Nash equilibrium by a finite indirect mechanism. We establish the following claim in Appendix A.1.

Claim 5 *No direct mechanism implements f in pure-strategy Nash equilibrium.*

Suppose to the contrary that there exists such a direct mechanism $(g, (\tau_i))_{i \in \mathcal{I}}$ (recall that $M_i = \Theta$) which implements f. Hence, we set $g(\alpha, \alpha) = a$, $g(\beta, \beta) = b$. We write $m \succeq_i^\theta m'$ if and only if $v_i(g(m), \theta) + \tau_i(m) \geq v_i(g(m'), \theta) + \tau_i(m')$ for any m and m' in Θ^2. We have self-selection conditions:

$$(\theta, \theta) \succeq_i^\theta (\theta', \theta) \text{ for } \theta, \theta' \in \{\alpha, \beta\} \text{ and } i \in \{A, B\}. \tag{18}$$

In addition, to guarantee that a false consensus is not an equilibrium, we need a whistle blower in each state. The following claim shows that agent A cannot be whistle blower when the true state is β and the announced false state is α, and agent B cannot be a whistle blower when the true state is α and the announced false state is β.

By Claim 5.1, for Nash implementation to be achievable by the direct mechanism, agent B must be the one who deviates to knock out (α, α) as an equilibrium in state β and agent A must be the one who deviates to knock out (β, β) in state α. However, Claim 5.2 shows that we cannot satisfy both simultaneously. By two claims, we therefore conclude that there exists some state θ and $\theta' \neq \theta$ such that (θ', θ') is an equilibrium in θ, i.e., it is impossible to implement f in pure-strategy Nash equilibrium by a direct mechanism.

Claim 5.1. We must have $(\alpha, \alpha) \succeq_A^\beta (\beta, \alpha)$ and $(\beta, \beta) \succeq_B^\alpha (\beta, \alpha)$.

Proof. We prove that $(\alpha, \alpha) \succeq_A^\beta (\beta, \alpha)$. The proof of $(\beta, \beta) \succeq_B^\alpha (\beta, \alpha)$ is similar and so omitted. Suppose to the contrary that $(\beta, \alpha) \succ_A^\beta (\alpha, \alpha)$. Without loss of generality, we write the allocation at (β, α) as a lottery $(p_a, p_b, p_c, p_d) \in \Delta(A)$ (where $p_{\tilde{a}}$ denotes the probability assigned to $\tilde{a}$) and transfer pair (t_A, t_B). Then, we have

$$p_a \bar{v} + p_b \times 0 + p_c \times \varepsilon + p_d \times (-\varepsilon) + t_A > \bar{v}. \tag{19}$$

The self-selection condition (18) for $i = A$ and $\theta = \alpha$ implies that

$$\underline{v} \geq p_a \bar{v} + p_b \times 0 + p_c \times (-\varepsilon) + p_d \times \varepsilon + t_A. \tag{20}$$

Summing up (19) and (20), we get

$$2(p_c - p_d)\varepsilon > (1 - p_a)(\bar{v} - \underline{v}). \tag{21}$$

Since $\varepsilon < (\bar{v} - \underline{v})/2$, it follows from (21) that $p_c - p_d > 1 - p_a$ which is a contradiction. ■

Claim 5.2. It is impossible to have $(\alpha, \beta) \succ_B^\beta (\alpha, \alpha)$ and $(\alpha, \beta) \succ_A^\alpha (\beta, \beta)$.

Proof. Suppose to the contrary that we have $(\alpha, \beta) \succ_B^\beta (\alpha, \alpha)$ and $(\alpha, \beta) \succ_A^\alpha (\beta, \beta)$. Again, we write the allocation at (α, β) as a lottery $(p_a, p_b, p_c, p_d) \in \Delta(A)$ and transfer pair (t_A, t_B). Since $(\alpha, \beta) \succ_B^\beta (\alpha, \alpha)$, we have

$$p_a \times 0 + p_b \times \underline{v} + p_c \times \varepsilon + p_d \times (-\varepsilon) + t_B > 0. \tag{22}$$

The self-selection condition (18) for $i = B$ and $\theta = \alpha$ implies that

$$p_a \times 0 + p_b \times \bar{v} + p_c \times (-\varepsilon) + p_d \times \varepsilon + t_B \leq 0. \tag{23}$$

Subtracting (23) from (22), we have

$$p_b(\underline{v} - \bar{v}) + 2(p_c - p_d)\varepsilon > 0. \tag{24}$$

Hence, $p_c - p_d > 0$. Similarly, $(\alpha, \beta) \succ_A^\alpha (\beta, \beta)$ and the self-selection condition (18) for $i = A$ and $\theta = \beta$ imply that $p_d - p_c > 0$. This is a contradiction. ■

A.2 Maskin Monotonicity and Maskin Monotonicity*

We provide an SCF which satisfies strict Maskin monotonicity but not strict Maskin monotonicity*.[23] This implies that rationalizable implementation is more restrictive than Nash implementation. Recall that in environments with transfers, strict Maskin monotonicity is equivalent to Maskin monotonicity and strict Maskin monotonicity* is also equivalent to Maskin monotonicity*. Let $A = \{a, b, c, d\}$, $\mathcal{I} = \{1, 2\}$, $X = \Delta(A) \times \mathbb{R}^2$, and $\Theta = \{\alpha, \beta, \gamma, \delta\}$. The agents' utility functions are given in the two tables below. Consider the following SCF $f(\alpha) = f(\beta) = f(\gamma) = a$ and $f(\delta) = b$. For simplicity of notation, we write $\tilde{a} \in A$ for $(a, 0, 0) \in X$ which is a degenerate allocation with no transfer to any agent.

v_A	α	β	γ	δ
a	3	2	2	2
b	2	3	1	3
c	1	1	3	1
d	0	0	0	0

v_B	α	β	γ	δ
a	3	2	2	2
b	1	0	1	1
c	2	1	3	3
d	0	3	0	0

Claim 6 *For every agent i and $\theta \in \Theta$, $\mathcal{SL}_i(a, \theta) \subset \mathcal{L}_i(a, \alpha)$.*

Proof. Observe that for any agent, any $\tilde{a} \in A \setminus \{a\}$, and any $\theta \in \Theta$, the utility difference between a and $\tilde{a}$ is larger at α than at θ, that is,

$$v_i(a, \alpha) - v_i(\tilde{a}, \alpha) \geq v_i(a, \theta) - v_i(\tilde{a}, \theta).$$

Hence, for any $x \in X$, we have $u_i(a, \theta) - u_i(x, \theta) \geq 0$ whenever $u_i(a, \tilde{\theta}) - u_i(x, \tilde{\theta}) \geq 0$. ■

Claim 7 *The SCF f violates strict Maskin monotonicity*.*

[23] This example is considered a two-agent version of the example in Appendix A of Jain (2017) which also accommodate the environments with lottery and transfers.

Proof. Consider an arbitrary partition finer than $\mathcal{P}_f = \{\{\alpha, \beta, \gamma\}, \{\delta\}\}$. Note that $\mathcal{P}(\delta) = \{\delta\}$ for any partition $\mathcal{P}$ finer than $\mathcal{P}_f$.

Case 1. $\alpha \in \mathcal{P}(\beta)$ and $\alpha \in \mathcal{P}(\gamma)$. In this case, $\mathcal{P} = \mathcal{P}_f$ and hence $\mathcal{P}(\alpha) = \{\alpha, \beta, \gamma\}$. Since $\mathcal{SL}_A(a, \beta) = \mathcal{SL}_A(a, \delta)$ and $\mathcal{SL}_B(a, \gamma) = \mathcal{SL}_B(a, \delta)$. Thus, $\mathcal{SL}_i(a, \mathcal{P}(\alpha)) \subset \mathcal{L}_i(a, \delta)$ but $f(\alpha) \neq f(\delta)$. Hence, f violates strict Maskin monotonicity* for such $\mathcal{P}$.

Case 2. $\alpha \notin \mathcal{P}(\beta)$ or $\alpha \notin \mathcal{P}(\gamma)$. We derive a contradiction for $\alpha \notin \mathcal{P}(\beta)$ and the argument for the case with $\alpha \notin \mathcal{P}(\gamma)$ is similar and so omitted. If $\alpha \notin \mathcal{P}(\beta)$, then by Claim 6, we have $\mathcal{SL}_i(a, \mathcal{P}(\beta)) \subset \mathcal{L}_i(a, \alpha)$. Then, f violates strict Maskin monotonicity* for $\mathcal{P}$ since $\mathcal{SL}_i(a, \mathcal{P}(\beta)) \subset \mathcal{L}_i(a, \alpha)$ and $\alpha \notin \mathcal{P}(\beta)$. ■

Claim 8 *The SCF f satisfies strict Maskin monotonicity.*

Proof. Indeed, observe that $b \in \mathcal{SL}_A(a, \alpha) \cap \mathcal{SU}_A(a, \delta)$, $c \in \mathcal{SL}_B(a, \beta) \cap \mathcal{SU}_B(a, \delta)$, $b \in \mathcal{SL}_A(a, \gamma) \cap \mathcal{SU}_A(a, \delta)$, $a \in \mathcal{SL}_A(b, \delta) \cap \mathcal{SU}_A(b, \alpha)$, $d \in \mathcal{SL}_B(b, \delta) \cap \mathcal{SU}_B(b, \beta)$, and $a \in \mathcal{SL}_A(b, \delta) \cap \mathcal{SL}_A(b, \gamma)$. ■

A.3 Proof of Theorem 3

Let $\theta \in \Theta$ be a true state. We prove the if-part of Theorem 3 in five steps.

Step 0: *Suppose f satisfies Maskin monotonicity* and let $\mathcal{P}$ denote the partition such that (i) $\mathcal{P}$ is weakly finer than $\mathcal{P}_f$; (ii) for any $\theta, \theta' \in \Theta$, whenever $\theta' \notin \mathcal{P}(\theta)$, there exists $i \in \mathcal{I}$ for whom*

$$\mathcal{L}_i(f(\theta), \mathcal{P}(\theta)) \cap \mathcal{SU}_i(f(\theta), \theta'_i) \neq \varnothing.$$

Then, for each $\theta \in \Theta$, we have $\mathcal{P}(\theta) = \times_{i\in\mathcal{I}} \mathcal{P}_i(\theta)$ where $\mathcal{P}_i(\theta) \subset \Theta_i$.

Proof. We prove it by contradiction. Suppose that there exist $\theta^i \subset \mathcal{P}(\theta)$ for every i and $(\theta^i_i)_{i\in\mathcal{I}} \notin \mathcal{P}(\theta)$, where $\theta^i_i \in \Theta_i$ is the projection of $\theta^i \in \Theta$ into Θ_i. Consider $\theta' \equiv (\theta^i_i)_{i\in\mathcal{I}}$. Since $\theta' \in \Theta$ and $\theta' \notin \mathcal{P}(\theta)$, it follows from Maskin monotonicity* that there exists an agent i^* such that

$$\mathcal{L}_{i^*}(f(\theta), \mathcal{P}(\theta)) \cap \mathcal{SU}_{i^*}(f(\theta), \theta'_{i^*}) \neq \varnothing.$$

Since $\theta^{i^*} \in \mathcal{P}(\theta)$, we have

$$\mathcal{L}_i\left(f(\theta), \theta^{i^*}_{i^*}\right) \cap \mathcal{SU}_i(f(\theta), \theta'_{i^*}) \neq \varnothing.$$

Finally, recall that $\theta' = (\theta_i^i)_{i\in\mathcal{I}}$. Hence, $\theta'_{i^*} = \theta_{i^*}^{i^*}$ which implies that

$$\mathcal{L}_i\left(f(\theta), \theta_{i^*}^{i^*}\right) \cap \mathcal{SU}_i(f(\theta), \theta_{i^*}^{i^*}) \neq \varnothing,$$

which is a contradiction. ■

Step 1: *For any agent i and any $m_i \in S_i^\infty(\mathcal{M}, \theta)$, there is some type profile θ'_{-i} such that $\left(m_i^1, \theta'_{-i}\right)$ identifies a state $\tilde{\theta} \in \mathcal{P}(\theta)$.*

Proof. Fix $i \in \mathcal{I}$ and $m_i \in S_i^\infty(\mathcal{M}, \theta)$. Then, there is a conjecture $\lambda_i \in \Delta\left(S_{-i}^\infty(\mathcal{M}, \theta)\right)$ against which m_i is a best reply. Assume that for any $m_{-i} \in$supp(λ_i), $e_{i,j}(m_i, m_j) = \varepsilon$. By (14) and (15), we have $m_i^1 = \theta_i$. Then, Step 1 follows by setting $\theta'_{-i} = \theta_{-i}$. Now suppose that there exists $m_{-i} \in S_{-i}^\infty(\mathcal{M}, \theta)$ with $\lambda_i(m_{-i}) > 0$ such that $e_{i,j}(m_i, m_j) = 0$, i.e., there exists $\tilde{\theta}$ such that

$$m^1 \text{ identifies some } \tilde{\theta} \in \Theta;$$
$$m_j^2 \sim m_j^3 \sim \tilde{\theta} \text{ and } B_{m_j^4(\tilde{\theta})}(\tilde{\theta}) = f(\tilde{\theta}), \forall j \in \mathcal{I}.$$

We claim that $\tilde{\theta} \in \mathcal{P}(\theta)$. Suppose on the contrary that $\tilde{\theta} \notin \mathcal{P}(\theta)$. Then, since f satisfies strict Maskin monotonicity*, there exists some agent $j \in \mathcal{I}$ for whom $B_{\theta_j}(\tilde{\theta}) \neq f(\tilde{\theta})$. By (12) we know that

$$u_j(C_{i,j}^\varepsilon(m), \theta_j) > u_j(f(\tilde{\theta}), \theta_j) \text{ if } m_i^3 = \tilde{\theta}. \tag{25}$$

Now we construct $\tilde{m}_j$ that is identical to m_j except that $\tilde{m}_j^4(\tilde{\theta}) = \theta_j$. In the following, we shall show that $\tilde{m}_j$ strictly dominates m_j against any $m_{-j} \in S_{-j}^\infty(\mathcal{M}, \theta)$, which contradicts the hypothesis that $m_j \in S_j^\infty(\mathcal{M}, \theta)$. Fix $m_{-j} \in S_{-j}^\infty(\mathcal{M}, \theta)$. We first observe that $e_{j,j}(\tilde{m}_j, \tilde{m}_j) = \varepsilon$ because we have $B_{\theta_j}(\tilde{\theta}) \neq f(\tilde{\theta})$ and $m_j^3 \sim \tilde{\theta}$.

Second, we know that (m_j, m_{-j}) and $(\tilde{m}_j, m_{-j})$ induce different allocations only when agents j and k are picked up and agent k announces a state m_k^3 which is equivalent to $\tilde{\theta}$. This happens with positive probability since we allow for $k = j$ and $\tilde{m}_j^3 = m_j^3 \sim \tilde{\theta}$. In that case, (m_j, m_{-j}) implements $f(\tilde{\theta})$, while $(\tilde{m}_j, m_{-j})$ implements $C_{k,j}^\varepsilon(\tilde{m}_j, m_{-j})$. By (25), agent j gets a strictly better payoff under $\tilde{m} = (\tilde{m}_j, m_{-j})$ than under $(m_j, \tilde{m}_{-j})$. Hence, $\tilde{m}_j$ strictly dominates m_j against any $m_{-j} \in S_{-j}^\infty(\mathcal{M}, \theta)$. ■

Step 2: *For any agent i and any $m_i \in S_i^\infty(\mathcal{M}, \theta)$, we have $m_i^2 \sim \tilde{\theta}$* where $\tilde{\theta} \in \mathcal{P}(\theta)$.

Proof. By Step 1, we know that for every $i \in \mathcal{I}$, if $m_i \in S_i^\infty(\mathcal{M}, \theta)$, then there exists $\hat{\theta}_{-i}$ such that $(m_i^1, \hat{\theta}_{-i})$ identifies some $\tilde{\theta} \in \mathcal{P}(\theta)$. Since the partition $\mathcal{P}$ has product structure by

Step 0, m^1 identifies some $\tilde{\theta} \in \mathcal{P}(\theta)$. Suppose by way of contradiction that $m_i^2 = \theta' \not\sim \theta$. Now, we construct $\tilde{m}_i = (m_i^1, \theta, m_i^3, m_i^4)$ which is identical to m_i except that $\tilde{m}_i^2 = \theta_i$. We claim that announcing $\tilde{m}_i$ is strictly better than announcing m_i against any $m_{-i} \in S_{-i}^{\infty}(\mathcal{M}, \theta)$.

Fix $m_{-i} \in S_{-i}^{\infty}(\mathcal{M}, \theta)$. Since $\theta' \not\sim \theta$, (m_i, m_{-i}) is "not" consistent. Consider two cases:

Case 1. If $e_{i,j}(\tilde{m}_i, m_j) = \varepsilon$, then $(\tilde{m}_i, m_{-i})$ and (m_i, m_{-i}) implement the same allocation since m_i and $\tilde{m}_i$ only differ in their second report. In terms of transfers incurred, $(\tilde{m}_i, m_{-i})$ avoids the penalty from $\tau_i^2(\cdot)$, while (m_i, m_{-i}) incurs the penalty from $\tau_i^2(\cdot)$. Hence, $\tilde{m}_i$ is a better reply than m_i against m_{-i}.

Case 2. If $e_{i,j}(\tilde{m}_i, m_{-i}) = 0$, then $(\tilde{m}_i, m_{-i})$ and (m_i, m_{-i}) implement the same allocation since m_i and $\tilde{m}_i$ only differ in their second component. Moreover, (m_i, m_{-i}) incurs the penalty η' to agent i, whereas $(\tilde{m}_i, m_{-i})$ does not. It follows that $\tilde{m}_i$ is a strictly better response than m_i against m_{-i}. This completes the proof of Step 2. ■

Step 3: *For any $i \in \mathcal{I}$ and $m_i \in S_i^{\infty}(\mathcal{M}, \theta)$, we have $m_i^3 \sim \tilde{\theta}$ where $\tilde{\theta} \in \mathcal{P}(\theta)$.*

Proof. By Step 2, we know that for every $i \in \mathcal{I}$ and $m_i \in S_i^{\infty}(\mathcal{M}, \theta)$, we have $m_i^2 \sim \theta$. Suppose on the contrary that $m_i^3 = \theta' \not\sim \theta$. Now, we construct $\tilde{m}_i = (m_i^1, m_i^2, \theta, m_i^4)$ which is identical to m_i except the third component of the message. We claim that $\tilde{m}_i$ strictly dominates m_i against any $m_{-i} \in S_{-i}^{\infty}(\mathcal{M}, \theta)$. Fix $m_{-i} \in S_{-i}^{\infty}(\mathcal{M}, \theta)$. Indeed, $(\tilde{m}_i, m_{-i})$ avoids the penalty η from $\tau_i^3(\cdot)$, while (m_i, m_{-i}) incurs the penalty η from $\tau_i^3(\cdot)$. The potential loss from $(\tilde{m}_i, m_{-i})$ rather than (m_i, m_{-i}) is bounded by $\eta + D$, which may happen when $(\tilde{m}_i, m_{-i})$ is not consistent without any challenge, while (m_i, m_{-i}) is with a challenge. It follows from (16) that $\tilde{m}_i$ is a better response against m_{-i} than m_i. This implies that $\tilde{m}_i$ strictly dominates m_i against any $m_{-i} \in S_{-i}^{\infty}(\mathcal{M}, \theta)$, which contradicts the hypothesis that $m_i \in S_i^{\infty}(\mathcal{M}, \theta)$. This completes the proof of Step 3. ■

Step 4: *$g(m) = f(\theta)$ and $\tau_i(m) = 0$ for every $i \in \mathcal{I}$ and $m \in S^{\infty}(\mathcal{M}, \theta)$.*

Proof. By Steps 1 through 3, for any $m \in S^{\infty}(\mathcal{M}, \theta)$, we have that m^1 identifies some $\tilde{\theta} \in \mathcal{P}(\theta)$ and $\tilde{\theta} \sim m_i^2 \sim m_i^3$ for every $i \in \mathcal{I}$. Moreover, since $\tilde{\theta} \in \mathcal{P}(\theta)$, if $B_{m_i^4(\tilde{\theta})}(\tilde{\theta}) \neq f(\tilde{\theta})$, then $B_{m_i^4(\tilde{\theta})}(\tilde{\theta})$ belongs to $\mathcal{SL}_i(f(\tilde{\theta}), \theta_i)$. By (13), player i is worse off by challenging $\tilde{\theta} \in \mathcal{P}(\theta)$. Hence, $B_{m_i^4(\tilde{\theta})}(\tilde{\theta}) = f(\tilde{\theta})$ for every $m_i \in S_i^{\infty}(\mathcal{M}, \theta)$. We thus conclude that for every $m \in S^{\infty}(\mathcal{M}, \theta)$ we have $e_{i,j}(m_i, m_j) = 0$ for any agents i and j, no transfer is incurred, and $f(\tilde{\theta})$ is implemented. Again, since $\tilde{\theta} \in \mathcal{P}(\theta)$, it follows that $g(m) = f(\theta)$. This completes the proof of Step 4. ■

A.4 Proof of Theorem 4

We propose a mechanism $\mathcal{M} = ((M_i), g, (\tau_i))_{i\in\mathcal{I}}$ that is used to prove the if-part of Theorem 4. We define the message space, allocation rule, and transfer rule below. Let Γ_i denote the set of functions from $F(\Theta)$ to Θ_i. Observe that Γ_i is a finite set when $F(\Theta)$ and Θ_i are both finite sets. Call each γ_i a *challenge function* of agent i which is viewed as a plan to challenge contingent on the possible outcome in $F(\theta)$.

A generic message of agent i is described as follows:

$$m_i = \left(m_i^1, m_i^2, m_i^3, m_i^4\right) \in M_i = M_i^1 \times M_i^2 \times M_i^3 \times M_i^4 = \Theta_i \times \Theta \times \Gamma_i \times (F(\Theta))^{\Theta}$$

where $m_i^4 : \Theta \to F(\Theta)$ satisfies $m_i^4[\theta] \in F(\theta)$ for each $\theta \in \Theta$. That is, agent i is asked to make (1) an announcements of agent i's own type (i.e., m_i^1); (2) an announcement of the state (i.e., m_i^2); (3) a challenge function (i.e., m_i^3); and (4) an announcement of a mapping from states to outcomes within the corresponding image of the SCC (i.e., m_i^4). For the ease of notation, as we do in the case of SCFs, we write $m_{i,j}^2 = \tilde{\theta}_j$ if agent i's report in m_i^2 that agent j's type is $\tilde{\theta}_j$.

For each $m \in M$, let

$$e_{i,j}(m_i, m_j) = \begin{cases} 0, & \text{if } m_i^2 = m_j^2 \text{ and } B_{m_j^3(m_i^4[m_i^2])}\left(m_i^2, m_i^4[m_i^2]\right) = m_i^4[m_i^2]; \\ \varepsilon, & \text{otherwise.} \end{cases}$$

The allocation rule is then defined as follows: for each $m \in M$,

Rule 1: If there exist $\tilde{\theta} \in \Theta$, $x \in F(\tilde{\theta})$, and $i \in \mathcal{I}$ such that $m_k^2 = \tilde{\theta}$ for every agent $k \in \mathcal{I}$, and $m_j^4[\tilde{\theta}] = x$ for all $j \neq i$, then

$$g(m) = \frac{1}{I(I-1)} \sum_{i\in\mathcal{I}} \sum_{j\in\mathcal{I}\setminus\{i\}} \left[e_{i,j}(m_i, m_j) \frac{1}{I} \sum_{k\in\mathcal{I}} y_k\left(m_k^1\right) + (1 - e_{i,j}(m_i, m_j))\, B_{m_j^3(x)}(\tilde{\theta}, x) \right]$$

where $y_k : \Theta \to X$ is the dictator lottery for agent k which is defined in Lemma 1.

Rule 2: Otherwise,

$$g(m) = \frac{1}{I(I-1)} \sum_{i\in\mathcal{I}} \sum_{j\in\mathcal{I}\setminus\{i\}} \left[e_{i,j}(m_i, m_j) \frac{1}{I} \sum_{k\in\mathcal{I}} y_k\left(m_k^1\right) + (1 - e_{i,j}(m_i, m_j))\, B_{m_j^3(m_i^4[m_i^2])}\left(m_i^2, m_i^4[m_i^2]\right) \right]$$

That is, the designer first chooses a pair of distinct agents (i, j) with equal probability. We distinguish two cases: (1) if the second reports of both agents i and j are consistent and agent j does not challenge agent i, then we implement $m_i^4[m_i^2]$; (2) if either the second reports of both agents i and j are inconsistent or agent j challenges agent i, then we consider two subcases: (2.1) if everyone reports a common state $\tilde{\theta}$, and $I-1$ agents agree on the allocation announced in their fourth report, say x, then we implement the compound lottery

$$\varepsilon \times \frac{1}{I} \sum_{k \in \mathcal{I}} y_k\left(m_k^1\right) + (1-\varepsilon) \times B_{m_j^3(x)}(\tilde{\theta}, x).$$

(2.2), Otherwise, we implement the compound lottery

$$C_{i,j}^{\varepsilon}(m) = \varepsilon \times \frac{1}{I} \sum_{k \in \mathcal{I}} y_k\left(m_k^1\right) + (1-\varepsilon) \times B_{m_j^3(m_i^4[m_i^2])}\left(m_i^2, m_i^4[m_i^2]\right).$$

For any message profile $m \in M$ and agent $i \in \mathcal{I}$, we specify the transfer from agent i as follows:

$$\tau_i(m) = \sum_{j \neq i}\left[\tau_{i,j}^1(m) + \tau_{i,j}^2(m)\right].$$

By strict Maskin monotonicity of the SCC F, we can choose $\varepsilon > 0$ sufficiently small such that for all $m \in M$, $i, j \in \mathcal{I}$, and $\theta \in \Theta$, we have

$$u_j(C_{i,j}^{\varepsilon}(m), \theta_j) > u_j(x, \theta_j) \text{ if } B_{m_j^3(x)}(m_i^2, x) \neq x \text{ where } m_i^4[m_i^2] = x \text{ and } m_j^3 = \theta_j \tag{26}$$

$$u_j(C_{i,j}^{\varepsilon}(m), \theta_j) < u_i(x, \theta_j) \text{ if } B_{m_j^3(x)}(m_i^2, x) \neq x \text{ where } m_i^4[m_i^2] = x \text{ and } m_i^2 = \theta. \tag{27}$$

We double the scale of transfer rule in Section 5.1.3 by replacing η with 2η and $-\eta$ with -2η in the definition of $\tau_{i,j}^1$ and $\tau_{i,j}^1$. Moreover, we add one more transfer rule as follows,

$$\tau_{i,j}^3(m) = \begin{cases} 0, & \text{if } B_{m_j^3(m_i^4[m_i^2])}\left(m_i^2, m_i^4[m_i^2]\right) = m_i^4[m_i^2]; \\ -\eta, & \text{if } B_{m_j^3(m_i^4[m_i^2])}\left(m_i^2, m_i^4[m_i^2]\right) \neq m_i^4[m_i^2]. \end{cases}$$

That is, agent i is asked to pay η if his reported outcome $m_i^4[m_i^2]$ are challenged by agent j (via agent j's challenge function m_j^3). Note that we still require that η be greater than the payoff difference D as in (10) in Section 5.1.3.

To prove Theorem 4, observe that Claims 1, 2 and 3 hold with exactly the same proof. In the following, we establish Claim 9 as the counterpart of Claim 4 in Theorem 2 in the modified mechanism above.

Claim 9 *No one challenges any social outcome at the common state announced in the second report, i.e., for any pair of agents i and j, $m_i \in$supp(σ_i), and $m_j \in$supp(σ_j), we have $B_{m_j^3(m_i^4[\tilde{\theta}])}(\tilde{\theta}, m_i^4[\tilde{\theta}]) = m_i^4[\tilde{\theta}]$.*

Proof. By Claim 3, denote by $\tilde{\theta}$ the common state announced in the second report. For each $l \in F(\Theta)$, we define the set of agents:

$$\mathcal{J}(l) \equiv \left\{ j \in \mathcal{I} : \mathcal{SL}_j(l, \tilde{\theta}_j) \cap \mathcal{U}_j(l, \theta_j) = \varnothing \right\}.$$

First, if $j \in \mathcal{J}\left(m_i^4[\tilde{\theta}]\right)$, then $B_{m_j^3(m_i^4[\tilde{\theta}])}(\tilde{\theta}, m_i^4[\tilde{\theta}]) \neq m_i^4[\tilde{\theta}]$ implies that $B_{m_j^3(m_i^4[\tilde{\theta}])}(\tilde{\theta}, m_i^4[\tilde{\theta}])$ is strictly worse than $m_i^4[\tilde{\theta}]$ under the true type of agent j. Hence, whenever m_j^3 triggers a challenge and the allocation $B_{m_j^3(m_i^4[\tilde{\theta}])}(\tilde{\theta}, m_i^4[\tilde{\theta}])$, agent j can profitably deviate to withdraw the challenge by annoucing $\tilde{m}_j = \left(m_j^1, m_j^2, \tilde{m}_j^3, m_j^4\right)$ such that $\tilde{m}_j^3(m_i^4[\tilde{\theta}]) = \tilde{\theta}_i$ and $\tilde{m}_j^3(l) = m_j^3(l)$ for any $l \neq m_i^4[\tilde{\theta}]$.

Hence, to establish the claim, it suffices to prove that $\mathcal{J}\left(m_i^4[\tilde{\theta}]\right) = \mathcal{I}$ for each message $m_i \in$supp(σ_i) and each agent i. Suppose to the contrary that for some agent i and some message $m_i \in$supp(σ_i), we have agent $j \notin \mathcal{J}(l)$. Then, by adding a small transfer to j, we have

$$\mathcal{SL}_j(m_i^4[\tilde{\theta}], \tilde{\theta}_j) \cap \mathcal{SU}_j(m_i^4[\tilde{\theta}], \theta_j) \neq \varnothing. \tag{28}$$

First, we claim that $B_{m_j^3(m_i^4[\tilde{\theta}])}(\tilde{\theta}, m_i^4[\tilde{\theta}]) \neq m_i^4[\tilde{\theta}]$ for every $m_j \in$supp(σ_j). Indeed, if $B_{m_j^3(m_i^4[\tilde{\theta}])}(\tilde{\theta}, m_i^4[\tilde{\theta}]) = m_i^4[\tilde{\theta}]$ for some $m_i^4[\tilde{\theta}]$, agent j can profitably deviate to announce $\tilde{m}_j = \left(m_j^1, m_j^2, \tilde{m}_j^3, m_j^4\right)$ such that $\tilde{m}_j^3(m_i^4[\tilde{\theta}]) = \theta_j$ and $\tilde{m}_j^3(l) = m_j^3(l)$ for any $l \in F(\tilde{\theta})$. This deviation results in the better allocation $B_{\theta_j}(\tilde{\theta}, m_i^4[\tilde{\theta}]) \in \mathcal{SU}_j(f(\tilde{\theta}), \theta_j)$.

Second, since $B_{m_j^3(m_i^4[\tilde{\theta}])}(\tilde{\theta}, m_i^4[\tilde{\theta}]) \neq m_i^4[\tilde{\theta}]$ for every $m_j \in$supp(σ_j), by playing m_i agent i suffers from the penalty η by $\tau_{i,j}^3$. We then derive a contradiction in each of the following two cases. Firstly, if there is some $l \in F\left(\tilde{\theta}\right)$ such that $\mathcal{J}(l) = \mathcal{I}$, then agent i can profitably deviate to announce $\tilde{m}_i$ where $\tilde{m}_i$ is identical to m_i except that $\tilde{m}_i^4[\tilde{\theta}] = l$. By doing so agent i avoids paying the penalty η since no agent j will challenge $m_i^4[\tilde{\theta}] = l$ to obtain an allocation in $\mathcal{U}_j(l, \theta_j)$. This contradicts to the assumption that σ is a Nash equilibrium. Secondly, suppose that $\mathcal{J}(l) \neq \mathcal{I}$ for every $l \in F\left(\tilde{\theta}\right)$. That is, for each $m_i \in$supp(σ_i), there is some agent $k \notin \mathcal{J}\left(m_i^4[\tilde{\theta}]\right)$. It follows that $B_{m_j^3(m_i^4[\tilde{\theta}])}\left(\tilde{\theta}, m_i^4[\tilde{\theta}]\right) \neq m_i^4[\tilde{\theta}]$ for every $m_k \in$supp(σ_k). In other words, for each $m_i \in$supp(σ_i), for some agent k we have $e_{i,k}(m_i, m_k) = \varepsilon$ for every $m_k \in$supp(σ_k). It then follows that for every message of every agent k, the dictator lottery is triggered with σ_{-k}-positive probability. Thus, by (4), each agent k has strict incentive to announce the true type in his first report, i.e., $m_k^1 = \theta_k$ for each $m_k \in$supp(σ_k) and agent $k \in \mathcal{I}$. By Claim 2, we conclude that $\tilde{\theta} = \theta$. This, together with (28), implies that $\mathcal{SL}_j(f(\theta), \theta_j) \cap \mathcal{U}_j(f(\theta), \theta_j) \neq \varnothing$, which is impossible. ■

It only remains to prove the existence of pure-strategy Nash equilibrium.

Claim 10 *For any $\theta \in \Theta$ and $x \in F(\theta)$, there exists a pure-strategy Nash equilibrium $m \in M$ of the game $\Gamma(\mathcal{M}, \theta)$ such that $g(m) = x$ and $\tau_i(m) = 0$ for any $i \in \mathcal{I}$.*

Proof. Fix an SCF $f : \Theta \to X$ with $f(\theta) = x$. We argue that truth-telling (i.e., $m_i = (\theta_i, \theta, \theta_i, f)$ for each $i \in \mathcal{I}$) constitutes a pure-strategy equilibrium. Under the message profile m, for any $i, j \in \mathcal{I}$, we have $B_{m_j^3}(\theta, x) = x$ and $e_{i,j}(m_i, m_j) = 0$. Firstly, reporting $\tilde{m}_i$ with $\tilde{m}_i^2 \neq \theta$ suffers the penalty of $\eta > D$ and hence cannot be a profitable deviation. Secondly, reporting $\tilde{m}_i$ with $\tilde{m}_i^2 = \theta$ and $\tilde{m}_i^3 \neq \theta_i$ either leads to $B_{m_j^3}(\theta) = x$ and results in no change in payoff or $B_{m_j^3}(\theta, x)$ which is strictly worse than x. Moreover, reporting $\tilde{m}_i$ with $\tilde{m}_i^2 = \theta$, $\tilde{m}_i^3 = \theta_i$, and either $\tilde{m}_i^1 \neq \theta_i$ or $\tilde{m}_i^4 \neq f$ does not affect the allocation or transfer. This completes the proof. ■

A.5 Proof of Theorem 5

Recall that in the mechanism which we use to prove Theorem 2, agent i's generic message is given $m_i = (m_i^1, m_i^2, m_i^3) \in \Theta_i \times \Theta \times \Theta_i$. We expand m_i^2 into $H + 2$ copies of Θ and define

$$m_i = (m_i^1, m_i^2, \ldots, m_i^{H+4}) \in \Theta_i \times \underbrace{\Theta \times \cdots \times \Theta}_{H+2 \text{ terms}} \times \Theta_i$$

where H is a positive integer to be chosen later. For each message profile $m \in M$, the allocation is defined as follows:

$$g(m) = \frac{1}{I(I-1)} \sum_{i \in \mathcal{I}} \sum_{j \neq i} \left[e_{i,j}(m_i, m_j) \frac{1}{I} \sum_{k \in \mathcal{I}} l_k\left(m_k^1\right) + \frac{1 - e_{i,j}(m_i, m_j)}{H+1} \left[\sum_{h=3}^{H+2} \rho\left(m^h\right) + B_{m_j^{H+4}}\left(m_i^{H+3}\right) \right] \right]$$

where $l_k : \Theta \to \Delta(A)$ is the dictator lottery for agent k defined in Lemma 2 and

$$e_{i,j}(m_i, m_j) = \begin{cases} 0, & \text{if } m_i^2 = m_j^2 = m_i^h = m_j^h \text{ and } B_{m_j^{H+4}}\left(m_i^{H+3}\right) = f(m_i^{H+3}), \ \forall h \in \{3, \ldots, H+3\}; \\ \varepsilon, & \text{otherwise.} \end{cases}$$

$$\rho\left(m^h\right)=\begin{cases} f(\tilde{\theta}), & \text{if } m_i^h=\tilde{\theta} \text{ for at least } I-1 \text{ agents;}\\ b, & \text{otherwise, where } b \text{ is an arbitrary outcome in } A.\end{cases}$$

We now define the transfer rule. For any message profile $m \in M$ and agent $i \in \mathcal{I}$, we specify the transfer to agent i as follows:

$$\tau_i(m)=\sum_{j\neq i}\left[\tau_{i,j}^{1,2}(m)+\tau_{i,j}^{2,2}(m)\right]+\sum_{h=3}^{H+3}\tau_i^h(m)+d_i(m^2,\ldots,m^{H+3})$$

where $\gamma, \eta, \xi > 0$ (its size is determined later)

$$\tau_{i,j}^{1,2}(m) = \begin{cases} 0, & \text{if } m_{i,j}^2=m_{j,j}^2;\\ -\gamma & \text{if } m_{i,j}^2\neq m_{j,j}^2 \text{ and } m_{i,j}^2\neq m_j^1;\\ \gamma & \text{if } m_{i,j}^2\neq m_{j,j}^2 \text{ and } m_{i,j}^2= m_j^1.\end{cases}$$

$$\tau_{i,j}^{2,2}(m) = \begin{cases} 0, & \text{if } m_{i,i}^2=m_{j,i}^2;\\ -\gamma, & \text{if } m_{i,i}^2\neq m_{j,i}^2;\end{cases}$$

moreover, for any $h \in \{3,\ldots,H+3\}$,

$$\tau_i^h(m)=\begin{cases} -\eta, & \text{if for some } \tilde{\theta},\ m_i^h\neq\tilde{\theta} \text{ but } m_j^h=\tilde{\theta} \text{ for all } j\neq i;\\ 0, & \text{otherwise,}\end{cases}$$

and

$$d_i(m^2,\ldots,m^{H+3})=\begin{cases} -\xi, & \begin{array}{l}\text{if } m_i^h\neq m_i^2 \text{ and } m_j^{h'}=m_j^2 \text{ for some } h\in\{3,\ldots,H+3\},\\ \text{for all } h'\in\{2,\ldots,h-1\} \text{ for all } j\neq i;\end{array}\\ 0, & \text{otherwise.}\end{cases}$$

Finally, we choose positive numbers γ, ξ, H, η,and ε such that

$$\bar{\tau} > \gamma + (H+1)\,\eta + \xi;$$

$$\begin{aligned}\gamma &> \xi+\varepsilon D\\ \eta &> \varepsilon D\\ \xi &> \frac{1}{H}D+\eta.\end{aligned}$$

More precisely, we first fix $\bar{\tau}$ and choose $\gamma < \frac{1}{3}\bar{\tau}$ and $\xi < \min\left\{\frac{1}{3}\bar{\tau},\gamma\right\}$. Second, we choose H large enough so that $\xi > \frac{1}{H}D$. Third, we choose η small enough such that $(H+1)\,\eta < \frac{1}{3}\bar{\tau}$ and $\xi > \frac{1}{H}D+\eta$. Fourth, we choose ε small such that $\gamma > \xi+\varepsilon D$ and $\eta > \varepsilon D$. We can now prove Theorem 5 following the three steps in the proof of Theorem 2.

A.5.1 Contagion of Truth

Claim 11 *If every agent j reports the truth in his first report (i.e., $m_j^1 = \theta_j$ for any $m_j \in$supp(σ_j)), then every agent j reports the truth in his 2nd,...,$(H+3)$th report. That is, $m_j^h = \theta$ for $h = 2, \ldots, H+3$.*

First, every agent j reports the state truthfully in his 2nd report. This follows from the proof of Claim 2 with only one minor difference: Here m_i^2 may affect agent i's payoff through $d_i(\cdot)$ while it will not affect the allocation. However, a similar argument follows, since we have $\gamma > \xi + \varepsilon D$. This step corresponds to Property (b) in Abreu and Matsushima (1994). Then, we can follow verbatim the argument on p. 12 of Abreu and Matsushima (1994) which shows that every agent j reports the state truthfully in his hth report for every $h = 2, ..., H+2$. Finally, since $\xi > \frac{1}{H}D + \eta$ and $m_i^h = \theta$ for all $h = 2, ..., H+2$ and for every agent i, it is the best response for agent j to choose $m_j^{H+3} = \theta$.

A.5.2 Consistency

Claim 12 *Everyone reports the same state from their 2nd report to the last report. That is, there exists $\tilde{\theta} \in \Theta$ such that, for any agent $i \in \mathcal{I}$ and any $m_i \in$ supp(σ_i), we have $m_i^h = \tilde{\theta}$ for $h = 2, \ldots, H+3$.*

Proof. We prove consistency in the three cases as in the proof of Claim 3. The proof for the first two cases remains the same. For the third case, suppose that only one agent, say i, tells a lie in the first report. For any $h = 2, \ldots, H+2$, as agent i believes that all the other agent report the same state $\tilde{\theta}$, by the rule $\rho\left(m^h\right)$ and $\tau_i^h\left(m^h\right)$, we know $m_i^h = \tilde{\theta}$. ■

A.5.3 No Challenge

Claim 13 *No one challenges the common state announced in the $(H+3)$th report, i.e., $B_{m_j^{H+4}}(\tilde{\theta}) = f(\tilde{\theta})$ for any $j \in \mathcal{I}$.*

The argument is the same as the proof of Claim 4.

A.6 Proof of Theorem 6

For simplicity, assume that A is a finite set. Suppose that Θ is a metric space. For any $l \in \Delta(A)$, we write $v_i(l, \theta_i) = l \cdot \bar{v}_i(\theta_i)$ where $\bar{v}_i(\theta_i)$ is a vector of utilities over A induced by

$v_i(\cdot, \theta_i)$. Let $D \equiv \max_{i,\theta\in\Theta,a,a'\in\Delta(A)\cup f(\Theta)} 2 \times |u_i(a,\theta) - u_i(a',\theta)|$. Let $\bar{X} \equiv \Delta(A) \times [-D, D]^I$ and identify $\bar{X}$ with a compact subset of $\mathbb{R}^{I+|A|}$. Let d be a metric on Θ, d_i be the metric on Θ_i, and $\rho : \bar{X} \times \bar{X} \to \mathbb{R}$ be a metric on the outcome space. We endow Θ and X with the Borel sigma-algebra. Moreover, say that the setting is *compact and continuous* if Θ is compact and $\bar{v}_i$ and f are continuous functions on Θ.

We introduce the following version of best challenge scheme. For agent i of type θ_i, an allocation x, and $\tilde{\theta}$, we construct a lottery,

$$l(x, \tilde{\theta}) = \frac{\bar{\rho}(x, \tilde{\theta})}{1 + \bar{\rho}(x, \tilde{\theta})} x + \frac{1}{1 + \bar{\rho}(x, \tilde{\theta})} f(\tilde{\theta}),$$

where $\bar{\rho}(x, \tilde{\theta}) \equiv \min_{y\in\mathcal{U}_i(f(\tilde{\theta}),\tilde{\theta}_i)} \rho(x, y)$ and define

$$B_x(\tilde{\theta}) = \begin{cases} l(x, \tilde{\theta}), & \text{if } x \in \mathcal{SL}_i(f(\tilde{\theta}), \tilde{\theta}_i); \\ f(\tilde{\theta}), & \text{otherwise.} \end{cases}$$

Say a function $\alpha : S \to Y$ between two metric spaces S and Y endowed with the Borel σ-algebra is *analytic* if its pre-image of any open set on Y is an analytic set. Since every analytic set is universally measurable, an analytic function is "almost" a measurable function (see pp. 498-499 of Stinchcombe and White (1992)). We show below that the mechanism which we are about to construct have analytic outcome function and transfer rule. Hence, whenever we fix a mixed-strategy Nash equilibrium σ which is a Borel probability measure on M, we can work with the σ-completion of the Borel sigma-algebra on M to make all the expected payoffs well defined.

Claim 14 *$B_x(\tilde{\theta})$ is an analytic function on $\bar{X} \times \Theta$. If the setting is compact and continuous, then $B_x(\tilde{\theta})$ is a continuous function on $\bar{X} \times \Theta$.*

Proof. It also follows from Theorem 2.17 of Stinchcombe and White (1992) that $\bar{\rho}(\cdot, \cdot)$ is analytic on $\bar{X} \times \Theta$ and hence $B_x(\tilde{\theta})$ is an analytic function on $\bar{X} \times \Theta$. Then, by the theorem of maximum, $\bar{\rho}(\cdot, \cdot)$ is jointly continuous on $\bar{X} \times \Theta$. Let $(x[n], \tilde{\theta}[n])$ be a sequence converging to $(x, \tilde{\theta})$. We show that $B_{x[n]}(\tilde{\theta}[n]) \to B_x(\tilde{\theta})$ in each of the following two cases.

Case 1. $x \in \mathcal{SL}_i(f(\tilde{\theta}), \tilde{\theta}_i)$.

In this case, $B_x(\tilde{\theta}) = l(x, f(\tilde{\theta}))$. Since f and u_i are both continuous, it follows that $x[n] \in \mathcal{SL}_i(f(\tilde{\theta}[n]), \tilde{\theta}_i[n])$ for large enough n. Thus, $B_{x[n]}(\tilde{\theta}[n]) = l(x[n], \tilde{\theta}[n])$. Hence, $B_{x[n]}(\tilde{\theta}[n]) \to l(x, f(\tilde{\theta}))$ as $(x[n], \tilde{\theta}[n]) \to (x, \tilde{\theta})$.

Case 2. $x \notin \mathcal{SL}_i(f(\tilde{\theta}), \tilde{\theta}_i)$.

In this case, $B_x(\tilde{\theta}) = f(\tilde{\theta})$. If there is some $\overline{n}$ such that $x[n] \notin \mathcal{SL}_i(f(\tilde{\theta}[n]), \tilde{\theta}_i[n])$ for every $n \geq \overline{n}$, then $B_{x[n]}(\tilde{\theta}[n]) = f(\tilde{\theta}[n])$. Since f is continuous and $\tilde{\theta}[n] \to \tilde{\theta}$, it follows that $B_{x[n]}(\tilde{\theta}[n] \to f(\tilde{\theta})$. Now suppose that there is a subsequence of $x[n], \tilde{\theta}[n]$, say itself, such that $x[n] \in \mathcal{SL}_i(f(\tilde{\theta}[n]), \tilde{\theta}_i[n])$ for every n. Then, we have $B_{x[n]}(\tilde{\theta}[n]) = l(x[n], f(\tilde{\theta}[n]))$. Since $x \notin \mathcal{SL}_i(f(\tilde{\theta}), \tilde{\theta}_i)$, it follows that $\bar{\rho}(x, \tilde{\theta}) = 0$. Since $\bar{\rho}$ is jointly continuous, we must have $\bar{\rho}(x[n], \tilde{\theta}[n]) \to 0$. By construction of $l(x[n], f(\tilde{\theta}[n]))$, $l(x[n], f(\tilde{\theta}[n])) \to f(\tilde{\theta})$. Hence, $B_{x[n]}(\tilde{\theta}[n]) \to f(\tilde{\theta})$. ■

Claim 15 *Suppose that Assumption 1 holds. For each $i \in \mathcal{I}$, there exists an analytic function $y_i : \Theta_i \to X$ such that for any $\theta_i, \theta_i' \in \Theta_i$ with $\theta_i \neq \theta_i'$,*

$$v_i(y_i(\theta_i), \theta_i) > v_i(y_i(\theta_i'), \theta_i),$$

and for any $x \in \bar{X}$

$$u_i\left(\frac{1}{I}\sum_{k \in \mathcal{I}} y_k(\theta_k'), \theta_i\right) < u_i(x, \theta_i).$$

Moreover, if the setting is compact and continuous, $y_i(\theta_i)$ is continuous on Θ_i.

Proof. We construct the dictator lotteries in the infinite state space. We construct

$$(l_i(\theta_i), t_1(\theta_i), ..., t_I(\theta_i)) \in \bar{X}$$

where $l_i(\theta_i)$ is a lottery over A and $t_k(\theta_i) \in [0, D]$ for any k, and let

$$y_i(\theta_i) = (l_i(\theta_i), t_1 - 2D, ..., t_I - 2D).$$

Hence, we obtain $u_i\left(\frac{1}{I}\sum_{k \in \mathcal{I}} y_k(\theta_k'), \theta_i\right) < u_i(x, \theta_i)$ for any θ_i and any $x \in \bar{X}$. Let l^* be the uniform lottery over A, i.e., $l^*[a] = 1/|A|$. Pick $r < 1/|A|$. Consider the maximization problem as follows:

$$\max_{(l,t)} (l, t) \cdot (\bar{v}_i(\theta_i), 1)$$
$$\text{s.t.} \quad \|(l, t) - (l^*, 1)\| \leq r$$

The Kuhn-Tucker condition for $(l_i(\theta_i), t_i(\theta_i))$ to be the solution is

$$\bar{v}_i(\theta_i) - \lambda_i(\theta_i)(2(l_i(\theta_i), t_i(\theta_i)) - (l^*, 1)) = 0$$

where $\lambda_i(\theta_i) > 0$ as $\bar{v}_i(\theta_i) > 0$. Hence, $(l_i(\theta_i), t_i(\theta_i)) = \frac{1}{2}\left(\frac{\bar{v}_i(\theta_i)}{\lambda_i(\theta_i)} + (l^*, 1)\right)$. For every $\theta_i \neq \theta_i'$, since $\bar{v}_i(\theta_i)$ is not an affine transform of $\bar{v}_i(\theta_i')$, it follows that $(l_i(\theta_i), t_i(\theta_i)) \neq (l_i(\theta_i'), t_i(\theta_i'))$. Finally, by the theorem of maximum, $l_i(\cdot)$ is a continuous function on Θ_i. ■

A.6.1 The Mechanism

A generic message of agent i is described as follows:

$$m_i = \left(m_i^1, m_i^2, m_i^3\right) \in M_i = M_i^1 \times M_i^2 \times M_i^3 = \Theta_i \times \Theta \times \bar{X}.$$

That is, agent i is asked to make (1) one announcement about agent i's type (i.e., m_i^1); and (2) one announcement about the state (i.e., m_i^2); and (3) one announcement about the allocation (i.e., m_i^3). As in the main text, we write $m_{i,j}^2 = \tilde{\theta}_j$ if agent i reports in m_i^2 that agent j's type is $\tilde{\theta}_j$.

A.6.1.1 Allocation Rule

For each message profile $m \in M$, the allocation is defined as follows:

$$g(m) = \frac{1}{I(I-1)} \sum_{i \in \mathcal{I}} \sum_{j \neq i} \left[e_{i,j}(m_i, m_j) \frac{1}{I} \sum_{k \in \mathcal{I}} y_k\left(m_k^1\right) + \left(1 - e_{i,j}(m_i, m_j)\right) B_{m_j^3}\left(m_i^2\right) \right]$$

where $y_k(\theta_k) = (l_k(\theta_k), t_1(\theta_k), ..., t_I(\theta_k))$ is the dictator lottery for agent k with type θ_k defined in Claim 15 and

$$e_{i,j}(m_i, m_j) \equiv \min\left\{ \max\left\{ d\left(m_i^2, m_j^2\right), \bar{\rho}\left(m_j^3, m_i^2\right)^3 \right\}, 1 \right\}.$$

For each m, let

$$C_{i,j}(m) \equiv e_{i,j}(m_i, m_j) \frac{1}{I} \sum_{k \in \mathcal{I}} y_k\left(m_k^1\right) + \left(1 - e_{i,j}(m_i, m_j)\right) B_{m_j^3}\left(m_i^2\right).$$

Thus, with probability $\frac{1}{I(I-1)}$ an ordered pair (i,j) is chosen, then $C_{i,j}(m)$ is implemented.

Claim 16 *$g : M \to X$ is an analytic function. Moreover, if the setting is compact and continuous, then g is a continuous function.*

Proof. It follows from Claims 14 and 15 that g is analytic; moreover, if the setting is compact and continuous, then g is continuous. ■

A.6.1.2 Transfer Rule

We now define the transfer rule.

For any message profile $m \in M$ and agent i chosen, we specify the transfer to agent i as follows:

$$\tau_i(m) = \sum_{j \neq i} \left[\tau^1_{i,j}(m) + \tau^2_{i,j}(m)\right]$$

Given a message profile m and agent j, let $\tilde{m}_i = \left(m^1_i, \left(m^1_j, m^2_{i,-j}\right), m^3_i\right)$ (which replaces $m^2_{i,j}$ in m_i by m^1_j), $\hat{m}_i = \left(m^1_i, \left(m^2_{j,j}, m^2_{i,-j}\right), m^3_i\right)$ (which replaces $m^2_{i,j}$ in m_i by $m^2_{j,j}$), and $\bar{m}_i = \left(m^1_i, \left(m^2_{j,i}, m^2_{i,-i}\right), m^3_i\right)$.

We define $\tau^1_{i,j}$ and $\tau^2_{i,j}$ as follows:

$$\begin{aligned} \tau^1_{i,j}(m) \quad = \quad & -\sup_{\tilde{\theta}_i} \left| u_i(g(m), \tilde{\theta}_i) - u_i(g(m/\tilde{m}_i), \tilde{\theta}_i) \right| \\ & + \sup_{\tilde{\theta}_i} \left| u_i(g(m/\hat{m}_i), \tilde{\theta}_i) - u_i(g(m/\tilde{m}_i), \tilde{\theta}_i) \right| \\ & + d_j\left(m^2_{j,j}, m^1_j\right) - d_j(m^2_{i,j}, m^1_j), \end{aligned} \tag{29}$$

where $m/\hat{m}_i \equiv (\hat{m}_i, m_{-i})$ and similarly $m/\tilde{m}_i \equiv (\tilde{m}_i, m_{-i})$. Observe that $\tau^1_{i,j}$ satisfies two important properties: (1) $u_i(g(m/\hat{m}_i), \tilde{\theta}_i) - u_i(g(m/\tilde{m}_i), \tilde{\theta}_i)$ remains constant regardless of agent i's choice of $m^2_{i,j}$ (2) $\tau^1_{i,j}(m) = 0$ if $m^2_{i,j} = m^2_{j,j}$.

$$\tau^2_{i,j}(m) = -\sup_{\tilde{\theta}_i} \left| u_i(g(m), \tilde{\theta}_i) - u_i(g(m/\bar{m}_i), \tilde{\theta}_i) \right| - d_i(m^2_{i,i}, m^2_{j,i}) \tag{30}$$

Claim 17 *$\tau_i : M \to \mathbb{R}$ is an analytic function. Moreover, if the setting is compact and continuous, then $\tau_i(\cdot)$ is a continuous function.*

Proof. It follows from Theorem 2.17 of Stinchcombe and White (1992) that τ_i is analytic. Suppose that the setting is compact and continuous. Then, by Claim 16, g is also continuous on M. Moreover, by the theorem of maximum, $\tau^1_{ij}(\cdot)$ and $\tau^2_{ij}(\cdot)$ are continuous on M. Hence $\tau^1_{ij}(\cdot)$ and $\tau^2_{ij}(\cdot)$ are both continuous. ■

Hence, with the claims above, we have that $\mathcal{M} = (M, g, \tau)$ is a mechanism with compact sets of messages, a continuous outcome function, and continuous transfer rules. Thus, in the complete-information game induced by the mechanism, there exists a mixed-strategy NE.

To show that implementation is achieved by the constructed mechanism, we only emphasize the differences from the argument in finite state space. Before we provide the main argument, we establish two lemmas which play an important role in the proof of Theorem 6.

Throughout the proof, we denote by θ the true state. First, we show that it is strictly worse for any agent to challenge the truth.

Lemma 4 *Let m be a message profile such that $m_k^2 = \theta$ for all k. Then, $u_j(C_{k,j}(m), \theta_j) < u_j(f(\theta), \theta_j)$ for all x with $B_x(\theta) \neq f(\theta)$.*

Proof. Note that whenever $B_x(\theta) \neq f(\theta)$, we have $u_i(B_x(\theta), \theta_i) < u_i(f(\theta), \theta_i)$. Moreover, since $u_i\left(\frac{1}{I}\sum_{k\in\mathcal{I}} y_k(\theta'_k), \theta_i\right) < u_i(f(\theta), \theta_i)$, we conclude that

$$u_j(C_{k,j}(m), \theta_j) < u_j(f(\theta), \theta_j) \text{ if } B_{m_j^3}(\theta) \neq f(\theta). \tag{31}$$

This completes the proof. ■

Second, whenever $\mathcal{SL}_j(f(\tilde{\theta}), \tilde{\theta}_i) \cap \mathcal{SU}_j(f(\tilde{\theta}), \theta_i) \neq \varnothing$, we show that it is strictly better for agent j to challenge.

Lemma 5 *Let m be a message profile, $\tilde{\theta}$ be a state such that $m_k^2 = \tilde{\theta}$ for all $k \in \mathcal{I}$ and $\mathcal{SL}_j(f(\tilde{\theta}), \tilde{\theta}_i) \cap \mathcal{SU}_j(f(\tilde{\theta}), \theta_i) \neq \varnothing$ for some $i, j \in \mathcal{I}$. Then, $u_j(C_{k,j}(m), \theta_j) > u_j(f(\tilde{\theta}), \theta_j)$ for some x with $B_x(\tilde{\theta}) \neq f(\tilde{\theta})$.*

Proof. We choose $m_j^3 = x \in \mathcal{SL}_j(f(\tilde{\theta}), \tilde{\theta}_i) \cap \mathcal{SU}_j(f(\tilde{\theta}), \theta_i) \cap \bar{X}$ and $\varepsilon \in (0, 1)$ such that

$$\varepsilon(-3D) + \left(1 - \varepsilon^3\right)\left[u_j(x, \theta_j) - u_j\left(f(\tilde{\theta}), \theta_j\right)\right] > 0. \tag{32}$$

and

$$\frac{\varepsilon^2}{1 - \varepsilon^2} < \bar{\rho}(x, \tilde{\theta}) < \varepsilon. \tag{33}$$

Now,

$$
\begin{aligned}
& u_j(C_{k,j}(m), \theta_j) - u_j(f(\tilde{\theta}), \theta_j) \\
= \; & e_{i,j}(m_i, m_j)\, u_j\left(\frac{1}{I}\sum_{k\in\mathcal{I}} y_k(m_k^1), \theta_j\right) + (1 - e_{i,j}(m_i, m_j))\, u_j\left(B_x\left(m_i^2\right), \theta_j\right) - u_j(f(\tilde{\theta}), \theta_j) \\
= \; & \bar{\rho}(x, \tilde{\theta})^3 u_j\left(\frac{1}{I}\sum_{k\in\mathcal{I}} y_k(m_k^1), \theta_j\right) + (1 - \bar{\rho}(x, \tilde{\theta})^3) u_j(l(x, f(\tilde{\theta})), \theta_j) - u_j(f(\tilde{\theta}), \theta_j) \\
> \; & \varepsilon^3 [u_j\left(\frac{1}{I}\sum_{k\in\mathcal{I}} y_k(m_k^1), \theta_j\right) - u_j\left(f(\tilde{\theta}), \theta_j)\right] \\
& + \left(1 - \varepsilon^3\right)\left[u_j\left(\frac{\bar{\rho}(x, \tilde{\theta})}{1 + \bar{\rho}(x, \tilde{\theta})} x + \frac{1}{1 + \bar{\rho}(x, \tilde{\theta})} f(\tilde{\theta}), \theta_j\right)\right] - u_j(f(\tilde{\theta}), \theta_j) \\
> \; & \varepsilon^3\left[u_j\left(\frac{1}{I}\sum_{k\in\mathcal{I}} y_k(m_k^1), \theta_j\right) - u_j(f(\tilde{\theta}), \theta_j)\right] + \\
& \left(1 - \varepsilon^3\right)\left[\frac{\bar{\rho}(x, \tilde{\theta})}{1 + \bar{\rho}(x, \tilde{\theta})} u_j(x, \theta_j) - \frac{\bar{\rho}(x, \tilde{\theta})}{1 + \bar{\rho}(x, \tilde{\theta})} u_j(f(\tilde{\theta}), \theta_j)\right] \\
> \; & \varepsilon^3\left[u_j\left(\frac{1}{I}\sum_{k\in\mathcal{I}} y_k(m_k^1), \theta_j\right) - u_j(f(\tilde{\theta}), \theta_j)\right] + \left(1 - \varepsilon^3\right)\varepsilon^2\left[u_j(x, \theta_j) - u_j(f(\tilde{\theta}), \theta_j)\right] \\
= \; & \varepsilon^2\left[\varepsilon\left(u_j\left(\frac{1}{I}\sum_{k\in\mathcal{I}} y_k(m_k^1), \theta_j\right) - u_j(f(\tilde{\theta}), \theta_j)\right) + \left(1 - \varepsilon^3\right)\left[u_j(x, \theta_j) - u_j(f(\tilde{\theta}), \theta_j)\right]\right] \\
> \; & \varepsilon^2\left[\varepsilon(-3D) + \left(1 - \varepsilon^3\right)\left[u_j(x, \theta_j) - u_j(f(\tilde{\theta}), \theta_j)\right]\right] \\
> \; & 0
\end{aligned}
$$

where the second equality follows from the fact that $e_{i,j}(m_i, m_j) = \bar{\rho}(x, \tilde{\theta})^3$ since $m_i^2 = m_j^2$; the third inequality is due to that $\bar{\rho}(x, \tilde{\theta}) < \varepsilon$ and $u_j(\frac{1}{I}\sum_{k\in\mathcal{I}} y_k(m_k^1), \theta_j) < u_j(l(x, f(\tilde{\theta})), \theta_j)$; the fifth inequality follows from that $\frac{\bar{\rho}(x,\tilde{\theta})}{1+\bar{\rho}(x,\tilde{\theta})} > \varepsilon^2$ due to inequality (33), and the last inequality follows from inequality (32). ■

A.6.2 Existence of Good Equilibrium

Consider an arbitrary true state $\theta = (\theta_i)_{i\in\mathcal{I}}$.

The proof consists of two parts. In the first part, we argue that truth-telling m where $m_i = (\theta_i, \theta, x)$ for each $i \in \mathcal{I}$ constitutes a pure-strategy equilibrium, where $B_x(\theta) = f(\theta)$. Under the message profile m, $e_{i,j}(m_i, m_j) = 0$. Firstly, reporting $\tilde{m}_i$ with either $\tilde{m}_{i,i}^2 \neq \theta_i$ or $\tilde{m}_{i,j}^2 \neq \theta_j$ suffers the penalty of $\tau_{i,j}^2(m)$ or $\tau_{i,j}^1(m)$ and hence cannot be a profitable deviation by Claim 18. Secondly, reporting $\tilde{m}_i$ with $\tilde{m}_i^2 = \theta$ and $\tilde{m}_i^3 = x' \neq x$ either leads to $B_{x'}(\theta) = f(\theta)$ and results in no change in payoff or $B_{x'}(\theta) \neq f(\theta)$ which is strictly worse than $f(\theta)$. By Lemma 4, this is not a profitable deviation. Finally, reporting $\tilde{m}_i$ with $\tilde{m}_i^2 = \theta$, $\tilde{m}_i^3 = \theta_i$, and $\tilde{m}_i^1 \neq \theta_i$ does not affect the allocation or transfer, since we still have $\tau_i(m) = 0$ and $e_{j,k}(m_j, m_k) = 0$ for every j and k.

In the second part, we show that for any Nash equilibrium σ of the game $\Gamma(\mathcal{M}, \theta)$ and any $m \in$supp(σ), $g(m) = f(\theta)$ and $\tau_i(m) = 0$ for any $i \in \mathcal{I}$. The proof of the second part is divided into three steps: (Step 1) *contagion of truth*: if agent j announces his type truthfully in his first report, then every agent must also report agent j's type truthfully in their second report; (Step 2) *consistency*: every agent reports the same state $\tilde{\theta}$ in the second report; and (Step 3) *no challenge*: no agent challenges the common reported state $\tilde{\theta}$, i.e., $B_{m_j^3}(\tilde{\theta}) = f(\tilde{\theta})$ for any $j \in \mathcal{I}$. Then, consistency implies that $\tau_i(m) = 0$ for any $i \in \mathcal{I}$, whereas no challenge is invoked and monotonicity of f together with Lemma 5 implies that $g(m) = f(\tilde{\theta}) = f(\theta)$. We now establish Steps 1 through 3.

A.6.3 Contagion of Truth

Claim 18 *We establish two results:*
(a) If agent j reports the truth in his first report (i.e., $m_j^1 = \theta_j$ for any $m_j \in$supp(σ_j)), then every agent $i \neq j$ must report agent j's type truthfully in his second report (i.e., $m_{i,j}^2 = \theta_j$ for any $m_i \in$supp(σ_i)).
(b) If every agent i reports a fixed type $\tilde{\theta}_j$ of agent j in his second report with probability one (i.e., $m_{i,j}^2 = \tilde{\theta}_j$ for any $m_i \in$supp(σ_i)), then agent j must report his own type truthfully in his second report (i.e., $m_{j,j}^2 = \theta_j$ for any $m_j \in$supp(σ_j)).

Proof. First, we prove part (a). That is, for any (m_i, m_{-i}), if $m_{i,j}^2 \neq m_j^1$ for some j, we show that

$$u_i(g(m/\tilde{m}_i), \theta_i) + \tau_i(m/\tilde{m}_i) > u_i(g(m), \theta_i) + \tau_i(m)$$

Notice that $\tau_{i,k}^2(m/\tilde{m}_i) = \tau_{i,k}^2(m)$ for any $k \neq i$ and $\tau_{i,k}^1(m/\tilde{m}_i) = \tau_{i,k}^1(m)$ for any $k \neq j$. Thus, in terms of transfers,

$$\begin{aligned}\tau_i(m/\tilde{m}_i) - \tau_i(m) &= \tau_{i,j}^1(m/\tilde{m}_i) - \tau_{i,j}^1(m) \\ &= \sup_{\theta_i'} |u_i(g(m), \theta_i') - u_i(g(m/\tilde{m}_i), \theta_i')| + d_j\left(m_{i,j}^2, m_j^1\right)\end{aligned}$$

Thus we have

$$\begin{aligned}
& \left\{u_i\left(g\left(m/\tilde{m}_i\right),\theta_i\right)+\tau_i\left(m/\tilde{m}_i\right)\right\}-\left\{u_i\left(g\left(m\right),\theta_i\right)+\tau_i\left(m\right)\right\} \\
= \; & u_i\left(g\left(m/\tilde{m}_i\right),\theta_i\right)-u_i\left(g\left(m\right),\theta_i\right)+\tau_{i,j}^1\left(m/\tilde{m}_i\right)-\tau_{i,j}^1\left(m\right) \\
= \; & u_i\left(g\left(m/\tilde{m}_i\right),\theta_i\right)-u_i\left(g\left(m\right),\theta_i\right) \\
& +\sup_{\theta_i'}\left|u_i\left(g\left(m\right),\theta_i'\right)-u_i\left(g\left(m/\tilde{m}_i\right),\theta_i'\right)\right| \\
& +d_j\left(m_{i,j}^2,m_j^1\right) \\
> \; & 0.
\end{aligned}$$

Second, we prove part (b). That is, for any m_i, m_{-i}, if $m_{j,j}^2 \neq m_{i,j}^2 = m_{k,j}^2$ for any $i,k \in \mathcal{I}\setminus\{j\}$, we show that

$$u_j\left(g\left(m/\bar{m}_j\right),\theta_j\right)+\tau_j\left(m/\bar{m}_j\right)>u_j\left(g\left(m\right),\theta_j\right)+\tau_j\left(m\right)$$

Notice that $\tau_{j,i}^1(m/\bar{m}_j)=\tau_{j,i}^1(m)$.

$$\begin{aligned}
\tau_j\left(m/\bar{m}_j\right)-\tau_j\left(m\right) &= \sum_{i\neq j}\left\{\tau_{j,i}^2\left(m/\bar{m}_j\right)-\tau_{j,i}^2\left(m\right)\right\} \\
&= \sum_{i\neq j}\left\{\begin{array}{c}\sup_{\theta_j'}\left|u_j\left(g\left(m\right),\theta_j'\right)-u_j\left(g\left(m/\tilde{m}_j\right),\theta_j'\right)\right| \\ +d_j\left(m_{j,j}^2,m_{i,j}^2\right)\end{array}\right\}
\end{aligned}$$

It suffices to show that for any $i\neq j$,

$$u_j\left(C_{j,i}\left(m/\bar{m}_j\right),\theta_j\right)+\tau_{j,i}^2\left(m/\bar{m}_j\right)>u_j\left(C_{j,i}\left(m\right),\theta_j\right)+\tau_{j,i}^2\left(m\right)$$

Thus we have

$$\begin{aligned}
& \left\{u_j\left(g\left(m/\bar{m}_j\right),\theta_j\right)+\tau_j\left(m/\bar{m}_j\right)\right\}-\left\{u_j\left(g\left(m\right),\theta_j\right)+\tau_j\left(m\right)\right\} \\
= \; & u_j\left(g\left(m/\bar{m}_j\right),\theta_j\right)-u_j\left(g\left(m\right),\theta_j\right)+\tau_{j,i}^2\left(m/\bar{m}_j\right)-\tau_{j,i}^2\left(m\right) \\
= \; & u_j\left(g\left(m/\bar{m}_j\right),\theta_j\right)-u_j\left(g\left(m\right),\theta_j\right) \\
& +\sum_{i\neq j}\left\{\begin{array}{c}\sup_{\theta_j'}\left|u_j\left(g\left(m\right),\theta_j'\right)-u_j\left(g\left(m/\tilde{m}_j\right),\theta_j'\right)\right| \\ +d_j\left(m_{j,j}^2,m_{i,j}^2\right)\end{array}\right\} \\
> \; & 0.
\end{aligned}$$

This completes the proof. ■

A.6.4 Consistency

The argument for consistency follows verbatim as in the counterpart proof of Theorem 2.

A.6.5 No Challenge

Claim 19 *No one challenges the common state announced in the second report, i.e.,* $B_{m_j^3}(\tilde{\theta}) = f(\tilde{\theta})$ *for any* $j \in \mathcal{I}$.

Proof. By Claim 4, it sufficies to show that $\mathcal{SL}_j(f(\tilde{\theta}), \tilde{\theta}_j) \cap \mathcal{SU}_j(f(\tilde{\theta}), \theta_j) = \varnothing$. Suppose to the contrary that $\mathcal{SL}_j(f(\tilde{\theta}), \tilde{\theta}_j) \cap \mathcal{SU}_j(f(\tilde{\theta}), \theta_j) \neq \varnothing$. Then, we first show that $B_{m_j^3}(\tilde{\theta}) \neq f(\tilde{\theta})$ for every $m_j \in$supp(σ_j). Indeed, by Lemma 5, there exists $x \in \mathcal{SL}_j(f(\tilde{\theta}), \tilde{\theta}_j) \cap \mathcal{SU}_j(f(\tilde{\theta}), \theta_j)$. If $B_{m_j^3}(\tilde{\theta}) = f(\tilde{\theta})$, then $\tilde{m}_j = \left(m_j^1, m_j^2, x\right)$ is a strictly profitable deviation from announcing m_j. This deviation results in the better allocation $B_{\tilde{m}_j^3}(\tilde{\theta}) \in \mathcal{SU}_j(f(\tilde{\theta}), \theta_j)$. Hence, we have $B_{m_j^3}(\tilde{\theta}) \neq f(\tilde{\theta})$ for every $m_j \in$supp(σ_j). Finally, it follows that the dictator lottery is triggered with positive probability. Thus, by (4), each agent i has strict incentive to announce the true type in his first report, i.e., $m_i^1 = \theta_i$ for each $m_i \in$supp(σ_i) and agent $i \in \mathcal{I}$. By Claim 18, we conclude that $\tilde{\theta} = \theta$ and hence $\mathcal{SL}_j(f(\theta), \theta_j) \cap \mathcal{SU}_j(f(\theta), \theta_j) \neq \varnothing$, which is impossible. ■

A.7 Proof of Theorem 7

A.7.1 Proof of the Only-If Part

In the proof, Claim C from Mezzetti and Renou (2012) plays an important role which is reproduced as follows:

Claim 20 *Suppose that* $L_i\left(f\left(\theta\right), \theta_i\right) \subset L_i\left(f\left(\theta\right), \theta_i'\right)$ *and* $SL_i\left(f\left(\theta\right), \theta_i\right) \subset SL_i\left(f\left(\theta\right), \theta_i'\right)$. *Then, given any cardinal representation* $v_i\left(\cdot, \theta_i\right)$ *of* $\succsim_i^{\theta_i}$, *there exists a cardinal representation* $v_i\left(\cdot, \theta'\right)$ *of* $\succsim_i^{\theta_i'}$ *such that* $v_i\left(a, \theta'\right) \leq v_i\left(a, \theta\right)$ *for all* $a \in A$ *and* $v_i\left(f\left(\theta\right), \theta'\right) = v_i\left(f\left(\theta\right), \theta\right)$

Suppose f is ordinally Nash implementable but not almost monotonic. That is, for each agent i, we have $L_i\left(f\left(\theta\right), \theta_i\right) \subset L_i\left(f\left(\theta\right), \theta_i'\right)$ and $SL_i\left(f\left(\theta\right), \theta_i\right) \subset SL_i\left(f\left(\theta\right), \theta_i'\right)$, but $f\left(\theta\right) \neq f\left(\theta'\right)$. By our hypothesis of implementation, we have that for any cardinal representation v_i, there exists pure-strategy Nash equilibrium m^* such that $g\left(m^*\right) = f\left(\theta\right)$. Since $f\left(\theta\right) \neq f\left(\theta'\right)$, m^* cannot be a Nash equilibrium at state θ' for any cardinal representation. Let u_i be the quasilinear preference induced by v_i. Then, there exists an agent i, and a message m_i such that

$$\begin{aligned} v_i\left(g\left(m_i^*, m_{-i}^*\right), \theta\right) + \tau_i\left(m_i^*, m_{-i}^*\right) &\geq v_i\left(g\left(m_i, m_{-i}^*\right), \theta\right) + \tau_i\left(m_i, m_{-i}^*\right); \\ v_i\left(g\left(m_i^*, m_{-i}^*\right), \theta'\right) + \tau_i\left(m_i^*, m_{-i}^*\right) &< v_i\left(g\left(m_i, m_{-i}^*\right), \theta'\right) + \tau_i\left(m_i, m_{-i}^*\right). \end{aligned}$$

Summing up the two inequalities, we obtain

$$v_i\left(g\left(m_i^*, m_{-i}^*\right), \theta\right) - v_i\left(g\left(m_i^*, m_{-i}^*\right), \theta'\right) > v_i\left(g\left(m_i, m_{-i}^*\right), \theta\right) - v_i\left(g\left(m_i, m_{-i}^*\right), \theta'\right). \quad (34)$$

Note that $g\left(m_i^*, m_{-i}^*\right) = f\left(\theta\right)$. By Claim 20, we can construct cardinal utility representation $v_i\left(\cdot, \theta'\right)$ of $\succcurlyeq_i^{\theta'_i}$such that $v_i\left(a, \theta'\right) \leq v_i\left(a, \theta\right)$ for all $a \in A$ and $v_i\left(f\left(\theta\right), \theta'\right) = v_i\left(f\left(\theta\right), \theta\right)$. Therefore, the left-hand side of (34) is zero, while the right-hand side is non-negative. This is a contradiction and hence we complete the proof.

A.7.2 Proof of the If Part

Let f be an SCF which is ordinally almost monotonic on Θ. Define $V^\theta = \times_{i \in \mathcal{I}} V_i^\theta$ with a generic element v^θ. Thanks to Assumption 3, $\theta \neq \theta'$ implies that $\succeq_i^{\theta_i} \neq \succeq_i^{\theta'_i}$ for some agent i. Hence, $\{V^\theta : \theta \in \Theta\}$ forms a partition of $\Theta^* \equiv \cup_{\theta \in \Theta} V^\theta$ which is the set of all cardinal utility profiles of agent i induced by Θ. Observe that Θ^* is a Polish space.[24] For notational simplicity, we write θ_i^* as a generic element in Θ_i^* and $\theta^* = \left(\theta_i^*\right)_{i \in \mathcal{I}}$. Let $f^* : \Theta^* \to A$ be the SCF on Θ^* induced by f such that $f^*\left(\theta^*\right) = f\left(\theta\right)$ if and only if $\theta^* \in V^\theta$.

We prove the if-part by establishing two claims: First, we show that f^* is Maskin-monotonic in Claim 21. Hence, Theorem 6 implies that f^* is implementable in Nash equilibrium on Θ^*. Second, it follows from Claim 22 that f is ordinally Nash implementable on Θ.

[24]Since any product or disjoint union of a countable family of Polish spaces remains a Polish space (see Proposition A.1(b) in p. 550 of Treves (2016)), it suffices to argue that V_i^θ is a Polish space. Indeed, let $V = [0,1]^{|A|}$ be the set of possible cardinalizations. We may write $V_i^\theta = \bigcap_{a \in A} V_{i,a}^\theta$ where for each $a \in A$, we set

$$V_{i,a}^\theta \equiv \bigcap_{\left\{b \in A : a \succ_i^{\theta_i} b\right\}} \{v \in V : v(a) > v(b)\} \bigcap \bigcap_{\left\{b \in A : a \sim_i^{\theta_i} b\right\}} \{v \in V : v(a) = v(b)\}.$$

It follows that V_i^θ is a finite intersection of open subsets and closed subsets of the Polish space V and hence remains a Polish space (see Proposition A.1(a)(c)(e) in p. 550 of Treves (2016)).

Claim 21 *If f is ordinally almost monotonic, then f^* is (strictly) Maskin-monotonic.*

Proof. Consider θ^* and $\tilde{\theta}^*$ in Θ^* such that $f^*(\theta^*) \neq f^*(\tilde{\theta}^*)$. Then, there must exist $\theta, \tilde{\theta} \in \Theta$ such that $f(\theta) \neq f(\tilde{\theta})$, $\theta^* \in V^\theta$, and $\tilde{\theta}^* \in V^{\tilde{\theta}}$, respectively. Since f is ordinal almost monotonic, there exist some agent i and some outcome a and a' such that either $a \in L_i(f(\tilde{\theta}), \tilde{\theta}_i) \cap SU_i(f(\tilde{\theta}), \theta_i)$ or $a' \in SL_i(f(\tilde{\theta}), \tilde{\theta}_i) \cap U_i(f(\tilde{\theta}), \theta_i)$. Then, choose either $t_i < 0$ such that $(a, (t_i, \mathbf{0})) \in \mathcal{SL}_i(f^*(\tilde{\theta}^*), \tilde{\theta}_i^*) \cap \mathcal{SU}_i(f^*(\tilde{\theta}^*), \theta_i^*)$ or $t_i' > 0$ such that $(a', (t_i', \mathbf{0})) \in \mathcal{SL}_i(f^*(\tilde{\theta}^*), \tilde{\theta}_i^*) \cap \mathcal{SU}_i(f^*(\tilde{\theta}^*), \theta_i^*)$ where $\mathbf{0} \in \mathbb{R}^{I-1}$ means zero transfer to any player $j \neq i$. Hence, f^* is strictly Maskin-monotonic on Θ^*. ■

Claim 22 *If f^* is implementable in Nash equilibrium, then f is ordinally Nash implementable.*

Proof. Suppose an SCF f^* is implementable in Nash equilibrium on Θ^*. Then, there exists a mechanism $\mathcal{M} = ((M_i), g, (\tau_i))_{i \in \mathcal{I}}$ such that for any state $\theta^* \in \Theta^*$ and $m \in M$, (i) there exists a pure-strategy Nash equilibrium in the game $\Gamma(\mathcal{M}, \theta^*)$; and (ii) $m \in \text{supp}\ (NE(\Gamma(\mathcal{M}, \theta^*))) \Rightarrow g(m) = f^*(\theta^*)$ and $\tau_i(m) = 0$ for every $i \in \mathcal{I}$. Thus, for any state $\theta^* \in V^\theta$, we must have (i) there exists a pure-strategy Nash equilibrium in the game $\Gamma(\mathcal{M}, \theta, v^{\theta^*})$; and (ii) $m \in \text{supp}\ (NE(\Gamma(\mathcal{M}, \theta, v^{\theta^*}))) \Rightarrow g(m) = f(\theta)$ and $\tau_i(m) = 0$ for every $i \in \mathcal{I}$. Hence, f is ordinally Nash implementable on Θ. ■

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