Maskin Meets Abreu and Matsushima*

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Abstract

The theory of full implementation has been criticized for using "unnatural" devices which admit no equilibrium (Jackson (1992)). To address this critique, we revisit the classical Nash implementation problem due to Maskin (1999) but allow for the use of lottery and monetary transfer as in Abreu and Matsushima (1992, 1994). We unify the two well-established but somewhat orthogonal approaches in full implementation theory. We show that Maskin monotonicity is a necessary and sufficient condition for (exact) mixed-strategy Nash implementation by a finite mechanism. In contrast to previous papers, our approach possesses the following appealing features: finite mechanisms (with no integer or modulo game) are used; mixed strategies are handled explicitly; neither transfer nor bad outcomes occur in equilibrium; our mechanism is robust to information perturbations; and the size of transfers (off the equilibrium) can be made arbitrarily small. Finally, our result can be extended to infinite/continuous settings and ordinal settings.

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Although the theory of implementation has been quite successful in identifying the social choice functions which can be implemented in different informational settings, a nagging criticism of the theory is that the mechanisms used in the general constructive proofs have "unnatural" features. A natural response to this criticism is that the mechanisms in the constructive proofs are designed to apply to a broad range of environments and social choice functions. Given this versatility, it is not surprising that the mechanisms possess questionable features. With this in mind, we would hope that for particular settings and social choice functions we could find "natural" mechanisms with desirable properties. To the extent that there are social choice functions which we can only implement using questionable mechanisms, the existing theory of implementation is inadequate. —Jackson (1992, pp. 757-758)

1 Introduction

Implementation theory can be seen as reverse engineering of game theory. Suppose that a society has decided on a social choice rule – a recipe for choosing the socially optimal alternatives on the basis of individuals' preferences over alternatives. To implement the social choice rule, a mechanism designer chooses a mechanism so that the equilibrium outcomes of the mechanism coincide with the social outcomes designated by the choice rule.

There are two prominent paradigms in the theory of implementation—partial implementation and full implementation. One critical difference between the two paradigms is that the former requires that *one* equilibrium outcome achieve the social choice rule, while the latter requires that *all* equilibrium outcomes be socially desirable. Conventional wisdom suggests that many fewer social choice rules can be fully implemented; even when they can be, this is often accomplished by invoking indirect mechanisms with "unnatural devices".

The historical development of these two paradigms, however, also leads to another important, and perhaps unexpected, difference: full implementation mainly focuses on general social choice environments, while partial implementation/mechanism design is explored usually in economic environments in which the designer can make use of both lotteries and monetary transfer.

In this paper, we study the full Nash implementation problem and allow for the use of lotteries and monetary transfer. We focus on the monotonicity condition (hereafter, Maskin monotonicity) which Maskin shows is necessary and "almost sufficient" for Nash implementation. We aim to implement *social choice functions* (henceforth, SCFs) that are Maskin-monotonic in (mixed-strategy) Nash equilibria by mechanisms without making use of the *integer game* or the *modulo game* device which prevails in the full implementation literature.

In the integer game, each agent announces some integer and the person who announces the highest integer gets to name his favorite outcome. When the agents' favorite outcomes differ, an integer game has no pure-strategy Nash equilibrium. This questionable feature is also shared by modulo games. The modulo game is considered a finite version of the integer game in which agents announce integers from a finite set. The agent whose identification matches the modulo of the sum of the integers gets to name the allocation. In order to "knock out" undesirable equilibria in general environments, most constructive proofs in the literature, following Maskin (1999), have either taken advantage of the fact that the integer/modulo game has no solution (in pure strategies) or assumed away unwanted mixed-strategy Nash equilibria.[1]

Instead of invoking integer/modulo games, we follow the "head-on approach" proposed by Jackson (2001, p. 695) to study Nash implementation in a restricted domain where the designer can invoke both lotteries and transfers in designing the implementing mechanism. We study a finite environment where a finite mechanism is to be anticipated yet mixed-strategy equilibrium needs to be handled.[2] Finite mechanisms are also bounded in the sense of Jackson (1992) and have no aforementioned questionable features. Indeed, Jackson (1992, Example 4) shows that without imposing any domain restriction on the environment, some Maskin-monotonic SCF is not implementable in mixed-strategy Nash equilibria by a finite

[1]We recall the following forceful argument made in (Moore, 1992, p. 210): "Unfortunately, from a positive perspective, these (integer/modulo game) devices seem esoteric. To my mind, we should be very wary of using them in our modeling – otherwise we may well be fooled by our own "success." A standard reply to this criticism is that they are used to prove general theorems, and it is to be expected that in an abstract environment the mechanism will also have to be abstract. But this begs the question: can the devices be dispensed with in specific applications? We have for too long relied on tricks that verge on the fanciful, and this has given implementation theory a bad name."

[2]More precisely, the implementing mechanism which we construct is finite as long as each agent has only finitely many possible preferences. We consider infinite environments in Section 5.3, in which we construct infinite yet well-behaved implementing mechanisms to achieve the same goal.

mechanism.[3] This raises the question as to whether every Maskin-monotonic SCF is mixed-strategy Nash implementable with domain restrictions imposed by lotteries and transfers.[4]

For environments with lotteries and transfers, Abreu and Matsushima (1992, 1994) obtain permissive full implementation results using finite mechanisms without the aforementioned questionable features.[5] However, Abreu and Matsushima (1992, 1994) do not investigate Nash implementation but rather appeal to a different notion of implementation: virtual implementation in Abreu and Matsushima (1992) or exact implementation under iterated weak dominance in Abreu and Matsushima (1994).[6] Virtual implementation means that the planner contents herself with implementing the SCF with arbitrarily high probability.[7] In contrast, by studying *exact* Nash implementation in the specific setting, we unify the two well-established but somewhat orthogonal approaches to implementation theory which are due to Maskin (1999) and to Abreu and Matsushima (1992, 1994). Our exercise is directly comparable to Maskin (1999) and highlights the pivotal trade-off between domain restrictions and the feature of implementing mechanisms. We consider this to be one step in advancing the research program proposed by Jackson (1992), cited in the beginning of this section.

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[3]Nevertheless, we show that the SCF which Jackson (1992) constructs can be implemented in mixed-strategy Nash equilibria in a finite mechanism with arbitrarily small transfers off the equilibrium (see Theorem 3 and footnote 24).

[4]Another direction is to characterize, without making any domain restriction, the subclass of Maskin-monotonic SCFs which can be implemented in mixed-strategy Nash equilibria in a finite mechanism. For this goal, our exercise serves to clarify whether in certain environments, the class of SCFs is as permissive as it can be.

[5]To be precise, Abreu and Matsushima (1992) do not need transfers and make use of lotteries only. "Reducing the probability of a favorable social choice outcome" in their setup plays the same role as "penalizing players by decreasing transfer" does in our setup.

[6]Iterated weak dominance in Abreu and Matsushima (1994) also yields the unique undominated Nash equilibrium outcome. For undominated Nash implementation by "well-behaved" mechanisms, see also Jackson, Palfrey, and Srivastava (1994) and Sjostrom (1994).

[7]Virtual implementation allows for the possibility that an outcome not allowed by the SCF is selected with positive probability even in equilibrium. As discussed in (Benoît and Ok, 2008, Section 3.3), this feature is problematic in situations in which the planner is free to renege. Specifically, if agents believe that the planner will not adopt a questionable outcome a when they know (according to the equilibrium) that a different outcome b is an element of the SCF, the equilibrium falls apart. On a related note, Bochet (2007) and Benoît and Ok (2008) also employ lottery mechanisms to achieve exact Nash implementation while their implementing mechanisms still invoke integer/modulo games, and they exclude mixed strategies.

Our main result (Theorem 1) shows that when the designer can make use of lotteries and transfers (off the equilibrium), Maskin monotonicity is indeed a necessary and sufficient condition for mixed-strategy Nash implementation by a finite mechanism. In that finite mechanism, each agent is asked only to report his preference/type and a preference/type profile which are all payoff-relevant information. The result relies on neither integer games nor refinements which are by far the standard way in the literature to handle mixed-strategy equilibria.

An alternative approach to handle mixed-strategy equilibria is to subscribe refinements such as undominated Nash equilibria or subgame-perfect equilibria. With such refinements, essentially every (Maskin-monotonic or non-Maskin-monotonic) SCF is implementable in a complete-information environment.[8] However, according to Chung and Ely (2003) and Aghion, Fudenberg, Holden, Kunimoto, and Tercieux (2012), if we were to achieve exact implementation in these refinements which are robust to a "small amount of incomplete information," then Maskin monotonicity would be restored as a necessary condition.[9] Their results, which are driven by the lack of the closed-graph property of the refinements, cast doubt on the success of addressing mixed-strategy equilibria by resorting to equilibrium refinements. In contrast, our Theorem 1 can be used to achieve exact and *robust* implementation in mixed-strategy equilibria to the maximal extent of implementing every Maskin-monotonic SCF (Proposition 3).[10]

We also provide several extensions of our main results. First, we extend Theorem 1

[8]See, for instance, Moore and Repullo (1988), Abreu and Sen (1991), Palfrey and Srivastava (1991), and Abreu and Matsushima (1994). In an economic environment similar to ours, Moore and Repullo (1988) construct a simple mechanism with no mixed-strategy "subgame-perfect" equilibrium, while Abreu and Matsushima (1994), Jackson, Palfrey, and Srivastava (1994), and Sjostrom (1994) construct a finite mechanism with no mixed-strategy "undominated" Nash equilibrium.

[9]Both Chung and Ely (2003, Theorem 2) and Aghion, Fudenberg, Holden, Kunimoto, and Tercieux (2012) establish the sufficiency result by using infinite mechanisms with integer games and restricting attention to pure-strategy equilibria. This raises the question as to whether their robustness test may be too demanding when it is applied to finite mechanisms such as the implementing mechanism of Jackson, Palfrey, and Srivastava (1994), that of Abreu and Matsushima (1994), or the simple mechanism in Section 5 of Moore and Repullo (1988), where mixed-strategy equilibria have to be taken seriously.

[10]Harsanyi (1973) shows that a mixed Nash equilibrium outcome may occur as the limit of a sequence of pure-strategy Bayesian Nash equilibria for "nearby games" in which players are uncertain about the exact profile of preferences. Hence, ignoring mixed-strategy equilibria would be particularly problematic if we were to achieve implementation which is robust to information perturbations.

to cover social choice *correspondences* (i.e., multi-valued social choice rules) which Maskin (1999) as well as many subsequent papers have studied. Formally, we show that when there are at least three agents, every Maskin-monotonic social choice correspondence is mixed-strategy Nash implementable (Theorem 2). As long as the social choice correspondence is finite-valued, our implementing mechanism remains finite. Second, we show that if there are at least three agents and the SCF satisfies Maskin monotonicity in the restricted domain without transfer, then it is implementable in mixed-strategy Nash equilibria by a finite mechanism in which the size of transfers remains zero on the equilibrium and can be made arbitrarily small off the equilibrium (Theorem 3).

Third, we consider an infinite setting in which the state space is a compact set, and the utility functions and the SCF are all continuous. In this setting, we show that Maskin monotonicity is a necessary and sufficient condition for mixed-strategy Nash implementation by a mechanism with a compact message space, a continuous outcome function, and a continuous transfer rule (Theorem 4). The extension covers many applications and verifies that our finite setting approximates settings with a continuum of states. To our knowledge, such an extension to an infinite setting has not appeared in the literature, even for virtual implementation.[11] Moreover, a compact and continuous mechanism allows each player to find a best response to every (possibly randomized) strategy profiles of the other players. Indeed, in discussing the notion of "well-behaved" mechanisms, Abreu and Matsushima (1992, footnote 8) also regard compactness and continuity as "plausible necessary desiderata".

Finally, the extension to an infinite setting yields another interesting extension. Specifically, in proving Theorem 1, we have assumed that each agent is an expected utility maximizer with a fixed cardinal utility function over pure alternatives. This raises the question as to whether our result is an artifact of the fixed finite set of cardinalizations. To answer the question, we adopt the concept of *ordinal* Nash implementation proposed by Mezzetti and Renou (2012). The notion requires that a *single* mechanism achieve mixed-strategy Nash implementation for *every* cardinal representation of preferences over lotteries. By making use of our implementing mechanism in the infinite setting, we show that ordinal almost monotonicity, as defined in Sanver (2006), is a necessary and sufficient condition for ordinal Nash implementation (Theorem 5). This extension also verifies that our implementation

[11]This was a prominent open question raised in Section 5 of Abreu and Matsushima (1992), and which, to the best of our knowledge, remains open.

result does not suffer from the critique raised by Jackson, Palfrey, Srivastava (1994. p. 490) to the dependence of Abreu and Matsushima (1994) on a finite set of cardinalizations.

The rest of the paper is organized as follows. In Section 2, we present the basic setup and definitions. Section 3 proves our main result and Section 4 provides its robustness to information perturbations. We discuss the extensions in Section 5 and conclude in Section 6. Appendix contains all proofs omitted from the main text.

2 Preliminaries

2.1 Environment

Consider a finite set of agents $\mathcal{I} = \{1, 2, ..., I\}$ with $I \geq 2$; a finite set of possible states Θ; and a set of pure alternatives A. We consider an environment with lotteries and transfers. Specifically, we work with the space of allocations/outcomes $X \equiv \Delta(A) \times \mathbb{R}^I$ where $\Delta(A)$ denotes the set of lotteries on A that have a countable support, and $\mathbb{R}^I$ denotes the set of transfers to the agents.

Each state $\theta \in \Theta$ induces a type $\theta_i \in \Theta_i$ for each agent $i \in \mathcal{I}$. Assume that Θ has no redundancy, i.e., whenever $\theta \neq \theta'$, we must have $\theta_i \neq \theta'_i$ for some agent i. Hence, we can identify a state θ with its induced type profile $(\theta_i)_{i\in\mathcal{I}}$ and Θ with a subset of $\times_{i=1}^{I}\Theta_i$. Moreover, we say that a type profile $(\theta_i)_{i\in\mathcal{I}}$ identifies a state θ' if $\theta_i = \theta'_i$ for every $i \in \mathcal{I}$. Each type $\theta_i \in \Theta_i$ induces a utility function $u_i(\cdot, \theta_i) : X \to \mathbb{R}$ which is quasilinear in transfers and has a bounded expected utility representation on $\Delta(A)$. That is, for each $x = \left(\ell, (t_i)_{i\in\mathcal{I}}\right) \in X$, we have $u_i(x, \theta_i) = v_i(\ell, \theta_i) + t_i$ for some bounded expected utility function $v_i(\cdot, \theta_i)$ over $\Delta(A)$. That is, we work with an environment with transferable utility (TU) restriction on agents' preferences which is absent in Maskin (1999). As in Abreu and Matsushima (1992), we will take for granted that distinct elements of Θ_i induce different preference orderings over $\Delta(A)$, and also that a player is never indifferent over all elements of A.

We focus on a *complete information* environment in which the state θ is common knowledge among the agents but unknown to a mechanism designer. Thanks to the complete-information assumption, it is indeed without loss of generality to assume that agents' values are private.[12] The designer's objective is specified by a *social choice function* $f : \Theta \to X$,

[12]This is no longer the case when we study information perturbations in Section 4. However, our result

namely, if the state is θ, the designer would like to implement the social outcome $f(\theta)$. We allow an SCF to be defined as a mapping from Θ to X only so as to keep its consistency with the range of the outcome function used in the implementing mechanism. We can define $f : \Theta \rightarrow \Delta(A)$ as a special case of SCFs, as long as the designer is still allowed to impose off-the-equilibrium transfers in the implementing mechanism.

2.2 Mechanism and Solution

A mechanism $\mathcal{M}$ is $((M_i, \tau_i)_{i\in\mathcal{I}}, g)$ where M_i is a nonempty finite *set of messages* available to agent i; $g : M \rightarrow X$ (where $M \equiv \times_{i=1}^{I} M_i$) is the *outcome function*; and $\tau_i : M \rightarrow \mathbb{R}$ is the *transfer rule* which specifies the payment to agent i. The environment and the mechanism together constitute a *game with complete information* at each state $\theta \in \Theta$, which we denote by $\Gamma(\mathcal{M}, \theta)$. Note that the restriction of M_i to being a finite set rules out the use of integer games à la Maskin (1999). We will consider infinite mechanisms in Section 5.3.

Let $\sigma_i \in \Delta(M_i)$ be a (possibly mixed) *strategy* of agent i in the game $\Gamma(\mathcal{M}, \theta)$. A strategy profile $\sigma = (\sigma_1, \ldots, \sigma_I) \in \times_{i\in\mathcal{I}}\Delta(M_i)$ is said to be a (mixed-strategy) *Nash equilibrium* of the game $\Gamma(\mathcal{M}, \theta)$ if, for all agent $i \in \mathcal{I}$, all messages $m_i \in$supp(σ_i) and $m_i' \in M_i$, we have

$$\begin{aligned} &\sum_{m_{-i}\in M_{-i}} \sigma_{-i}(m_{-i}) \left[u_i(g(m_i, m_{-i}); \theta_i) + \tau_i(m_i, m_{-i})\right] \\ \geq &\sum_{m_{-i}\in M_{-i}} \sigma_{-i}(m_{-i}) \left[u_i(g(m_i', m_{-i}); \theta_i) + \tau_i(m_i', m_{-i})\right], \end{aligned}$$

where $\sigma_{-i}(m_{-i})$ denotes the probability that m_{-i} is played under σ_{-i}. A pure-strategy Nash equilibrium is a Nash equilibrium σ such that for each agent i, we have $\sigma_i(m_i) = 1$ for some $m_i \in M_i$.

Let $NE(\Gamma(\mathcal{M}, \theta))$ denote the set of Nash equilibria of the game $\Gamma(\mathcal{M}, \theta)$. We also denote by supp $(NE(\Gamma(\mathcal{M}, \theta)))$ as the set of message profiles that can be played with positive probability under some Nash equilibrium $\sigma \in NE(\Gamma(\mathcal{M}, \theta)$, i.e.,

$$\text{supp } (NE(\Gamma(\mathcal{M}, \theta))) = \{m \in M : \text{there exists } \sigma \in NE(\Gamma(\mathcal{M}, \theta)) \text{ such that } \sigma(m) > 0\}.$$

We now propose our concept of Nash implementation.[13]

there (Proposition 3) only relies on the closed-graph property of the correspondence of (Bayesian) Nash equilibrium which holds even with interdependent values.

[13]We adopt the definition of mixed-strategy Nash implementation in Maskin (1999). Mezzetti and Renou

Definition 1 *An SCF f is **implementable in mixed-strategy Nash equilibria** if there exists a mechanism $\mathcal{M} = ((M_i, \tau_i)_{i\in\mathcal{I}}, g)$ such that for every state $\theta \in \Theta$, (i) there exists a pure-strategy Nash equilibrium in the game $\Gamma(\mathcal{M}, \theta)$; and (ii) $m \in supp\ (NE(\Gamma(\mathcal{M}, \theta))) \Rightarrow g(m) = f(\theta)$ and $\tau_i(m) = 0$ for every $i \in \mathcal{I}$.*

2.3 Maskin Monotonicity

For $(x, \theta_i) \in X \times \Theta_i$, we use $\mathcal{L}_i(x, \theta_i)$ to denote the lower-contour set at allocation x in X for type θ_i, i.e.,

$$\mathcal{L}_i(x, \theta_i) = \{x' \in X : u_i(x, \theta_i) \geq u_i(x', \theta_i)\}.$$

We use $\mathcal{SU}_i(x, \theta_i)$ to denote the strict upper-contour set of $x \in X$ for type θ_i, i.e.,

$$\mathcal{SU}_i(x, \theta_i) = \{x' \in X : u_i(x', \theta_i) > u_i(x, \theta_i)\}.$$

We now state the definition of Maskin monotonicity which Maskin (1999) proposes for Nash implementation.

Definition 2 *An SCF f satisfies **Maskin monotonicity** if, for every pair of states $\tilde{\theta}$ and θ with $f(\tilde{\theta}) \neq f(\theta)$, there is some agent $i \in \mathcal{I}$ such that*

$$\mathcal{L}_i(f(\tilde{\theta}), \tilde{\theta}_i) \cap \mathcal{SU}_i(f(\tilde{\theta}), \theta_i) \neq \varnothing. \tag{1}$$

The agent i in Definition 2 is called a "whistle-blower" or a "test agent"; likewise, an allocation in $\mathcal{L}_i(f(\tilde{\theta}), \tilde{\theta}_i) \cap \mathcal{SU}_i(f(\tilde{\theta}), \theta_i)$ is called a "test allocation" for agent i and the ordered pair of states $(\tilde{\theta}, \theta)$.

To see the idea of Maskin monotonicity, suppose that f is implemented in Nash equilibria by a mechanism. When $\tilde{\theta}$ is the true state, there exists a pure-strategy Nash equilibrium σ in $\Gamma(\mathcal{M}, \tilde{\theta})$ which induces $f(\tilde{\theta})$. If $f(\tilde{\theta}) \neq f(\theta)$ but θ is the true state, the strategy profile σ cannot be a Nash equilibrium, i.e., there exists some agent i ("test agent") who has a profitable deviation. Suppose this deviation induces outcome x ("test allocation"), i.e., agent i strictly prefers x to $f(\tilde{\theta})$ at state θ. However, since σ is a Nash equilibrium at state $\tilde{\theta}$, such a deviation cannot be profitable, and hence, agent i weakly prefers $f(\tilde{\theta})$ to x at state

(2012) propose another definition of Nash implementation that keeps requirement (ii) but weakens (i) in requiring only the existence of mixed-strategy Nash equilibria (which by Nash's theorem holds for our finite implementing mechanism).

$\tilde{\theta}$. In other words, x belongs to the lower contour set at $f(\tilde{\theta})$ for type $\tilde{\theta}_i$, as well as to the strict upper-contour set at $f(\tilde{\theta})$ for type θ_i. Therefore, Maskin monotonicity is a necessary condition for Nash implementation.

Defining the range of SCFs to be X makes our result more general, as some Maskin-monotonic SCF $f : \Theta \to X$ need not be Maskin-monotonic once we restrict the range of f to $\Delta(A)$ with no transfer. The only cost we need to pay is that Maskin monotonicity defined over $\Delta(A)$ is more stringent than that defined over $X = \Delta(A) \times \mathbb{R}^I$.

3 Main Result

In this section, we present our main result which shows that Maskin monotonicity is necessary and sufficient for mixed-strategy Nash implementation. We formally state the result as follows:

Theorem 1 *An SCF f is implementable in mixed-strategy Nash equilibria if and only if it satisfies Maskin monotonicity.*

We will establish Theorem 1 and discuss related issues in the rest of the section. In particular, Sections 3.1.2 provides the key building block for our implementing mechanism which we construct in Section 3.1. Based on the implementing mechanism, Section 3.2 provides a proof of our main result. Section 3.3 illustrates two special cases in which our implementing mechanism can be made a direct mechanism where each agent reports a state. In Section 3.4, we discuss the necessity of domain restrictions used in proving Theorem 1.

3.1 The Mechanism

We now construct a mechanism $\mathcal{M} = ((M_i, \tau_i)_{i \in \mathcal{I}}, g)$ which will be used to prove Theorem 1. The mechanism shares a number of features of the implementing mechanisms in Maskin (1977, 1999) and in Abreu and Matsushima (1992, 1994) which we summarize at the end of the subsection. The construction involves two major building blocks which we call best challenge scheme and dictator lotteries, respectively. We will rule, and then define in turn the message space, allocation rule, and transfer rule of our implementing mechanism.

3.1.1 Dictator Lotteries

Let $\tilde{X} \equiv A \cup \bigcup_{i\in\mathcal{I},\theta_i\in\Theta_i,\tilde{\theta}\in\Theta} x(\tilde{\theta},\theta_i)$. Since $v_i(\cdot,\theta_i)$ is bounded and Θ is finite, we choose $\eta' > 0$ as an upper bound on the monetary value of a change in the selection of an alternative in $\tilde{X}$, that is,

$$\eta' > \sup_{i\in\mathcal{I},\theta_i\in\Theta_i,x,x'\in\tilde{X}} \left|u_i(x,\theta_i) - u_i(x',\theta_i)\right|, \tag{2}$$

where we abuse notation to identify A with a subset of X, i.e., each $a \in A$ is identified with the allocation $(a, 0, ..., 0)$ in X. That is, we identify a with an allocation that consists of a degenerate lottery (which puts probability one) on a and zero transfer for all agents.

Given distinct elements of Θ_i induce different preference orderings over $\Delta(A)$, we have the following lemma. Condition (3) shows that under dictator lotteries, each agent has a strict incentive to reveal his true type, whereas Condition (4) says that these dictator lotteries are strictly less preferred than any alternative a or test allocations in $\tilde{X}$.

Lemma 1 *For each agent $i \in \mathcal{I}$, there exists a function $y_i : \Theta_i \to X$ such that for every types θ_i and θ_i' with $\theta_i \neq \theta_i'$, we have*

$$u_i\left(y_i\left(\theta_i\right),\theta_i\right) > u_i\left(y_i\left(\theta_i'\right),\theta_i\right); \tag{3}$$

moreover, for each type θ_j' of agent $j \in \mathcal{I}$, we also have for every $x \in \tilde{X}$

$$u_i(y_j(\theta_j'),\theta_i) < u_i(x,\theta_i). \tag{4}$$

From Abreu and Matsushima (1992) we can prove the existence of lotteries $\{y_i'(\cdot)\} \subset \Delta(A)$ which satisfy Condition (3). To satisfy Condition (4), we simply add a penalty of η' to each outcome of the lotteries $\{y_i'(\theta_i)\}_{\theta_i\in\Theta_i}$. More precisely, for each $\theta_i \in \Theta_i$, we set

$$y_i(\theta_i) = (y_i'(\theta_i), -\eta', \dots, -\eta') \in X$$

We call the resulting lotteries the *dictator lotteries* for agent i and denote them by $\{y_i(\cdot)\}$.

3.1.2 Best Challenge Scheme

We now define a notion called *the best challenge scheme*, which plays a crucial role in proving Theorem 1. First, a *challenge scheme* for an SCF f is a collection of (pre-assigned) test allocations $\{x(\tilde{\theta},\theta_i)\}$, one for each pair of state $\tilde{\theta}$ and type θ_i of agent i, such that

$$\begin{aligned}
\text{if } \mathcal{L}_i(f(\tilde{\theta}),\tilde{\theta}_i) \cap \mathcal{SU}_i(f(\tilde{\theta}),\theta_i) &\neq \varnothing, \text{ then } x(\tilde{\theta},\theta_i) \in \mathcal{L}_i(f(\tilde{\theta}),\tilde{\theta}_i) \cap \mathcal{SU}_i(f(\tilde{\theta}),\theta_i);\\
\text{if } \mathcal{L}_i(f(\tilde{\theta}),\tilde{\theta}_i) \cap \mathcal{SU}_i(f(\tilde{\theta}),\theta_i) &= \varnothing, \text{ then } x(\tilde{\theta},\theta_i) = f(\tilde{\theta}).
\end{aligned}$$

When $\mathcal{L}_i(f(\tilde{\theta}), \tilde{\theta}_i) \cap \mathcal{SU}_i(f(\tilde{\theta}), \theta_i) \neq \varnothing$, we may think of state $\tilde{\theta}$ as an announcement made by another agent(s) which agent i could challenge (as a whistle-blower) when agent i has true type θ_i. The following lemma shows that there is a challenge scheme in which each whistle-blower i facing state announcement $\tilde{\theta}$ has a weak incentive to report his true type θ_i to challenge $\tilde{\theta}$.

Lemma 2 *There is a challenge scheme $\{x(\tilde{\theta}, \theta_i)\}$ for an SCF f such that for every state $\tilde{\theta}$ and type θ_i,*

$$u_i(x(\tilde{\theta}, \theta_i), \theta_i) \geq u_i(x(\tilde{\theta}, \theta_i'), \theta_i), \forall \theta_i' \in \Theta_i. \tag{5}$$

We relegate a formal proof to Appendix A.1.[14] In defining the implementing mechanism, we shall invoke a challenge scheme which satisfies (5). We call such a challenge scheme *the best challenge scheme*. The existence of the best challenge scheme proved in Lemma 2 demonstrates that the designer's twin goals of allowing for whistle-blowing (as in Maskin (1977, 1999)) and eliciting the truth (from the dictator lotteries as in Abreu and Matsushima (1992, 1994)) can be aligned with the test allocations pre-specified at the outset. Hence, Maskin meets Abreu and Matsushima.

3.1.3 Message Space

A generic message of agent i is described as follows:

$$m_i = \left(m_i^1, m_i^2\right) \in M_i = M_i^1 \times M_i^2 = \Theta_i \times \left[\times_{j=1}^{I} \Theta_j\right].$$

That is, agent i is asked to make (1) a report of his own type (which we denote by m_i^1); and (2) a report of a type profile (which we denote by m_i^2). To simplify the notation, we write $m_{i,j}^2 = \tilde{\theta}_j$ if agent i reports in m_i^2 that agent j is of type $\tilde{\theta}_j$. Recall that agents have complete information about the true state, say θ. We say that agent i sends a truthful first report if $m_i^1 = \theta_i$ and a truthful second report if $m_i^2 = (\theta_j)_{j \in \mathcal{I}}$.

3.1.4 Allocation Rule

For each message profile $m \in M$, the allocation is determined as follows:

$$g(m) = \frac{1}{I(I-1)} \sum_{i \in \mathcal{I}} \sum_{j \neq i} \left[e_{i,j}(m_i, m_j) \left(\frac{1}{2} \sum_{k=i,j} y_k(m_k^1) \right) \oplus (1 - e_{i,j}(m_i, m_j))\, x(m_i^2, m_j^1) \right],$$

[14] We owe special thanks to Phil Reny for suggesting the lemma which simplifies the implementing mechanism adopted in an earlier version of our paper.

where $y_k : \Theta_k \to X$ is the dictator lottery for agent k obtained from Lemma 1 and $\alpha x \oplus (1-\alpha)\, x'$ denotes the outcome which corresponds to the compound lottery that with probability α, outcome x occurs, and with probability $1-\alpha$, outcome x' occurs;[15] moreover, we define

$$e_{i,j}(m_i, m_j) = \begin{cases} 0, & \text{if } m_i^2 \in \Theta,\ m_i^2 = m_j^2, \text{ and } x(m_i^2, m_j^1) = f(m_i^2); \\ \varepsilon, & \text{if } m_i^2 \in \Theta, \text{ and } [m_i^2 \neq m_j^2 \text{ or } x(m_i^2, m_j^1) \neq f(m_i^2)]; \\ 1, & \text{if } m_i^2 \notin \Theta. \end{cases}$$

That is, the designer first chooses a pair of distinct agents (i, j) with equal probability. The order of the pair matters, since for the pair (i, j), the designer will use agent j's report to check agent i's report in determining the allocation. In what follows, we say that the second reports of agent i and agent j are *consistent* iff $m_i^2 = m_j^2$ *and* the common type profile identifies a state in Θ; moreover, we say that agent j *does not challenge* agent i iff $x(m_i^2, m_j^1) = f(m_i^2)$.[16]

In words, the outcome function distinguishes three cases: (1) if the second reports of agent i and agent j are consistent *and* agent j does not challenge agent i, then we implement $f\left(m_i^2\right)$; (2) if agent i reports a type profile which does not identify a state in Θ, then we implement the dictator lottery $\frac{1}{2}\sum_{k=i,j} y_k\left(m_k^1\right)$; (3) otherwise, we implement the compound lottery:

$$C_{i,j}^{\varepsilon}(m_i, m_j) \equiv \varepsilon \left(\frac{1}{2} \sum_{k=i,j} y_k\left(m_k^1\right) \right) \oplus (1-\varepsilon)\, x(m_i^2, m_j^1).$$

That is, $C_{i,j}^{\varepsilon}(m_i, m_j)$ is an $(\varepsilon, 1-\varepsilon)$-combination between the two dictator lotteries $y_i\left(m_i^1\right)$ and $y_j\left(m_j^1\right)$ with equal probability and the allocation specified by the best challenge scheme $x(m_i^2, m_j^1)$.

By (2), we can choose $\varepsilon > 0$ sufficiently small, and $\eta > 0$ sufficiently large such that (i) we have

$$\eta > \sup_{i \in \mathcal{I}, \theta_i \in \Theta_i, m, m' \in M} \left| u_i(g\left(m\right), \theta_i) - u_i(g\left(m'\right), \theta_i) \right|; \tag{6}$$

[15] More precisely, if $x = \left(\ell, (t_i)_{i \in \mathcal{I}}\right)$ and $x' = \left(\ell', (t'_i)_{i \in \mathcal{I}}\right)$ are two outcomes in X, we identify $\alpha x \oplus (1-\alpha)\, x'$ with the outcome $\left(\alpha \ell \oplus (1-\alpha)\, \ell', (\alpha t_i + (1-\alpha)\, t'_i)_{i \in \mathcal{I}}\right)$. For simplicity, we also write the compound lottery $\frac{1}{2} y_i\left(m_i^1\right) \oplus \frac{1}{2} y_j\left(m_j^1\right)$ as $\frac{1}{2}\sum_{k=i,j} y_k\left(m_k^1\right)$.

[16] Observe that we make the first report of both agents i and j effective (through affecting the compound lottery $\frac{1}{2}\sum_{k=i,j} y_k\left(m_k^1\right)$), regardless of whether pair (i, j) or pair (j, i) is picked. This construction will be used in proving Claim 1, which, in turn, is used to prove Claim 4.

(ii) it does not disturb the "effectiveness" of agent j's challenge, i.e.,

$$\begin{aligned} x(m_i^2, m_j^1) &\neq f(m_i^2) \Rightarrow \\ u_j(C^{\varepsilon}_{i,j}(m_i, m_j), m^2_{i,j}) &< u_j(f(m_i^2), m^2_{i,j}) \text{ and } u_j(C^{\varepsilon}_{i,j}(m_i, m_j), m^1_j) > u_j(f(m_i^2), m_j^1). \end{aligned} \quad (7)$$

In other words, we have $C^{\varepsilon}_{i,j}(m_i, m_j) \in \mathcal{SL}_j\left(f\left(m_i^2\right), m^2_{i,j}\right) \cap \mathcal{SU}_j\left(f\left(m_i^2\right), m_j^1\right)$ whenever $x(m_i^2, m_j^1) \neq f(m_i^2)$. This means that whenever agent j challenges agent i, the lottery $C^{\varepsilon}_{i,j}(m_i, m_j)$ is strictly worse than $f\left(m_i^2\right)$ for agent j, when agent i tells the truth about agent j's preference in m_i^2; moreover, the lottery $C^{\varepsilon}_{i,j}(m_i, m_j)$ is strictly better than $f\left(m_i^2\right)$ for agent j, when agent j tells the truth in m_j^1 (therefore agent i tells a lie about agent j's preference).

3.1.5 Transfer Rule

We now define the transfer rule. For every message profile $m \in M$ and every agent $i \in \mathcal{I}$, we specify the transfer received by agent i as follows:

$$\tau_i(m) = \sum_{j \neq i} \left[\tau^1_{i,j}(m_i, m_j) + \tau^2_{i,j}(m_i, m_j)\right],$$

where for each agent $j \neq i$, we define

$$\tau^1_{i,j}\left(m_i, m_j\right) = \begin{cases} 0, & \text{if } m^2_{i,j} = m^2_{j,j}; \\ -\eta, & \text{if } m^2_{i,j} \neq m^2_{j,j} \text{ and } m^2_{i,j} \neq m^1_j; \\ \eta, & \text{if } m^2_{i,j} \neq m^2_{j,j} \text{ and } m^2_{i,j} = m^1_j. \end{cases} \quad (8)$$

$$\tau^2_{i,j}\left(m_i, m_j\right) = \begin{cases} 0, & \text{if } m^2_{i,i} = m^2_{j,i}; \\ -\eta, & \text{if } m^2_{i,i} \neq m^2_{j,i}. \end{cases} \quad (9)$$

Recall that $\eta > 0$ is larger than the maximal utility difference from the outcome function $g(\cdot)$; see (6). The transfer rule can be summarized using the following table:

Transfer to agents	$m^2_{i,j} = m^2_{j,j}$	$m^2_{i,j} \neq m^2_{j,j}$	
	$m^2_{i,j} = m^1_j$ or $m^2_{i,j} \neq m^1_j$	$m^2_{i,j} = m^1_j$	$m^2_{i,j} \neq m^1_j$
$\left(\tau^1_{i,j}\left(m_i, m_j\right), \tau^2_{j,i}\left(m_j, m_i\right)\right)$	$(0, 0)$	$(\eta, -\eta)$	$(-\eta, -\eta)$

Note that we have $\left(\tau^1_{i,j}, \tau^2_{j,i}\right)$ in the table above where $\tau^1_{i,j}$ is the first part of i's transfer and $\tau^2_{j,i}$ is the second part of j's transfer.

In words, for each pair of agents (i, j), if their second reports on agent j's type coincide, then no transfer will be made; if their second reports on agent j's type differ, then we distinguish between the following two subcases: (i) if agent i's report matches agent j's first report, then agent j pays η to agent i; (ii) if agent i's report does not match agent j's first report either, then both agents pay η to the designer. Note that the first report m_i^1 has no effect on the transfer of agent i.

3.1.6 A Key Property of the Implementing Mechanism

As a consequence of Lemmas 1 and 2, the mechanism has the following crucial property of which we will make use in establishing the implementation.

Claim 1 *Let σ be a Nash equilibrium of the game $\Gamma(\mathcal{M}, \theta)$. If $m_i^1 \neq \theta_i$ for some $m_i \in supp(\sigma_i)$, then for every agent $j \neq i$, we have $e_{i,j}(m_i, m_j) = e_{j,i}(m_j, m_i) = 0$ with σ_j-probability one.*

The claim essentially follows from Lemmas 1 and 2. Indeed, the two lemmas together imply that agents must have strict incentive to tell the truth in their first report, as long as switching from a lie to truth affects the allocation with positive probability. A detailed verification of the claim, however, is tedious as it involves different cases of functions $e_{i,j}(\cdot)$ and $e_{j,i}(\cdot)$. We relegate its formal proof to Appendix A.4.

To see why Claim 1 is useful, observe that once every agent sends a truthful first report with probability one, as in Abreu and Matsushima (1994), our transfer rule will ensure that everyone also announces the type profile truthfully in their second report (see Section 3.2.1 for a formal argument) which no one would challenge.[17] Hence, the difficulty occurs only when an agent tells a lie in the first report with positive probability. In this situation, the key issue is whether and how we are able to reach the desirable situation in Maskin (1977, 1999), i.e., all the reports are consistent and there is no challenge. In this situation, Maskin monotonicity implies that the socially desirable outcome must be achieved in Nash equilibrium.

[17] This is the feature shared by our transfer rule and the transfer rule in Abreu and Matsushima (1994). However, our transfer rule satisfies an additional property: as long as the reports become consistent, no transfer will occur to any agent, independently of whether their first reports are truthful or not. Abreu and Matsushima (1994) do not need to satisfy this property since their first report must be truthful, as a result of their solution concept of iterated weak dominance.

Had agent i reported a lie with probability one in equilibrium, Claim 1 would have immediately implied that reports are consistent and there is no challenge. This implication turns out to be stronger and enough to ensure consistency and no challenge, even when agent i randomizes between truthful and untruthful first reports. We provide the details in Sections 3.2.2 and 3.2.3.

3.2 Proof of Theorem 1

As we argue in Section 2.3, Maskin monotonicity is a necessary condition for Nash implementation. We therefore focus on the "if" part of the proof. Fix an arbitrary true state θ throughout the proof. Recall that θ_i stands for agent i's type at state θ and $(\theta_i)_{i\in\mathcal{I}}$ denotes the true type profile.

First, we argue that the truth-telling message profile m (i.e., $m_i = (\theta_i, \theta)$ for each agent i) constitutes a pure-strategy Nash equilibrium. Since m is truth-telling, for every agents i and j, we have $e_{i,j}(m_i, m_j) = 0$ (consistency and no challenge), and $\tau_i(m) = 0$. Consider a possible deviation $\tilde{m}_i$ of agent i. First, misreporting $\tilde{m}^2_{i,j} = \theta'_j \neq \theta_j$ for some j induces the penalty of η from rule $\tau^1_{i,j}(\cdot)$ if $j \neq i$ and rule $\tau^2_{i,j}(\cdot)$ if $j = i$. As a result, reporting $\tilde{m}_i$ is strictly worse than reporting m_i.

Second, misreporting $\tilde{m}^1_i \neq \theta_i$ and holding $\tilde{m}^2_i = \theta$ lead either to $x(\theta, \tilde{m}^1_i) = f(\theta)$ and thereby the same payoff, or to $x(\theta, \tilde{m}^1_i) \neq f(\theta)$. In the latter case, such message $\tilde{m}_i$ results in the outcome $C^{\varepsilon}_{i,j}(\tilde{m}_i, m_j)$ which, by (7), is strictly worse than $f(\theta)$ induced by m_i. Furthermore, deviating from m_i to $\tilde{m}_i$ does not affect the transfer of agent i. Therefore, the truth-telling message profile m constitutes a pure-strategy Nash equilibrium.

We next show that for every Nash equilibrium σ of the game $\Gamma(\mathcal{M}, \theta)$ and every message profile m reported with positive probability under σ, we must achieve the socially desirable outcome, i.e., $g(m) = f(\theta)$ and $\tau_i(m) = 0$ for every agent i. The proof is divided into three steps.

Step 1: *Contagion of truth.* If agent j announces his type truthfully in his first report with probability one, then everyone must also report agent j's type truthfully in their second report;

Step 2: *Consistency.* Every agent reports the same state $\tilde{\theta}$ in the second report;

Step 3: *No challenge.* No agent challenges the common reported state $\tilde{\theta}$, i.e., $x(\tilde{\theta}, m^1_j) = f(\tilde{\theta})$ for every agent j.

Consistency implies that $\tau_i(m) = 0$ for every agent i, whereas no challenge together with Maskin monotonicity of the SCF f implies that $g(m) = f(\tilde{\theta}) = f(\theta)$. This completes the proof of Theorem 1. We now proceed to establish these three steps. In the rest of the proof, we fix σ as an arbitrary mixed-strategy Nash equilibrium of the game $\Gamma(\mathcal{M}, \theta)$.

3.2.1 Contagion of Truth

Claim 2 *The following two statements hold:*
(a) If agent j sends a truthful first report with σ_j-probability one, then every agent $i \neq j$ must report agent j's type truthfully in his second report with σ_i-probability one.
(b) If every agent $i \neq j$ reports the same type $\tilde{\theta}_j$ of agent j in his second report with σ_i-probability one, then agent j must also report the type $\tilde{\theta}_j$ in his second report with σ_j-probability one.

Proof. We first prove (a). Suppose instead that there exists some agent i, and some message m_i played with σ_i-positive probability which misreports agent j's type in the second report, i.e., $m^2_{i,j} \neq \theta_j$. Let $\tilde{m}_i$ be a message that differs from m_i only in reporting j's type truthfully $\tilde{m}^2_{i,j} = \theta_j$. Such a change in which agent i has influence only on $\tau^1_{i,j}(\cdot)$. For every m_{-i} played with σ_{-i}-positive probability, we consider the following two cases.

Case 1: $m^2_{j,j} = \theta_j$

Since agent j sends a truthful first report with σ_j-probability one, due to the construction of $\tau^1_{i,j}(\cdot)$, we have $\tau^1_{i,j}(m_i, m_{-i}) = -\eta$ whereas $\tau^1_{i,j}(\tilde{m}_i, m_{-i}) = 0$.

Case 2: $m^2_{j,j} \neq \theta_j$

Since agent j sends a truthful first report with σ_j-probability one, according to the construction of $\tau^1_{i,j}(\cdot)$, we have $\tau^1_{i,j}(m_i, m_{-i})$ is either 0 or $-\eta$ whereas $\tau^1_{i,j}(\tilde{m}_i, m_{-i}) = \eta$.

Thus, in terms of transfers, the gain from reporting $\tilde{m}_i$ rather than m_i is at least η, which is larger than the maximal utility loss from the outcome function $g(\cdot)$ by (6). Hence, $\tilde{m}_i$ is a profitable deviation from m_i. As this contradicts the hypothesis that $m_i \in \text{supp}(\sigma_i)$, we have established (a).

We now prove (b). Suppose, on the contrary, that there exists some message m_j played with σ_j-positive probability and agent j reports $m^2_{j,j} \neq \tilde{\theta}_j$. Let $\tilde{m}_j$ be a message that is identical to m_j except that $\tilde{m}^2_{j,j} = \tilde{\theta}_j$. Such a change has influence only on $\tau^2_{j,i}(\cdot)$. According to the construction of $\tau^2_{j,i}(\cdot)$ and since every agent $i \neq j$ reports $\tilde{\theta}_j$ in the second report with σ_i-probability one, agent j saves the penalty of $(I-1)\eta$ from reporting $\tilde{m}_j$ instead of m_j.

Again, since η is greater than the maximal utility difference by (6), we conclude that $\tilde{m}_j$ is a profitable deviation from m_j. This contradicts the hypothesis that m_j is an equilibrium message. Hence, we prove (b). ■

3.2.2 Consistency

Claim 3 shows that in equilibrium, all agents must announce the same state $\tilde{\theta}$ with probability one.

Claim 3 *There exists a state $\tilde{\theta} \in \Theta$ such that every agent announces $\tilde{\theta}$ in their second report with probability one.*

Proof. We consider the following two cases:

Case 1: *Everyone tells the truth in the first report with probability one, i.e., $m_i^1 = \theta_i$ with σ_i-probability one for every agent i.*

It follows directly from Claim 2 that $m_i^2 = \theta$ with σ_i-probability one for every agent i.

Case 2: *Some agent, say, agent i, tells a lie in the first report with σ_i-positive probability.*

That is, there exists $m_i \in$supp(σ_i) such that $m_i^1 \neq \theta_i$. By Claim 1, (m_i, m_{-i}) is consistent with σ_{-i}-probability one. In particular, there exists $\tilde{\theta} \in \Theta$ such that every agent $j \neq i$ must report

$$m_j^2 = m_i^2 = \tilde{\theta} \text{ with } \sigma_j\text{-probability one.} \tag{10}$$

Hence, by Claim 2(b), for every $\tilde{m}_i \in$supp(σ_i), we have

$$\tilde{m}_{i,i}^2 = m_{i,i}^2 = \tilde{\theta}_i. \tag{11}$$

We now prove that for every $\tilde{m}_i \in$supp(σ_i), we have $\tilde{m}_i^2 = m_i^2 = \tilde{\theta}$, which would complete the proof. We prove it by contradiction, i.e., suppose there exists $\tilde{m}_i \in$supp(σ_i) such that

$$\tilde{m}_i^2 \neq m_i^2. \tag{12}$$

Furthermore, (10) and (12) imply that for every agent $j \neq i$, $e_{j,i}(m_j, \tilde{m}_i) = \varepsilon$ with σ_j-probability one, and hence, by Claim 1, agent j must tell the truth in the first report, i.e., $m_j^1 = \theta_j$ with σ_j-probability one, for every $j \neq i$. As a result, Claim 2(a) implies for every agent $j \neq i$

$$\tilde{m}_{i,j}^2 = m_{i,j}^2 = \theta_j \text{ with } \sigma_i\text{-probability one.} \tag{13}$$

Finally, (11) and (13) imply $\tilde{m}_i^2 = m_i^2$, contradicting (12). ■

3.2.3 No Challenge

By Claim 3, there exists a common state $\tilde{\theta} \in \Theta$ with σ_i-probability one for every agent i. We now show in Claim 4 that no one challenges the common state $\tilde{\theta}$.

Claim 4 *No agent challenges with positive probability the common state $\tilde{\theta}$ announced in the second report.*

Proof. Suppose by way of contradiction that $x(\tilde{\theta}, m_i^1) \neq f(\tilde{\theta})$ for some message $m_i \in$supp(σ_i). By Claim 3, we have $x(m_j^2, m_i^1) \neq f(m_j^2)$ for every message $m_j \in$supp(σ_j) and every agent $j \neq i$. It follows that we have $e_{j,i}(m_j, m_i) = \varepsilon$ with σ_j-probability one. By Claim 1, we have $m_j^1 = \theta_j$ with σ_j-probability one and $m_i^1 = \theta_i$. Thus, we obtain $x(\tilde{\theta}, \theta_i) \neq f(\tilde{\theta})$. By the construction of the best challenge scheme, we also have $x(\tilde{\theta}, \theta_i) \in \mathcal{L}_i(f(\tilde{\theta}), \tilde{\theta}_i) \cap \mathcal{SU}_i(f(\tilde{\theta}), \theta_i)$. Then, by (7), every message $\bar{m}_i$ with $x(\tilde{\theta}, \bar{m}_i^1) = f(\tilde{\theta})$ cannot be a best response against σ_{-i}. Indeed, since $x(\tilde{\theta}, \theta_i) \in \mathcal{SU}_i(f(\tilde{\theta}), \theta_i)$, it is a profitable deviation to replace $\bar{m}_i^1$ by θ_i. Hence, $x(\tilde{\theta}, \tilde{m}_i) \neq f(\tilde{\theta})$ and $e_{j,i}(m_j, \tilde{m}_i) = \varepsilon$ for every $\tilde{m}_i \in$supp(σ_i). Then, by Claim 1, we have $\tilde{m}_i^1 = \theta_i$ with σ_i-probability one. Therefore, every agent's first report is truthful with probability one. By Claim 2, we conclude that $\tilde{\theta} = \theta$. Since $x(\tilde{\theta}, \theta_i) \neq f(\tilde{\theta})$, it follows that $x(\tilde{\theta}, \theta_i)$ belongs to the empty intersection $\mathcal{L}_i(f(\theta), \theta_i) \cap \mathcal{SU}_i(f(\theta), \theta_i)$ which is impossible. ■

3.3 Implementation in a Direct Mechanism

In this subsection we illustrate two special cases where our implementing mechanism can be made a direct mechanism. A direct mechanism is a mechanism $((M_i), g, (\tau_i))_{i \in \mathcal{I}}$ in which (i) agents are asked to report the state (i.e., $M_i = \Theta$ for every agent i), and (ii) a unanimous report leads to the socially desirable outcome with no transfers (i.e., $g(\theta, ..., \theta) = f(\theta)$ and $\tau_i(\theta) = 0$ for every agent i and for each state θ). Both cases require three or more agents.

Although direct mechanisms invoke a simpler message space relative to the augmented mechanisms used in the full implementation literature, the literature on partial implementation has attempted to construct mechanisms that are simpler or easier to implement than direct mechanisms, allowing lotteries and transfers. See, for example, Dasgupta and Maskin (2000), Perry and Reny (2002), Ausubel (2004), Edelman, Ostrovsky, and Schwarz (2007), and Milgrom (2007). While our result complements these papers, our main focus is to study full (Nash) implementation in mixed-strategy equilibrium without making use of devices which have no equilibrium such as integer/modulo games.

The first case shows that every Maskin-monotonic SCF is implementable in pure-strategy Nash equilibria in a direct mechanism. Pure-strategy Nash implementation means that we only require that each pure-strategy Nash equilibrium achieve desirable outcomes, i.e., condition (ii) of Definition 1 holds only for pure-strategy Nash equilibrium. Indeed, one might expect that by penalizing disagreement with transfers, the designer can easily obtain a unanimous state announcement without using integer/modulo games. Once there is a unanimous state announcement in equilibrium, Maskin monotonicity will ensure implementation, as it does in Maskin (1999). The following proposition, whose proof can be found in Appendix A.2, formalizes this idea of "penalizing disagreement."

Proposition 1 *Suppose that there are at least three agents and the SCF f satisfies Maskin monotonicity. Then, f is implementable in pure-strategy Nash equilibria in a direct mechanism.*

The idea of "penalizing disagreement" becomes problematic once we consider mixed-strategy equilibria. Indeed, the direct mechanism which we construct in proving Proposition 1 is reminiscent of modulo games. It is well known that modulo games admit unwanted mixed-strategy equilibria; hence, it should come at no surprise that our direct mechanism also admits unwanted mixed-strategy equilibria.

The second case establishes mixed-strategy Nash implementation in direct mechanisms by considering a state space which has a product structure, i.e., $\Theta = \times_{i=1}^{I} \Theta_i$. We state the following result whose proof is relegated to Appendix A.3:

Proposition 2 *Suppose that there are at least three agents, $\Theta = \times_{i=1}^{I} \Theta_i$, and the SCF f satisfies Maskin monotonicity. Then, f is implementable in mixed-strategy Nash equilibria in a direct mechanism.*

Proposition 2 demonstrates an extreme case where mixed-strategy Nash implementation can be achieved not only without "unnatural devices" but also in a direct mechanism. Product state space naturally arises in a Bayesian setup with a full-support common prior. While such a full-support prior is precluded by the complete information assumption, it is consistent with "almost complete information" which we are about to introduce in Section 4.

Remark: Our notion of direct (revelation) mechanism is adopted by, for example, Dutta and Sen (1991) and (Osborne and Rubinstein, 1994, Definition 179.2) but different from the notion adopted by Dasgupta, Hammond, and Maskin (1979) in which agents report only their own types/preferences. In particular, Theorem 7.1.1 of Dasgupta, Hammond, and Maskin (1979) shows that only strategy-proof SCFs are "partially" implemented in Nash equilibrium by the latter notion of direct mechanisms.

3.4 Implementation without Transfer

The following example illustrates that without any domain restriction such as quasilinear preferences with transfers, it is impossible to achieve mixed-strategy Nash implementation in any finite mechanism:

Example 1 (Example 4 of Jackson (1992)) *Consider the environment with two agents* 1 *and* 2*. Suppose that there are four alternatives* $a, b, c,$ *and* d *and two states* θ *and* θ'*. Suppose that agent 1 has the state-independent preference* $a \succ_1 b \succ_1 c \sim_1 d$ *and agent* 2 *has the preference* $a \succ_2^\theta b \succ_2^\theta d \succ_2^\theta c$ *at state* θ *and preference* $b \succ_2^{\theta'} a \succ_2^{\theta'} c \sim_2^{\theta'} d$ *at state* θ'*. Consider the SCF* f *such that* $f(\theta) = a$ *and* $f(\theta') = c$*.*

Jackson (1992) argues that for every finite mechanism which implements f in pure-strategy equilibria, there must also exist a "bad" mixed-strategy Nash equilibrium such that at state θ' the equilibrium outcome differs from c with positive probability.[18] Since f satisfies Maskin monotonicity, this example shows that it is impossible to implement any Maskin-monotonic SCF in mixed-strategy equilibria by a finite mechanism without imposing any domain restrictions on the environment. However, by making use of the construction in Theorem 1 and insight of Abreu and Matsushima (1992, 1994), the SCF f in Jackson's example can actually be implemented in mixed-strategy equilibria with arbitrarily small transfers off the equilibrium (see Theorem 3 and footnote 24).

[18] We briefly recap the argument here. Let $\mathcal{M}$ be a finite mechanism which implements the SCF f in pure-strategy Nash equilibria. Consider a mechanism which restricts the message space of $\mathcal{M}$ such that, against any message of agent i, the opponent agent j can choose a message that induces either outcome a or b. The restricted set of messages is nonempty since the equilibrium message profile at state θ leads to outcome a. It follows that at state θ', the game induced by the restricted mechanism must have a mixed-strategy Nash equilibrium. Moreover, the equilibrium outcome must be a or b with positive probability; otherwise, agent 2 can deviate to induce outcome a or b with positive probability. Since c and d are ranked lowest by both agents at state θ', this mixed-strategy equilibrium must remain an equilibrium at state θ' in the game induced by $\mathcal{M}$; moreover, the equilibrium fails to achieve $f(\theta') = c$.

4 Robustness to Information Perturbations

4.1 Settings and Result

Chung and Ely (2003) and Aghion, Fudenberg, Holden, Kunimoto, and Tercieux (2012) consider a designer who not only wants all equilibria of her mechanism to yield a desirable outcome under complete information, but is also concerned about the possibility that agents may entertain small doubts about the true state. They argue that such a designer should insist on implementing the SCF in the closure of a solution concept as incomplete information about the state vanishes. Chung and Ely (2003) adopt undominated Nash equilibrium and Aghion, Fudenberg, Holden, Kunimoto, and Tercieux (2012) adopt subgame-perfect equilibrium as a solution concept in studying the robustness issue.

To allow for information perturbations, suppose that the agents do not observe the state directly but are informed of the state via signals. The set of agent i's signals is denoted as S_i, which is identified with Θ, i.e., $S_i \equiv \Theta$.[19] A signal profile is an element $s = (s_1, ..., s_I) \in S \equiv \times_{i \in \mathcal{I}} S_i$. When the realized signal profile is s, agent i observes only his own signal s_i. Let s_i^θ denote the signal which corresponds to state θ and we write $s^\theta = \left(s_i^\theta\right)_{i \in I}$. State and signals are drawn from some prior distribution over $\Theta \times S$. In particular, complete information can be modelled as a prior μ such that $\mu(\theta, s) = 0$ whenever $s \neq s^\theta$. Such a μ will be called a *complete-information prior*. We assume that for each agent i, the marginal distribution on i's signals places a strictly positive weight on each of i's signals, that is, $\text{marg}_{S_i} \mu(s_i) > 0$ for every $s_i \in S_i$, so that the posterior belief given every signal is well defined. For every prior ν, we also write $\nu(\cdot|s_i)$ for the conditional distribution of ν on signal s_i.

The distance between two priors is measured by the uniform metric. That is, for every two priors μ and ν, we have $d(\mu, \nu) \equiv \max_{\theta, s} |\mu(\theta, s) - \nu(\theta, s)|$. Write $\nu^\varepsilon \to \mu$ if $d(\nu^\varepsilon, \mu) \to 0$ as $\varepsilon \to 0$. A prior ν together with a mechanism $\mathcal{M} = ((M_i, \tau_i)_{i \in \mathcal{I}}, g)$ induces an incomplete-information game which we denote by $\Gamma(\mathcal{M}, \nu)$. A (mixed-)strategy of agent i is now a mapping $\sigma_i : S_i \to \Delta(M_i)$.

Generally, the designer may resort to a solution concept $\mathcal{E}$ for the game $\Gamma(\mathcal{M}, \nu)$ (such

[19] We adopt this formulation from Chung and Ely (2003) and Aghion, Fudenberg, Holden, Kunimoto, and Tercieux (2012). Our result holds for any alternative formulation under which the (Bayesian) Nash equilibrium correspondence has a closed graph.

as BNE) which induces a set of mappings from $\Theta \times S$ to X, which we call *acts*, following Chung and Ely (2003). For instance, each BNE σ induces the act α_σ with $\alpha_\sigma(\theta, s) \equiv \sigma(s) \circ \left(g, (\tau_i)_{i\in\mathcal{I}}\right)^{-1}$ where we abuse the notation to identify the finite-support distribution $\sigma(s) \circ \left(g, (\tau_i)_{i\in\mathcal{I}}\right)^{-1}$ on X with an allocation in X. We denote the set of acts induced by the solution concept $\mathcal{E}$ as $\mathcal{E}(\mathcal{M}, \nu)$. Also here we endow X with a topology with respect to which the utility function u_i is continuous on X.[20] We now define $\overline{\mathcal{E}}$-implementation as follows.

Definition 3 *An SCF f is $\overline{\mathcal{E}}$-implementable under the complete-information prior μ if there exists a mechanism $\mathcal{M} = ((M_i), g, (\tau_i))_{i\in\mathcal{I}}$ such that for every $(\theta, s) \in$supp(μ) and every sequence of priors $\{\nu^n\}$ converging to μ, the following two requirements hold: (i) there is a sequence of acts $\{\alpha_n\}$ with $\alpha_n \in \mathcal{E}(\mathcal{M}, \nu_n)$ such that $\alpha_n(\theta, s) \to f(\theta)$; (ii) for every sequence of acts $\{\alpha_n\}$ with $\alpha_n \in \mathcal{E}(\mathcal{M}, \nu_n)$, we have $\alpha_n(\theta, s) \to f(\theta)$.*

Chung and Ely (2003) and Aghion, Fudenberg, Holden, Kunimoto, and Tercieux (2012) show that Maskin monotonicity is a necessary condition for $\overline{UNE}$-implementation and $\overline{SPE}$-implementation, respectively.[21] The result of Chung and Ely (2003) implies that implementation of a non-Maskin-monotonic SCF in undominated Nash equilibria such as the result in Abreu and Matsushima (1994) is necessarily vulnerable to information perturbations. Moreover, both Chung and Ely (2003, Theorem 2) and Aghion, Fudenberg, Holden, Kunimoto, and Tercieux (2012) establish the sufficiency result by using infinite mechanisms with integer games and restricting attention to pure-strategy equilibria. This raises the question as to whether their robustness test may be too demanding when it is applied to finite mechanisms such as the implementing mechanism of Jackson, Palfrey, and Srivastava (1994), that of Abreu and Matsushima (1994), or the simple mechanism in Section 5 of Moore and Repullo (1988), where mixed-strategy equilibria have to be taken seriously.

The canonical mechanism which we propose in the proof of Theorem 1 is indeed finite, and we show that this finite mechanism implements every Maskin-monotonic SCF in mixed-strategy Nash equilibria. Since the solution concept of Bayesian Nash equilibrium, viewed

[20] For instance, this is the case if A is a (Hausdorff) topological space, $v_i(a, \theta)$ is bounded and continuous in a, and $\Delta(A)$ is endowed with the weak*-topology. Then, $X \equiv \Delta(A) \times \mathbb{R}^I$, endowed with the product topology, is also a Hausdorff topological space.

[21] Aghion, Fudenberg, Holden, Kunimoto, and Tercieux (2012) adopt sequential equilibrium as the solution concept for the incomplete-information game $\Gamma(\mathcal{M}, \nu)$.

as a correspondence on priors, has a closed graph, this finite mechanism also achieves $\overline{NE}$-implementation. We now obtain the following result as a corollary of Theorem 1 in our setup with lotteries and transfers.

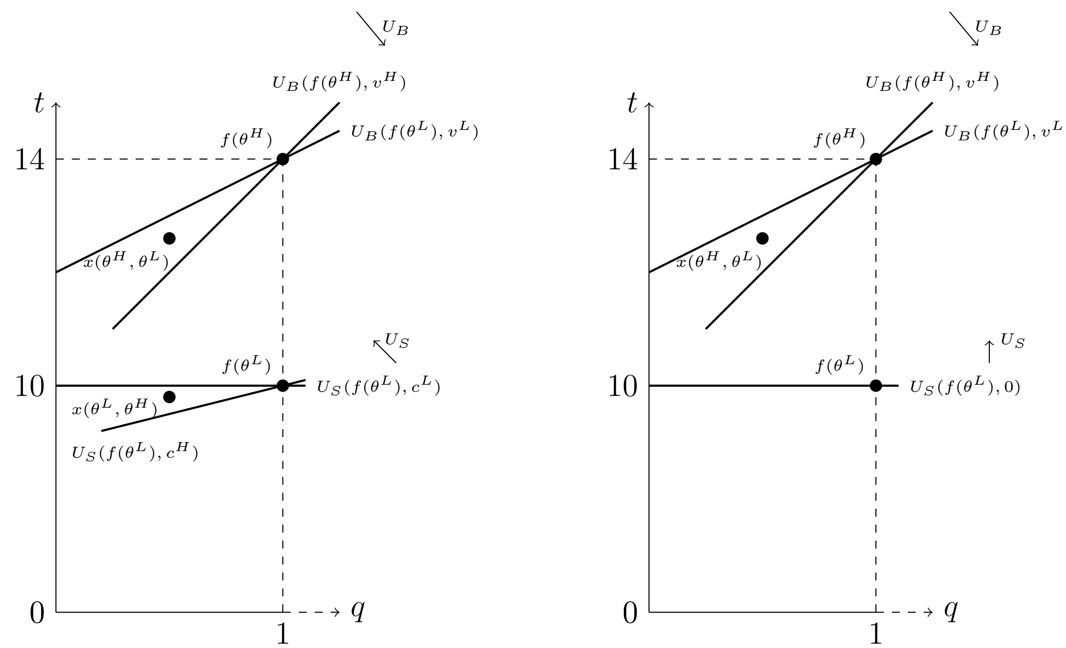
Proposition 3 *Let $\mathcal{E}$ be a solution concept such that $\varnothing \neq \mathcal{E}(\mathcal{M}, \mu) \subseteq NE(\mathcal{M}, \mu)$ for each finite mechanism $\mathcal{M}$ and a complete-information prior μ. Then, every Maskin-monotonic SCF f is $\overline{\mathcal{E}}$-implementable.*

The condition $\varnothing \neq \mathcal{E}(\mathcal{M}, \mu) \subseteq NE(\mathcal{M}, \mu)$ is satisfied for virtually every refinement of Nash equilibrium, because we allow for mixed-strategy equilibrium and $\Gamma(\mathcal{M}, \mu)$ is a finite game.

4.2 Application: The Hart-Moore Example

As an application of our result, we revisit the following example from Hart and Moore (2003) which is also recapped in Aghion, Fudenberg, Holden, Kunimoto, and Tercieux (2012). A seller S has a divisible unit of an object to sell to a buyer B. The object is worth v to the buyer and c to the seller. The pair (v, c) is commonly observed by both parties but unverifiable to the designer. The designer can impose transfers and hence each outcome is a triplet (q, t_B, t_S) with $q \in [0, 1]$ representing the quantity of the good being traded, t_B the price paid by the buyer, and t_S the payment received by the seller. Given every outcome (q, t_B, t_S), the buyer's utility is $u_B = qv + t_B$, and the seller's utility is $u_S = t_S - qc$. We identify state θ^H with the pair $\left(v^H, c^H\right)$ and state θ^L with $\left(v^L, c^L\right)$.

Suppose that the designer seeks to implement an efficient allocation rule according to which the good is always traded with prices $\left(t^L, t^H\right)$, where t^L (resp. t^H) stands for the price which the buyer pays to the seller at state θ^L (resp. state θ^H). That is, the SCF can be written as $f\left(\theta^L\right) = \left(1, -t^L, t^L\right)$ and $f\left(\theta^H\right) = \left(1, -t^H, t^H\right)$. In the Hart-Moore example, they set $v^H = 14$, $v^L = 10$, and $c^H = c^L = 0$; moreover, $t^H = v^H$ and $t^L = v^L$. That is, the buyer pays his value and all the surplus goes to the seller. Clearly, the resulting SCF in Hart and Moore (2003) is not Maskin-monotonic. Indeed, while the socially desirable outcomes differ, there is no whistle-blower for the misreported θ^H at state θ^L: if at state θ^L the buyer prefers $f\left(\theta^L\right)$ to an allocation, then the buyer will still prefer $f\left(\theta^L\right)$ to the allocation at state θ^H; moreover, the seller has the same preference in both states.



(a) Maskin-monotonic

(b) non-Maskin-monotonic

Figure 1: Hart-Moore Example

Figure 1a shows that the non-Maskin-monotonicity of the SCF actually represents a knife-edge case. To see this, suppose that we now have $c^H > c^L = 0$, while keeping $v^H > v^L$. The buyer can serve as a whistle-blower for the misreported θ^H at state θ^L with the test allocation $x\left(\theta^H, \theta^L\right)$. Likewise, the seller can serve as a whistle-blower for the misreported θ^L at state θ^H with the allocation $x\left(\theta^L, \theta^H\right)$. As a result, f becomes Maskin-monotonic and in fact it remains so for every t^L between c^L and v^L, and every t^H between c^H and v^H. It is also clear from Figure 1b that if $c^H = c^L = 0$ instead, then we can no longer find the test allocation $x\left(\theta^L, \theta^H\right)$, which explains why f is not Maskin-monotonic in the Hart-Moore example.

To implement a non-Maskin-monotonic SCF, the literature appeals to implementation with refinements of Nash equilibrium. For instance, the well-known Irrelevance Theorem of nonverifiable information due to Maskin and Tirole (1999) is based on the implementation in subgame-perfect equilibria via the mechanism proposed by Moore and Repullo (1988). In contrast, Theorem 3 of Aghion, Fudenberg, Holden, Kunimoto, and Tercieux (2012) shows that only a Maskin-monotonic SCF can be implemented in subgame-perfect equilibria in a manner that is robust to information perturbations. While this appears to be troublesome at first glance, we observe here that once we move from the knife-edge case to the case in which $c^H > c^L$, our Theorem 1 implies that the Maskin-monotonic SCF f can be implemented in mixed-strategy Nash equilibria in a finite mechanism. Moreover, our Proposition 3 shows that the implementation is robust to every small information perturbations in the sense of Aghion, Fudenberg, Holden, Kunimoto, and Tercieux (2012).

Our argument here is also related to the idea of virtual implementation. It is well known from the result of Abreu and Matsushima (1992) that if the designer can make use of lotteries, then every arbitrary SCF (such as the one in the Hart-Moore example) can be perturbed slightly to satisfy Maskin monotonicity and the perturbed Maskin-monotonic SCF is implemented in mixed-strategy Nash equilibria using a finite mechanism. In this vein, the virtual implementation results due to Abreu and Matsushima (1992) can be recast as proving mixed-strategy Nash implementation for a suitably perturbed Maskin-monotonic SCF. However, their results cannot be applied to an arbitrary Maskin monotonic SCF such as one (with $c^H > c^L$) in our revised Hart-Moore example. Here we complement their result by proving that *every* Maskin-monotonic SCF can be implemented in mixed-strategy Nash equilibria in a finite mechanism, with the help of lotteries and transfers.[22]

5 Extensions

We now establish several extensions of our main result. In Section 5.1, we extend our result to the case of social choice correspondences (henceforth, SCCs). Section 5.2 shows how the designer can modify the implementing mechanism to make the size of transfers arbitrarily small. In Section 5.3, we extend our results to a setting with an infinite state space. Finally, by making use of the infinite state space extension, we study the ordinal approach to Nash implementation in Section 5.4. To focus on studying each of the extensions, we will not discuss any combination of multiple extensions. For instance, we will study the case of SCC only in Section 5.1 but still restrict attention to the case of SCFs in the remaining sections.

The proofs of these extensions share ideas similar to our main result but involve more

[22]Arguably, our solution concept of mixed-strategy equilibrium is stronger than iterated deletion of strictly dominated strategies invoked by Abreu and Matsushima (1992). In Chen, Kunimoto, Sun, and Xiong (2020), we show that with lotteries and transfers, rationalizable implementation can be achieved by a finite mechanism if and only if the SCF satisfies a stronger monotonicity condition called Maskin monotonicity* introduced by Bergemann, Morris, and Tercieux (2011).

technical details. We relegate them to the appendix; moreover, to minimize the technicality, hereafter we assume that the set A (of pure alternatives) is finite.

5.1 Social Choice Correspondences

A large portion of the implementation literature strives to deal with social choice correspondences (hereafter, SCCs), i.e., multi-valued social choice rules. In this section, we extend our Nash implementation result to cover the case of SCCs. We suppose that the designer's objective is specified by an SCC $F : \Theta \rightrightarrows X$; and for simplicity, we assume that $F(\theta)$ is a finite set for each state $\theta \in \Theta$. This includes the special case where the co-domain of F is A. Following Maskin (1999), we first define the notion of Nash implementation for SCCs.

Definition 4 *An SCC F is **implementable in mixed-strategy Nash equilibria** if there exists a mechanism $\mathcal{M} = ((M_i, \tau_i)_{i\in\mathcal{I}}, g)$ such that for every state $\theta \in \Theta$, the following two conditions are satisfied: (i) for every $x \in F(\theta)$, there exists a pure-strategy Nash equilibrium m in the game $\Gamma(\mathcal{M}, \theta)$ such that $g(m) = x$ and $\tau_i(m) = 0$ for every agent $i \in \mathcal{I}$; and (ii) for every $m \in supp\ (NE(\Gamma(\mathcal{M}, \theta)))$, we have $supp(g(m)) \subseteq F(\theta)$ and $\tau_i(m) = 0$ for every agent $i \in \mathcal{I}$.*

Second, we state the definition of Maskin monotonicity for SCCs.

Definition 5 *Say an SCC F satisfies **Maskin monotonicity** if, for each pair of states $\tilde{\theta}$ and θ and $x \in F(\tilde{\theta})\backslash F(\theta)$, there is some agent $i \in \mathcal{I}$ such that*

$$\mathcal{L}_i(x, \tilde{\theta}_i) \cap \mathcal{SU}_i(x, \theta_i) \neq \varnothing.$$

We now state our Nash implementation result for SCCs and relegate the proof to Appendix 2.[23]

Theorem 2 *Suppose there are at least three agents. An SCC F is implementable in mixed-strategy Nash equilibria if and only if it satisfies Maskin monotonicity.*

[23] When there are only two agents, we can still show that every Maskin-monotonic SCC F is **weakly** implementable in Nash equilibria, namely that there exists a mechanism which has a pure-strategy Nash equilibrium and satisfies requirement (ii) in Definition 4.

In comparison with our result for SCFs, the proof of Theorem 2 needs to overcome additional difficulties. In the case of SCFs, when the agents' second reports are consistent at a common state $\tilde{\theta}$, they will be associated with a single outcome $f(\tilde{\theta})$. Hence, if agent i's second report is challenged, then *every* second report which is played by any agent with positive probability must also be challenged in equilibrium. That implies (by Claim 1) that every agent must tell the truth in their first and second report, which leads to the contradiction in the proof of Claim 4.

In the case of SCCs, however, each allocation $x \in F(\theta)$ has to be implemented as the outcome of some pure-strategy equilibrium. Hence, each agent must also report an allocation to be implemented. It also follows that a challenge scheme for an SCC must be defined for a type θ_i to challenge a pair $(\tilde{\theta}, x)$ with $x \in F(\tilde{\theta})$. As a result, even when the agents' second reports are consistent at state $\tilde{\theta}$ (still true by Claim 3), they might still be randomizing between two allocation x and x' in $F(\tilde{\theta})$ such that $(\tilde{\theta}, x)$ is challenged and yet $(\tilde{\theta}, x')$ is not. Hence, we cannot follow a similar argument as in Claim 1 to derive a contradiction. We build on the implementing mechanism in Section 3.1 so that agent i will not report $(\tilde{\theta}, x)$ which can be challenged/whistle-blown either by (i) agent $j \neq i$ or by (ii) agent i himself. We ensure Case (i) by imposing in the situation a large penalty on agent i, whereas we deal with Case (ii) by allowing agent i to challenge himself without penalty.

Remark. Mezzetti and Renou (2012b) also consider deterministic social choice correspondences in a separable environment considered by Jackson, Palfrey, and Srivastava (1994) and identify a condition (top-D inclusiveness) under which an SCC is mixed Nash implementation in finite mechanisms if and only if it satisfies *set-monotonicity* proposed by Mezzetti and Renou (2012a). There are several differences between our Theorem 2 and their result. First, Mezzetti and Renou (2012b) require only the existence of mixed strategy equilibria but we follow Maskin (1999) in requiring the existence of a pure strategy equilibrium. Second, Mezzetti and Renou (2012b) consider an ordinal setup, while we consider a cardinal setup. This is the reason why Mezzetti and Renou (2012b) use set-monotonicity as a relevant necessary condition for characterizing their ordinal Nash implementation (which we study for the case of SCF in Section 5.4). Third, our quasilinear environments with transfers are more restrictive than separable environments considered by Mezzetti and Renou (2012b). Finally, Mezzetti and Renou (2012b) need top D-inclusiveness as an additional efficiency condition, which requires that there exist at least one agent for whom the social choice correspondence contains this agent's best outcome within the range of the social choice correspondence for every state of the world, whereas we impose no condition beyond Maskin-monotonicity in the quasilinear environments with transfers.

5.2 Small Transfer

One potential deficiency of the mechanism we propose for Theorem 1 is that the size of transfers may be large. However, since we allow lottery allocations, we can use the technique introduced by Abreu and Matsushima (1994) to show that if the SCF satisfies Maskin monotonicity in the restricted domain without any transfer, then it is Nash-implementable with arbitrarily small transfers.

We first propose a notion of Nash implementation with bounded transfers off the equilibrium and still no transfer on the equilibrium.

Definition 6 *An SCF $f : \Theta \to \Delta(A)$ is implementable in mixed-strategy Nash equilibria* ***with transfers bounded by*** $\bar{\tau}$ *if there exists a mechanism $\mathcal{M} = ((M_i, \tau_i)_{i\in\mathcal{I}}, g)$ such that for every state $\theta \in \Theta$ and $m \in M$, (i) there exists a pure-strategy Nash equilibrium in the game $\Gamma(\mathcal{M}, \theta)$; (ii) for each m in supp $(NE(\Gamma(\mathcal{M}, \theta)))$, we have $g(m) = f(\theta)$ and $\tau_i(m) = 0$ for every agent $i \in \mathcal{I}$; and (iii) $|\tau_i(m)| \leq \bar{\tau}$ for every $m \in M$ and every agent $i \in \mathcal{I}$.*

Next, we propose a notion of Nash implementation in which there are no transfers on the equilibrium and only arbitrarily small transfers off the equilibrium.

Definition 7 *An SCF f is implementable in mixed-strategy Nash equilibria* ***with arbitrarily small transfers*** *if for every $\bar{\tau} > 0$, the SCF f is implementable in Nash equilibria with transfer bounded by $\bar{\tau}$.*

We say that an SCF f satisfies Maskin monotonicity in the restricted domain $\Delta(A)$ if $f(\tilde{\theta}) \neq f(\theta)$ implies that there are an agent i and some lottery $x(\tilde{\theta}, \theta_i)$ in $\Delta(A)$ such that $x(\tilde{\theta}, \theta_i)$ belongs to $\mathcal{L}_i(f(\tilde{\theta}), \tilde{\theta}_i) \cap \mathcal{SU}_i(f(\tilde{\theta}), \theta_i)$. Here, for $(\ell, \theta_i) \in \Delta(A) \times \Theta_i$, we use $\mathcal{L}_i(\ell, \theta_i)$ to denote the lower-contour set at allocation ℓ in $\Delta(A)$ for type θ_i, i.e.,

$$\mathcal{L}_i(\ell, \theta_i) = \{\ell' \in \ell : v_i(\ell, \theta_i) \geq v_i(\ell', \theta_i)\}.$$

In a similar fashion, $\mathcal{SU}_i$ is defined. Clearly, Maskin monotonicity in the restricted domain $\Delta(A)$ is stronger than Maskin monotonicity in the domain X, as the former requires that the test allocation be a lottery over alternative without transfer. In Appendix A.6, we assume there are at least three agents, and prove the following result.[24]

Theorem 3 *Suppose there are at least three agents. An SCF $f_A : \Theta \to \Delta(A)$ is implementable in mixed-strategy Nash equilibria with arbitrarily small transfers if f_A satisfies Maskin monotonicity in the restricted domain.*

5.3 Infinite State Space

One significant assumption we have made in this paper is that the state space is finite. In Appendix A.7, we extend Theorem 1 to an infinite state space in which the agents' utility functions are continuous. A similar extension was raised as an open question for virtual implementation in Abreu and Matsushima (1992) (see their Section 5) and it has not been answered to our knowledge.

In appendix A.7, we construct an extension of the implementing mechanism for mixed-strategy Nash implementation which accommodates an infinite state space. We state the result as follows:

Theorem 4 *Suppose that Θ is a Polish space. Then, an SCF f satisfies Maskin monotonicity if and only if there exists a mechanism which implements f in mixed-strategy Nash equilibria. Moreover, if Θ is compact and both the utility function $\{v_i(a, \cdot)\}_{a \in A}$ and the SCF are continuous functions on Θ, then the implementing mechanism has a compact message space together with a continuous outcome function and continuous transfer rules.*

One notable feature of this extension is that as long as the setting is compact and continuous, the resulting implementing mechanism will also be compact and continuous. This feature ensures that best responses are always well defined in our mechanism; hence, it differentiates our construction from the traditional approach of invoking integer games.

The proof of Theorem 4 needs to overcome two difficulties. First, in a finite state space, the transfer rules $\tau^1_{i,j}$ and $\tau^2_{i,j}$ which we define in (8) and (9) impose either a large

[24] In the case with only two agents, Theorem 3 still holds if there exists an alternative $w \in A$ which is the worst alternative for any agent at any state. In this case, we can simply modify the "voting rule" ϕ in the proof of Theorem 3 to be $\phi(m^h) = f(\tilde{\theta})$ if both agents announce a common type profile which identifies a state $\tilde{\theta}$ in m^h; and $\phi(m^h) = w$ otherwise. In particular, $w = c$ in Example 4 of Jackson (1992) and thus the SCF can be implemented with arbitrarily small transfer.

penalty and/or a large reward as long as the designer sees a discrepancy in the agents' announcements. With a continuum of states/types, however, such a drastic change in transfer scale is precluded by the continuity requirement. Hence, our first challenge is to suitably define $\tau^1_{i,j}$ and $\tau^2_{i,j}$ so that they vary continuously yet still incentivize truth-telling.

Second, unlike the case with a finite Θ, in an infinite setting we know of no way to construct a challenge scheme by pre-selecting a test allocation in a continuous manner. As a result, we cannot have the agents report their type, let alone the true type, to cast a challenge to state $\tilde{\theta}$. Instead, we will restore continuity of the outcome function by asking them to report a test allocation x directly. Despite the change, we will establish a counterpart of Condition (7) as Lemmas 5 and 6 in Appendix A.7.

5.4 The Ordinal Approach

So far we have assumed that the agents are expected utility maximizers. This leaves open the issue as to whether, and to what extent, our implementation result depends on the designer's knowledge about the cardinalization of the agents' preferences over lotteries. To address the issue, we discuss here how our result can accommodate an ordinal setting.

First, we introduce the notion of *ordinal Nash implementation*. The notion requires that the mixed-strategy Nash implementation is obtained for *any* cardinal representation of the ordinal preferences over the finite set of alternatives A. Formally, we follow the approach proposed by Mezzetti and Renou (2012).

Suppose that at state θ, agents only have common knowledge about their ordinal rankings over the set of pure alternatives A. We write the induced ordinal preference profile at state θ by $(\succeq_i^\theta)_{i\in\mathcal{I}}$. We also assume no redundancy, i.e., i.e., whenever $\theta \neq \theta'$, we must have $\succeq_i^\theta \neq \succeq_i^{\theta'}$ for some agent i. It is taken for granted that distinct pair of $\succeq_i^\theta$and $\succeq_i^{\theta'}$induce different preference orderings over A, and also that a player is never indifferent over all elements of A. We denote a profile of indices $(v_i)_{i\in\mathcal{I}}$ (defined on A) as a *cardinal representation* of $(\succeq_i^\theta)_{i\in\mathcal{I},\theta\in\Theta}$, i.e., for each pair of alternatives a and a', agent i, and state θ, we have

$$v_i(a;\theta_i) \geq v_i(a';\theta_i) \Leftrightarrow a \succeq_i^\theta a'.$$

We assume that each function v_i takes a value in $[0,1]$. Again, each cardinal representation v_i induces an expected utility function on $\Delta(A)$ which, by abusing the notation, we also denote by v_i. We denote by V_i^θ the set of all cardinal representations $v_i(\cdot,\theta_i)$ of $\succeq_i^\theta$. Following

Mezzetti and Renou (2012), we focus our discussion on the case of a deterministic SCF, i.e., $f : \Theta \to A$. We say an SCF f is ordinally Nash implementable if it is implementable in mixed-strategy Nash equilibria independently of the cardinal representation. We formalize this idea in the following definition.

Definition 8 *An SCF f is said to be ordinally Nash implementable if there exists a mechanism $\mathcal{M}$ such that, for every $\theta \in \Theta$ and every profile of cardinal representations $v = (v_i)_{i\in\mathcal{I}}$ of $(\succeq_i^\theta)_{i\in\mathcal{I},\theta\in\Theta}$, the following two conditions are satisfied: (i) there exists a pure-strategy Nash equilibrium m in the game $\Gamma(\mathcal{M}, \theta, v)$ such that $g(m) = f(\theta)$ and $\tau_i(m) = 0$ for every $i \in \mathcal{I}$; and (ii) for every $m \in supp(NE(\Gamma(\mathcal{M}, \theta, v)))$, we have $supp(g(m)) = f(\theta)$ and $\tau_i(m) = 0$ for every $i \in \mathcal{I}$.*

In Appendix A.8, we introduce the notion of *ordinal almost monotonicity* proposed by Sanver (2006). Roughly speaking, an SCF satisfies ordinal almost monotonicity if whenever the SCF designates different outcomes at states θ and θ', there must be an agent with a *deterministic* test allocation which displays a suitable preference reversal with respect to the socially desirable outcome at the two states. We obtain that an SCF f is ordinally Nash implementable if and only if it satisfies ordinal almost monotonicity. Thus, we obtain the following result:

Theorem 5 *An SCF f is ordinally Nash implementable if and only if it satisfies ordinal almost monotonicity.*

Somewhat surprisingly, Theorem 4 provides the key to prove this result. Indeed, we show that ordinal almost monotonicity of an SCF implies Maskin monotonicity of the SCF consistently extended from Θ (now the set of ordinal preference profiles) to the set of cardinalizations. The latter becomes an infinite state space and the extension renders Theorem 4 applicable.

Instead of relying on ordinal almost monotonicity, Mezzetti and Renou (2012) propose a notion called set-monotonicity for SCCs. They show that the notion of set-monotonicity is weaker than Maskin monotonicity and is necessary and "almost sufficient" in their notion of implementation in mixed-strategy Nash equilibria. There are three further differences between the results of Mezzetti and Renou (2012) and ours. First, Mezzetti and Renou (2012) require only the existence of mixed-strategy equilibria but we follow Maskin (1999) in requiring the existence of a pure-strategy equilibrium. The difference makes our ordinal implementation notion more demanding than that of Mezzetti and Renou (2012). Second, we use monetary transfers, while Mezzetti and Renou (2012) do not. Indeed, ordinal almost monotonicity is weaker than set-monotonicity. Yet, since we allow transfers, we are able to characterize our stronger notion of ordinal mixed-strategy Nash implementation à la Maskin (1999) for the case of SCFs by means of the weaker condition of ordinal almost monotonicity. Finally, Mezzetti and Renou (2012) also study the case of SCCs which we omit here.

6 Concluding Remarks

Despite its tremendous success, implementation theory has also been criticized on various fronts. A major criticism is that the mechanisms used to achieve full implementation are not "natural," as reflected in the quote from Jackson (1992) at the beginning of this paper. To address such criticism, Jackson (1992) proposes that we restrict attention to "natural mechanisms" and study which SCFs can be fully implemented, even at the cost of restricting attention to specific environments.

We consider our results as an important step in advancing the Jackson program. Specifically, we propose to recast an implementation problem by requiring that the implementing mechanism be finite/well-behaved and have no unwanted mixed-strategy equilibrium. Such requirements are to be anticipated, when the implementation setup of interest is indeed finite/well-behaved to start with. We prove a first set of benchmark results on mixed-strategy Nash implementation by considering environments with lotteries and transfers. We also show that our results are robust to information perturbations and amenable to prominent extensions such as SCCs, small transfers, infinite settings, and ordinal settings.

There have been some past attempts at tackling implementation in Bayesian (incomplete-information) environments, such as Mookherjee and Reichelstein (1990) and d'Aspremont, Crémer, and Gérard-Varet (2003). Both papers construct an indirect yet well-behaved mechanism which achieves full implementation in pure-strategy Bayesian Nash equilibrium in quasilinear environments with transfers. What has remained unknown is how one can also handle mixed strategies by a well-behaved mechanism in a Bayesian environment. There is also an attempt at tackling robust implementation in a well-behaved mechanism. Bergemann and Morris (2009) characterize when robust full implementation is possible in the direct mechanism (a well-behaved mechanism) in what they call the single-crossing monotone aggregator environments. In a more general environment, the focus on direct mechanisms exhibits a loss of generality for robust full implementation. Therefore, our results also invite extensions to a Bayesian setup (Jackson (1991)) and a robust setup (Bergemann and Morris (2009)).

Our implementation results are achieved by imposing transfers off the equilibrium. This feature is intimately related to the burgeoning literature on repeated implementation, such as Lee and Sabourian (2011) and Mezzetti and Renou (2017), in which continuation values can serve as transfers in our construction.[25] We leave the extensions and the precise connections for future research.

A Appendix

In this Appendix, we provide the proofs omitted from the main body of the paper.

A.1 Proof of Lemma 2

First, we elaborate the proof of Lemma 2 here.

Proof. Consider a challenge scheme $\bar{x}(\cdot,\cdot)$. First, we show that we can modify $\bar{x}(\cdot,\cdot)$ into a new challenge scheme $x(\cdot,\cdot)$ such that

$$x(\tilde{\theta},\theta_i) \neq f(\tilde{\theta}) \text{ and } x(\tilde{\theta},\theta_i') \neq f(\tilde{\theta}) \Rightarrow u_i(x(\tilde{\theta},\theta_i),\theta_i) \geq u_i(x(\tilde{\theta},\theta_i'),\theta_i). \tag{14}$$

To construct $x(\cdot,\cdot)$, for each player i, we distinguish two cases: (a) if $\bar{x}(\tilde{\theta},\theta_i) = f(\tilde{\theta})$ for all $\theta_i \in \Theta_i$, then set $x(\tilde{\theta},\theta_i) = \bar{x}(\tilde{\theta},\theta_i) = f(\tilde{\theta})$; (b) if $\bar{x}(\tilde{\theta},\theta_i) \neq f(\tilde{\theta})$ for some $\theta_i \in \Theta_i$, then define $x(\tilde{\theta},\theta_i)$ as the most preferred allocation of type θ_i in the finite set

$$X(\tilde{\theta}) = \left\{\bar{x}(\tilde{\theta},\theta_i') : \theta_i' \in \Theta_i \text{ and } \bar{x}(\tilde{\theta},\theta_i') \neq f(\tilde{\theta})\right\}.$$

Since $\bar{x}(\tilde{\theta},\theta_i') \in \mathcal{L}_i(f((\tilde{\theta}),\tilde{\theta}_i)$, we have $u_i(x(\tilde{\theta},\theta_i),\tilde{\theta}_i) \leq u_i(f(\tilde{\theta}),\tilde{\theta}_i)$; moreover, since $x(\tilde{\theta},\theta_i)$ as the most preferred allocation of type θ_i in $X(\tilde{\theta})$ and $\bar{x}(\tilde{\theta},\theta_i) \in \mathcal{SU}_i(f(\tilde{\theta}),\theta_i)$, it follows that $u_i(x(\tilde{\theta},\theta_i),\theta_i) > u_i(f\left(\tilde{\theta}\right),\theta_i)$. In other words, $x(\cdot,\cdot)$ remains a challenge scheme. Moreover, $x(\cdot,\cdot)$ satisfies (14) by construction.

[25] We thank Hamid Sabourian for drawing our attention to this point.

Next, for each state $\tilde{\theta}$ and type θ_i, we show that $x(\cdot, \cdot)$ satisfies (5). We proceed by considering the following two cases. First, suppose that $x(\tilde{\theta}, \theta_i) \neq f(\tilde{\theta})$. Then, by (14), it suffices to consider type θ_i' with $x(\tilde{\theta}, \theta_i') = f(\tilde{\theta})$. Since $x(\tilde{\theta}, \theta_i') = f(\tilde{\theta})$ and $x(\tilde{\theta}, \theta_i) \neq f(\tilde{\theta})$, then it follows from $x(\tilde{\theta}, \theta_i) \in \mathcal{SU}_i(f(\tilde{\theta}), \theta_i)$ that $u_i(x(\tilde{\theta}, \theta_i), \theta_i) > u_i(x(\tilde{\theta}, \theta_i'), \theta_i)$. Hence, (5) holds. Second, suppose that $x(\tilde{\theta}, \theta_i) = f(\tilde{\theta})$. Then, it suffices to consider type θ_i' with $x(\tilde{\theta}, \theta_i') \neq f(\tilde{\theta})$. Since $x(\tilde{\theta}, \theta_i) = f(\tilde{\theta})$, we have $\mathcal{L}_i(f(\tilde{\theta}), \tilde{\theta}_i) \cap \mathcal{SU}_i(f(\tilde{\theta}), \theta_i) = \varnothing$. Moreover, $x(\tilde{\theta}, \theta_i') \neq f(\tilde{\theta})$ implies that $x(\tilde{\theta}, \theta_i') \in \mathcal{L}_i(f(\tilde{\theta}), \tilde{\theta}_i)$. Hence, we must have $x(\tilde{\theta}, \theta_i') \notin \mathcal{SU}_i(f(\tilde{\theta}), \theta_i)$. That is, $u_i(x(\tilde{\theta}, \theta_i), \theta_i) \geq u_i(x(\tilde{\theta}, \theta_i'), \theta_i)$, i.e., (5) holds. ■

A.2 Proof of Proposition 1

To facilitate the comparison with Maskin (1977, 1999), we assume that there are three or more agents and define the following direct mechanism, denoted by $\mathcal{M}^{\mathrm{D}}$, according to three rules:

Rule 1. If there exists state $\tilde{\theta}$ such that every agent announces $\tilde{\theta}$, then implement the outcome $f(\tilde{\theta})$.

Rule 2. If there exists state $\tilde{\theta}$ such that everyone except agent i announces $\tilde{\theta}$ and agent i announces $\tilde{\theta}'$, then implement a test allocation $x(\tilde{\theta}, \tilde{\theta}_i')$ for agent i and the ordered pair of states $(\tilde{\theta}, \tilde{\theta}')$; and if there is no such test allocation, implement $f(\tilde{\theta})$. Moreover, charge agent $i+1 \pmod{I}$ a large penalty 2η, where the scale η dominates any difference in utility from allocation.

Rule 3. Otherwise, implement $f(m_1)$. Moreover, charge each agent i a penalty of η if i reports a state which is not reported by the unique majority (i.e., $\{m_i\} \neq \arg\max_{\tilde{\theta}} |\{j \in \mathcal{I} : m_j = \tilde{\theta}\}|$).[26]

Now let the true state be θ.

It follows from Rule 2 that since θ is the true state, $x(\theta, \tilde{\theta}_i') \neq f(\theta)$ implies that $x(\theta, \tilde{\theta}_i') \in \mathcal{L}_i(f(\theta), \theta_i)$. Hence, everyone reporting the true state constitutes a pure-strategy Nash equilibrium.

Now fix an arbitrary pure-strategy Nash equilibrium m. First, we claim that m cannot trigger Rule 2. To see this, suppose that Rule 2 is triggered, and let agent i be the odd man out. Then, agent $i+1$ finds it strictly profitable to deviate to announce m_i. After such a deviation, since $I \geq 3$, either Rule 3 is triggered or it remains in Rule 2, but agent i is

[26]Note that Rule 3 penalizes every agent by η, if each of them reports a different state.

no longer the odd man out. Thus, agent $i+1$ saves at least η (from paying 2η to paying η or 0). Such a deviation may also change the allocation selected by the outcome function $g(\cdot)$, which induces utility change less than η. Hence, agent $i+1$ strictly prefers deviating to announce m_i, which contradicts the hypothesis that m is a Nash equilibrium.

Second, we claim that m cannot trigger Rule 3 either. To see this, suppose that Rule 3 is triggered. Pick an arbitrary state reported by some (not necessarily unique) majority of agents, i.e., $\hat{\theta} \in \arg\max_{\tilde{\theta}} |\{j \in I : m_j = \tilde{\theta}\}|$. Let $\mathcal{I}_{\hat{\theta}}$ be the set of agents who report $\hat{\theta}$. Clearly, $\mathcal{I}_{\hat{\theta}} \subsetneq \mathcal{I}$, because Rule 3 (rather than Rule 1) is triggered. Then, we can find an agent $i^* \in \mathcal{I}_{\hat{\theta}}$ such that agent i^*+1 (mod I) is not in $I_{\hat{\theta}}$. Since agent i^*+1 does not belong to the unique majority, he must pay η under m. Then, agent i^*+1 will strictly prefer deviating to announce $m_{i^*} = \hat{\theta}$. After such a deviation, either Rule 3 is triggered, and agent i^*+1 falls in the unique majority who reports $\hat{\theta}$; or Rule 2 is triggered, but agent i^* cannot be the odd man out. Thus, agent i^*+1 saves η (from paying η to paying 0) and η' is larger than the maximal utility change induced by different allocations in $g(\cdot)$. The existence of profitable deviation of agent i^*+1 contradicts the hypothesis that m is a Nash equilibrium.

Hence, we conclude that m must trigger Rule 1. It follows that $f(\tilde{\theta}) = f(\theta)$. Otherwise, by Maskin monotonicity, a whistle blower can deviate to trigger Rule 2.

A.3 Proof of Proposition 2

The proof is based on modifying the implementing mechanism and the proof of Theorem 1. We only provide a sketch here. Set $M_i = M_i^1 \times M_i^2$ where $M_i^1 = \Theta_i$ and $M_i^2 = \times_{j \neq i} \Theta_j$. Since $I \geq 3$, the type of each agent is reported by at least two agents in their second report. For each message profile $m = (m_i)_{i=1}^I$, denote by $\tilde{\Theta}(m)$ the set of state induced from the agents' second report, namely that $\tilde{\theta} \in \tilde{\Theta}(m)$ iff for every $i \in \mathcal{I}$, we have $\tilde{\theta}_i = m_{j,i}^2$ for some agent $j \in \mathcal{I}$ (possibly $j = i$). Then, we modify the outcome function:

$$g(m) = \frac{1}{I\left|\tilde{\Theta}(m)\right|} \sum_{i \in \mathcal{I}} \sum_{\tilde{\theta} \in \tilde{\Theta}(m)} \left[e(m) \frac{1}{I} \sum_{j \in \mathcal{I}} y_j(m_j^1) \oplus (1 - e(m)) x(\tilde{\theta}, m_i^1) \right]$$

where $e(m) = 0$ if (i) $\tilde{\Theta}(m)$ contains a unique state (consistency); and (ii) $x(\tilde{\theta}, m_i^1) = f(\tilde{\theta})$ for every agent i and every $\tilde{\theta} \in \tilde{\Theta}(m)$ (no challenge); otherwise, $e(m) = \varepsilon$.[27] For the transfer

[27] Here we do not have the case with $e(m) = 1$ since $\Theta = \times_{i=1}^I \Theta_i$ implies that $\tilde{\Theta}(m) \subseteq \Theta$.

rule, we define

$$\hat{\tau}^1_{i,j}(m_i, m_{-i}) = \begin{cases} 0 & \text{if } m^2_{i,j} = m^2_{k,j} \text{ for all } k \in \mathcal{I}\backslash\{i,j\}; \\ -\eta & \text{if } m^2_{i,j} \neq m^2_{k,j} \text{ for some } k \in \mathcal{I}\backslash\{i,j\} \text{ and } m^2_{i,j} \neq m^1_j; \\ \eta & \text{if } m^2_{i,j} \neq m^2_{k,j} \text{ for some } k \in \mathcal{I}\backslash\{i,j\} \text{ and } m^2_{i,j} = m^1_j. \end{cases}$$

Set $\tau_i(m) = \sum_{j \neq i} \hat{\tau}^1_{i,j}(m)$. As the agents no longer report their own type in the second report, we do not need to define $\tau^2_{i,j}(\cdot)$.

The proof of implementation follows the same steps as the proof of Theorem 1 and we only highlight the difference. First, for the contagion of truth argument, we can only establish Claim 2(a) because in this modified mechanism, the agents no longer report their own type in the second report so that we do not have rule $\tau^2_{i,j}(\cdot)$. For the consistency argument, it turns out that Claim 2(a) suffices. Specifically, consider an arbitrary message $m_i \in$supp(σ_i) such that $m^1_i \neq \theta_i$. The same argument as in the proof of Claim 3 implies that (m_i, m_{-i}) is consistent for every $m_{-i} \in$supp(σ_{-i}). To show that $(\tilde{m}_i, m_{-i})$ is consistent for any other $\tilde{m}_i \in$supp(σ_i), we make use of the assumption that we have three or more agents. In particular, since (m_i, m_{-i}) is consistent for every $m_{-i} \in$supp(σ_{-i}), if $(\tilde{m}_i, m_{-i})$ is inconsistent, it must be $\tilde{m}^2_{i,k} \neq m^2_{j,k}$ for some $j \neq i$, $k \neq i$, and $k \neq j$. By Claim 1 agent k must report his true type with probability one. Then, it follows from Claim 2(a) that $\tilde{m}^2_{i,k} = m^2_{j,k}$ with probability one and we have reached a contradiction. The argument for no challenge remains the same.

A.4 Proof of Claim 1

Suppose that $m^1_i \neq \theta_i$ for some $m_i \in$supp(σ_i). Consider a message $\tilde{m}_i$ which differs from m_i only in sending a truthful first report, i.e., $\tilde{m}^1_i = \theta_i$ and $\tilde{m}^2_i = m^2_i$. We prove the claim by showing that $\tilde{m}_i$ is not a strictly better response than m_i against m_j only when $e_{i,j}(m_i, m_j) = e_{j,i}(m_j, m_i) = 0$. Recall that the first report of agent i has no effect on his own transfer.

We consider first the case that the designer uses agent j's report to check agent i's report. In this situation, the first report of agent i has no effect on the function $e_{i,j}(\cdot, m_j)$ for every m_j. Hence, we have $e_{i,j}(\tilde{m}_i, m_j) = e_{i,j}(m_i, m_j)$. Moreover, if $m^2_i \notin \Theta$, then $e_{i,j}(\tilde{m}_i, m_j) = e_{i,j}(m_i, m_j) = 1$; thus, by Lemma 1, $\tilde{m}_i$ is a strictly better response than m_i against m_j. Hence, we may assume $m^2_i \in \Theta$ and consider the following two cases:

Case 1.1. $e_{i,j}(\tilde{m}_i, m_j) = e_{i,j}(m_i, m_j) = \varepsilon$.

It follows from Lemmas 1 and 2 that

$$u_i(C^{\varepsilon}_{i,j}(\tilde{m}_i, m_j), \theta_i) - u_i(C^{\varepsilon}_{i,j}(m_i, m_j), \theta_i) > 0.$$

Hence, $\tilde{m}_i$ is a strictly better response than m_i against m_j.

Case 1.2. $e_{i,j}(\tilde{m}_i, m_j) = e_{i,j}(m_i, m_j) = 0$.

In this case, since $m_i^2 = \tilde{m}_i^2$, both (m_i, m_j) and $(\tilde{m}_i, m_j)$ lead to the same outcome $x(m_i^2, m_j^1) = x(\tilde{m}_i^2, m_j^1) = f(m_i^2)$.

Next, suppose that the designer uses agent i's report to check agent j's report. Again, if $m_j^2 \notin \Theta$, then $e_{j,i}(m_j, \tilde{m}_i) = e_{j,i}(m_j, m_i) = 1$; thus, by Lemma 1, $\tilde{m}_i$ is a strictly better response than m_i against m_j. Hence, we may assume $m_j^2 \in \Theta$ and consider the following four cases:

Case 2.1. $e_{j,i}(m_j, m_i) = \varepsilon$ and $e_{j,i}(m_j, \tilde{m}_i) = 0$.

It follows from (4) and Lemma 2 that

$$u_i(f(m_j^2), \theta_i) - u_i(C^{\varepsilon}_{j,i}(m_j, m_i), \theta_i) > 0,$$

where $f(m_j^2)$ is the outcome induced by $(m_j, \tilde{m}_i)$.

Case 2.2. $e_{j,i}(m_j, m_i) = 0$ and $e_{j,i}(m_j, \tilde{m}_i) = \varepsilon$.

Since $e_{j,i}(m_j, m_i) = 0$, we have $m_i^2 = \tilde{m}_i^2 = m_j^2$. Hence, $e_{j,i}(m_j, \tilde{m}_i) = \varepsilon$ implies that $x(m_j^2, \tilde{m}_i^1) = x(m_j^2, \theta_i) \neq f(m_j^2)$. Thus, it follows from (7) that

$$u_i(C^{\varepsilon}_{j,i}(m_j, \tilde{m}_i), \theta_i) - u_i(f(m_j^2), \theta_i) > 0,$$

where $f(m_j^2)$ is the outcome induced by (m_j, m_i).

Case 2.3. $e_{j,i}(m_j, m_i) = e_{j,i}(\tilde{m}_j, m_i) = \varepsilon$.

It follows from Lemmas 1 and 2 that

$$u_i(C^{\varepsilon}_{j,i}(m_j, \tilde{m}_i), \theta_i) - u_i(C^{\varepsilon}_{j,i}(m_j, m_i), \theta_i) > 0.$$

Case 2.4. $e_{j,i}(m_j, m_i) = e_{j,i}(\tilde{m}_j, m_i) = 0$.

In this case, both (m_j, m_i) and $(m_j, \tilde{m}_i)$ lead to the same outcome $x(m_j^2, \tilde{m}_i^1) = x(m_j^2, m_i^1) = f(m_j^2)$.

To sum up, as long as $e_{i,j}(m_i, m_j) = \varepsilon$ or $e_{j,i}(m_j, m_i) = \varepsilon$ (Case 1.1 and Cases 2.1-2.3), $\tilde{m}_i$ is a strictly better response than m_i against m_j. Hence, in order for $\tilde{m}_i$ not to be a profitable deviation, we must have $e_{i,j}(m_i, m_j) = e_{j,i}(m_j, m_i) = 0$.

A.5 Proof of Theorem 2

We first extend the notion of a *challenge scheme* to the case of SCCs. Fix agent i of type θ_i. For each state $\tilde{\theta} \in \Theta$, and $x \in F(\tilde{\theta})$, if $\mathcal{L}_i(x, \tilde{\theta}_i) \cap \mathcal{SU}_i(x, \theta_i) \neq \varnothing$, we select some $x(\tilde{\theta}, x, \theta_i) \in \mathcal{L}_i(x, \tilde{\theta}_i) \cap \mathcal{SU}_i(x, \theta_i)$; otherwise, we set $x(\tilde{\theta}, x, \theta_i) = x$. In the sequel, we define $F(\Theta) \equiv \bigcup_{\theta \in \Theta} F(\theta)$. Observe that $F(\Theta)$ is a finite set since each $F(\theta)$ is assumed to be finite.

As in the case of SCFs, the following lemma shows that there is a challenge scheme under which truth-telling induces the best allocation. In addition, here we choose the challenge scheme such that for every agent i, type θ_i, and state $\tilde{\theta}$ under which the challenge is effective (i.e. $x(\tilde{\theta}, x, \theta_i) \neq x$), no type $\theta_i'' \in \Theta_i$ is indifferent between $x(\tilde{\theta}, x, \theta_i)$ and an allocation x' in $F(\Theta)$. This property can be achieved by adding a small transfer to $x(\tilde{\theta}, x, \theta_i)$ with $x(\tilde{\theta}, x, \theta_i) \neq x$, thanks to the finiteness of $F(\Theta)$ and Θ_i.

Lemma 3 *There is a challenge scheme $\{x(\tilde{\theta}, x, \theta_i)\}$ for an SCC F such that for every state $\tilde{\theta}$, every $x \in F(\tilde{\theta})$ and type θ_i,*

$$u_i(x(\tilde{\theta}, x, \theta_i), \theta_i) \geq u_i(x(\tilde{\theta}, x, \theta_i'), \theta_i), \forall \theta_i' \in \Theta_i; \tag{15}$$

moreover, whenever, $x(\tilde{\theta}, x, \theta_i) \neq x$, we have

$$u_i(x(\tilde{\theta}, x, \theta_i), \theta_i'') \neq u_i(x', \theta_i''), \forall \theta_i'' \in \Theta_i, \forall x' \in F(\Theta). \tag{16}$$

Proof. We first prove (16) by constructing a challenge scheme $\{x(\tilde{\theta}, x, \theta_i)\}$. Fix agent i of type θ_i. For each state $\tilde{\theta} \in \Theta$ and $x \in F(\tilde{\theta})$, if $\mathcal{L}_i(x, \tilde{\theta}_i) \cap \mathcal{SU}_i(x, \theta_i) = \varnothing$, we let $x(\tilde{\theta}, x, \theta_i) = x$. We define

$$\mathcal{S}(i, x, \tilde{\theta}, \theta) = \left\{ x'' \in X : u_i(x'', \tilde{\theta}_i) < u_i(x, \tilde{\theta}_i) \text{ and } u_i(x'', \theta_i) > u_i(x, \theta_i) \right\}.$$

Thus, $\mathcal{S}(i, x, \tilde{\theta}, \theta)$ is a nonempty open set. Now consider

$$\begin{aligned} & \mathcal{S}^*(i, x, \tilde{\theta}, \theta) \\ \equiv \; & \mathcal{S}(i, x, \tilde{\theta}, \theta) \backslash \left\{ x'' \in X : u_i(x'', \theta_i'') = u_i(x', \theta_i'') \text{ for some } \theta_i'' \in \Theta_i \text{ and some } x' \in F(\Theta) \right\}. \end{aligned}$$

Thanks to the finiteness of $F(\Theta)$ and Θ_i, $\mathcal{S}^*(i, x, \tilde{\theta}, \theta)$ is still a nonempty open set when we delete a finite set of elements. Now we choose an element $x(\tilde{\theta}, x, \theta_i) \in \mathcal{S}^*(i, x, \tilde{\theta}, \theta)$. Hence, we obtain (16). The proof of (15) is the same as the proof of Lemma 2 applied to the challenge scheme $\{x(\tilde{\theta}, x, \theta_i)\}$. ■

Next, we propose a mechanism $\mathcal{M} = ((M_i), g, (\tau_i))_{i\in\mathcal{I}}$ which will be used to prove the if-part of Theorem 2. We define the message space, allocation rule, and transfer rule below. First, a generic message of agent i is described as follows:

$$m_i = \left(m_i^1, m_i^2, m_i^3\right) \in M_i = M_i^1 \times M_i^2 \times M_i^3 = \Theta_i \times \left[\times_{j=1}^{I}\Theta_j\right] \times F(\Theta) \text{ s.t.}$$
$$m_i^2 \in \Theta \Rightarrow m_i^3 \in F(m_i^2).$$

That is, agent i is asked to make (1) an announcements of agent i's own type (which we denote by m_i^1); (2) an announcement of a type profile (which we denote by m_i^2); (3) an allocation which must belong to $F(m_i^2)$ if m_i^2 is a state. As we do in the case of SCFs, we write $m_{i,j}^2 = \tilde{\theta}_j$ if agent i's report in m_i^2 that agent j's type is $\tilde{\theta}_j$.

First, the allocation rule is defined as follows. For each $m \in M$, we apply exactly one of the following two rules:

Rule 1: If there exist $\tilde{\theta} \in \Theta$ and $x \in F(\tilde{\theta})$ such that every agent reports state $\tilde{\theta}$ in his second report and $I-1$ agents report lottery $m_i^3(\tilde{\theta}) = x$ according to their third announcement, then

$$g(m) = \frac{1}{I^2}\sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{I}}\left[e_{i,j}\left(m_i, m_j\right)\frac{1}{I}\sum_{k\in\mathcal{I}} y_k\left(m_k^1\right) \oplus \left(1 - e_{i,j}\left(m_i, m_j\right)\right)x(\tilde{\theta}, x, m_j^1)\right]$$

where $y_k : \Theta \to A$ is the dictator lottery for agent k as defined in Lemma 1 and

$$e_{i,j}\left(m_i, m_j\right) = \begin{cases} 0, & \text{if } m_i^2 \in \Theta,\ m_i^2 = m_j^2, \text{ and } x(m_i^2, m_i^3, m_j^1) = m_i^3; \\ \varepsilon, & \text{if } m_i^2 \in \Theta, \text{ and } [m_i^2 \neq m_j^2 \text{ or } x(m_i^2, m_i^3, m_j^1) \neq m_i^3]; \\ 1, & \text{if } m_i^2 \notin \Theta. \end{cases}$$

Rule 2: Otherwise,

$$g\left(m\right) = \frac{1}{I^2}\sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{I}}\left[e_{i,j}\left(m_i, m_j\right)\frac{1}{I}\sum_{k\in\mathcal{I}} y_k\left(m_k^1\right) \oplus \left(1 - e_{i,j}\left(m_i, m_j\right)\right)x(m_i^2, m_i^3, m_j^1)\right].$$

Next, we define

$$C_{i,j}^{\varepsilon}(m) \equiv \varepsilon \times \frac{1}{I}\sum_{k\in\mathcal{I}} y_k\left(m_k^1\right) \oplus (1-\varepsilon) \times x(m_i^2, m_i^3, m_j^1).$$

For every message profile m and agent j, we can choose $\varepsilon > 0$ sufficiently small such that (i) it does not disturb the "effectiveness" of agent j's challenge, i.e.,

$$\begin{aligned} x(m_i^2, m_i^3, m_j^1) &\neq m_j^3 \Rightarrow \\ u_j(C_{i,j}^{\varepsilon}(m), m_{i,j}^2) &< u_j(m_i^3, m_{i,j}^2) \text{ and } u_j(C_{i,j}^{\varepsilon}(m), m_j^1) > u_j(m_i^3, m_j^1); \end{aligned} \tag{17}$$

moreover, (ii) an "effective self-challenge" of agent j induces a generic outcome such that at each state, no agent is indifferent between the resulting outcome and any outcome in $F(\Theta)$, i.e.,

$$\begin{aligned} x(m_j^2, m_j^3, m_j^1) &\neq m_j^3 \Rightarrow \\ \frac{1}{I} u_j(C_{j,j}^{\varepsilon}(m), \theta_j) + \left(1 - \frac{1}{I}\right) u_j(m_j^3, \theta_j) &\neq \frac{1}{I} u_j(x, \theta_j) + \left(1 - \frac{1}{I}\right) u_j(x, \theta_j) \\ \text{for any } \theta \text{ and any } x &\in F(\Theta). \end{aligned} \tag{18}$$

Observe that (ii) is possible because inequality (16) holds in Lemma 3; moreover, by (4) in Lemma 1, $u_j(C_{j,j}^{\varepsilon}(m), \theta_j)$ is a strictly decreasing function in ε.

Second, the transfer rule from agent i is specified as follows:

$$\tau_i(m) = \sum_{j \neq i} \left[\overline{\tau}_{i,j}^1(m) + \overline{\tau}_{i,j}^2(m) + \tau_{i,j}^3(m)\right]$$

where $\overline{\tau}_{i,j}^1(m)$ double the scale of $\tau_{i,j}^1(m)$ and $\tau_{i,j}^2(m)$ defined in Section 3.1.5 (i.e., $\overline{\tau}_{i,j}^1(m) = 2\tau_{i,j}^1(m)$ and $\overline{\tau}_{i,j}^2(m) = 2\tau_{i,j}^2(m)$); moreover, we introduce one more transfer rule:

$$\tau_{i,j}^3(m) = \begin{cases} 0, & \text{if } x(m_i^2, m_i^3, m_j^1) = m_i^3; \\ -\eta, & \text{if } x(m_i^2, m_i^3, m_j^1) \neq m_i^3. \end{cases}$$

That is, agent i is asked to pay η if his reported outcome m_i^3 is challenged by agent $j \neq i$. Note that we still require that η be greater than the payoff difference by (6) in Section 3.1.5.

To prove Theorem 2, we first observe that we have a stronger statement than Claim 1 since each agent i's dictator lotteries are triggered whenever there is a pair of agents (j, k) such that $e_{j,k}(m_j, m_k) = \varepsilon$ where $j, k \in \mathcal{I}$. The proof of the additional claim is identical to the proof of Case 1.1 in Claim 1.

Claim 5 *Let σ be a Nash equilibrium of the game $\Gamma(\mathcal{M}, \theta)$. If $m_i^1 \neq \theta_i$ for some $m_i \in supp(\sigma_i)$, then for each agent $j \neq i$, we have $e_{i,j}(m_i, m_j) = e_{j,i}(m_j, m_i) = 0$ with σ_j-probability one.*

In addition, for any each agents $j, k \in \mathcal{I} \setminus \{i\}$, $m_j \in$*supp*(σ_j), *and* $m_k \in$*supp*(σ_k), *we have* $e_{j,k}(m_j, m_k) = e_{k,j}(m_k, m_j) = 0$.

Then, Claims 2 and 3 hold with exactly the same proof. Again, we denote the true state by θ and (by Claim 3) the common state announced in the agents' second report by $\tilde{\theta}$. In the following, we establish Claim 6 as the counterpart of Claim 4 in Theorem 1 in the modified mechanism above.

Claim 6 *No one challenges an allocation announced in the third report of the other agents, i.e., for every pair of agents* i *and* j *where* $i \neq j$, $m_i \in$*supp*(σ_i), *and* $m_j \in$*supp*(σ_j), *we have* $x(\tilde{\theta}, m_i^3, m_j^1) = m_i^3$.

Proof. For each $x \in F(\Theta)$, we define the following set of agents:

$$\mathcal{J}(x) \equiv \left\{ j \in \mathcal{I} : \mathcal{L}_j(x, \tilde{\theta}_j) \cap \mathcal{SU}_j(x, \theta_j) = \varnothing \right\}.$$

First, if $j \in \mathcal{J}(m_i^3)$, then $x(\tilde{\theta}, m_i^3, m_j^1) \neq m_i^3$ implies that $x(\tilde{\theta}, m_i^3, m_j^1)$ is weakly worse than m_i^3 under the true type of agent j. In addition, the dictator lotteries are triggered when $x(\tilde{\theta}, m_i^3, m_j^1) \neq m_i^3$. By (4) in Lemma 1, the dictator lotteries deliver worse allocations than every alternative in $F(\Theta)$, we know that agent j can profitably deviate from m_j to $\tilde{m}_j = (\theta_j, m_j^2, m_j^3)$. Hence, to establish the claim, it suffices to prove that $\mathcal{J}(m_i^3) = \mathcal{I}$ for each message $m_i \in$supp(σ_i) and each agent $i \in \mathcal{I}$.

Suppose to the contrary that for some agent i and some message $m_i \in$supp(σ_i), we have agent $j \notin \mathcal{J}(m_i^3)$ and $j \neq i$. That is, we have

$$\mathcal{L}_j(m_i^3, \tilde{\theta}_j) \cap \mathcal{SU}_j(m_i^3, \theta_j) \neq \varnothing. \tag{19}$$

First, we claim that agent j will challenge m_i^3 with probability one, i.e., $x(\tilde{\theta}, m_i^3, m_j^1) \neq m_i^3$ for every $m_j \in$supp(σ_j). To see this, observe that if $x(\tilde{\theta}, m_i^3, m_j^1) = m_i^3$ for some m_i^3, agent j can deviate to announce $\tilde{m}_j = (\theta_j, m_j^2, m_j^3)$. This deviation is profitable, since $x(\tilde{\theta}, m_i^3, \theta_j) \in \mathcal{SU}_j(f(\tilde{\theta}), \theta_j)$.

Second, we claim that if agent j will challenge m_i^3 with probability one, then by playing m_i agent i suffers from the penalty η by $\tau_{i,j}^3$. We derive a contradiction by showing that agent i can profitably deviate in each of the following two cases. Firstly, if there is some $x \in F(\tilde{\theta})$ such that $x(\tilde{\theta}, x, m_k^1) = x$ with σ_k-probability one for agent $k \neq i$, then agent i can profitably deviate to $\tilde{m}_i$ which is identical to m_i except that $\tilde{m}_i^3 = x$. By doing so agent i avoids paying the penalty η for being challenged by agent $k \neq i$. By (6), this is a profitable deviation.

Secondly, suppose that for each $x \in F(\tilde{\theta})$, there is some agent $k_x \neq i$ such that $x(\tilde{\theta}, x, m_{k_x}^1) \neq x$ for some $m_{k_x} \in$supp(σ_{k_x}). In other words, for every $m_k \in$supp(σ_k) with $m_k^3 = x$, there exists some agent $k_x \neq i$ with a message $m_{k_x} \in$supp(σ_{k_x}) such that $e_{k,k_x}(m_k, m_{k_x}) = \varepsilon$. Then, by Claim 5, we know that $m_k^1 = \theta_k$ with σ_k-probability one. By Claim 2, we conclude that $\tilde{\theta} = \theta$. This, together with (19), implies that $\mathcal{L}_j(m_i^3, \theta_j) \cap \mathcal{SU}_j(m_i^3, \theta_j) \neq \varnothing$, which is impossible. ■

Claim 7 *No one challenges an allocation announced in his own third report, i.e., for every agent i, $m_i \in$supp(σ_i), we have $x(\tilde{\theta}, m_i^3, m_i^1) = m_i^3$.*

Proof. Note that all agents report a common state $\tilde{\theta}$ by Claim 3. Suppose to the contrary that there exist agent i and some message $\tilde{m}_i \in$supp(σ_i) such that $x(\tilde{\theta}, m_i^3, \tilde{m}_i^1) \neq \tilde{m}_i^3$. It follows that

$$\mathcal{L}_i(\tilde{m}_i^3, \tilde{\theta}_j) \cap \mathcal{SU}_i(\tilde{m}_i^3, \theta_i) \neq \varnothing. \tag{20}$$

By Claim 5, for every $\tilde{m}_i \in$supp(σ_i) such that $x(\tilde{\theta}, m_i^3, \tilde{m}_i^1) \neq \tilde{m}_i^3$, we have $\tilde{m}_i^1 = \theta_i$, and $m_k^1 = \theta_k$ for each $m_k \in$supp(σ_k) and agent $k \neq i$. We further conclude that for every $\tilde{m}_i \in$supp(σ_i) , we have $x(\tilde{\theta}, m_i^3, \tilde{m}_i^1) \neq \tilde{m}_i^3$. Suppose on the contrary that there exists some $m_i \in$supp(σ_i) such that $x(\tilde{\theta}, m_i^3, m_i^1) = m_i^3$. For every $m_{-i} \in$supp(σ_{-i}), we consider four different situations: (i) when (j, i) with $j \neq i$ is chosen, by Claim 6, both $(\tilde{m}_i, m_{-i})$ and (m_i, m_{-i}) induce the outcome m_j^3; (ii) when (j, j) with $j \neq i$ is chosen, since we have $m_j^1 = \theta_j$ with σ_j-probability one, it follows from Claim 2 that $m_{j,j}^2 = \theta_j$ with σ_j-probability one. Hence, we know that agent j does not challenge himself. Thus, $(\tilde{m}_i, m_{-i})$ and (m_i, m_{-i}) both induce the outcome m_j^3; (iii) when (i, j) with $j \neq i$ is chosen, by Claim 6 again, the outcome induced by $(\tilde{m}_i, m_{-i})$ is $\tilde{m}_i^3$ and the outcome induced by (m_i, m_{-i}) is m_i^3; and (iv) when (i, i) is chosen, $(\tilde{m}_i, m_{-i})$ induces the outcome $C_{i,i}^{\varepsilon}(\tilde{m}_i, m_{-i})$ and (m_i, m_{-i}) induces m_i^3. To summarize, against every $m_{-i} \in$supp(σ_{-i}) the payoff difference between m_i and $\tilde{m}_i$, is as follows,

$$
\begin{aligned}
& \frac{1}{I^2}u_i(C^{\varepsilon}_{i,i}(\tilde{m}_i, m_{-i}), \theta_i) + \frac{1}{I^2}\sum_{j\neq i} u_i(\tilde{m}^3_i, \theta_i) \\
& - \left[\frac{1}{I^2}u_i(m^3_i, \theta_i) + \frac{1}{I^2}\sum_{j\neq i} u_i(m^3_i, \theta_i)\right] \\
= \; & \frac{1}{I^2}u_i(C^{\varepsilon}_{i,i}(\tilde{m}_i, m_{-i}), \theta_i) + \frac{I-1}{I^2}u_i(\tilde{m}^3_i, \theta_i) \\
& - \left[\frac{1}{I^2}u_i(x, \theta_i) + \frac{I-1}{I^2}u_i(x, \theta_i)\right] \\
\neq \; & 0
\end{aligned}
$$

where the last non-equality follows from (18), i.e., agent i is never indifferent between the outcomes induced by m_i and $\tilde{m}_i$. It contradicts to the fact that $m_i \in$supp(σ_i). Hence, for every $\tilde{m}_i \in$supp(σ_i), we have $\tilde{m}^1_i = \theta_i$. Hence, $m^1_j = \theta_j$ for each $m_j \in$supp(σ_k) and agent $j \in \mathcal{I}$. By Claim 2, we conclude that $\tilde{\theta} = \theta$. This is a contradiction to (20). ■

It only remains to prove the existence of pure-strategy Nash equilibrium.

Claim 8 *For every $\theta \in \Theta$ and $x \in F(\theta)$, there exists a pure-strategy Nash equilibrium $m \in M$ of the game $\Gamma(\mathcal{M}, \theta)$ such that $g(m) = x$ and $\tau_i(m) = 0$ for every $i \in \mathcal{I}$.*

Proof. Fix an arbitrary allocation $x \in F(\theta)$. We argue that truth-telling (i.e., $m_i = (\theta_i, \theta, x)$ for each $i \in \mathcal{I}$) constitutes a pure-strategy equilibrium of the game $\Gamma(\mathcal{M}, \theta)$. Note that reporting $\tilde{m}_i$ with $\tilde{m}^1_i = \theta_i$, $\tilde{m}^2_i = \theta$, and $\tilde{m}^3_i \neq x$ instead of m_i affects neither the allocation nor the transfer. The argument for proving that either $\tilde{m}^1_i \neq \theta_i$ or $\tilde{m}^2_i \neq \theta$ cannot be a profitable unilateral deviation for every agent $i \in \mathcal{I}$ is identical to the proof of Theorem 1.
■

A.6 Proof of Theorem 3

Recall that in the mechanism which we use to prove Theorem 1, agent i's generic message is $m_i = (m^1_i, m^2_i) \in \Theta_i \times \left[\times^I_{j=1}\Theta_j\right]$. We expand m^2_i into H copies of $\left[\times^I_{j=1}\Theta_j\right]$ and define

$$
m_i = (m^1_i, m^2_i, \ldots, m^{H+1}_i) \in \Theta_i \times \underbrace{\left[\times^I_{j=1}\Theta_j\right] \times \cdots \times \left[\times^I_{j=1}\Theta_j\right]}_{H \text{ terms}}
$$

where H is a positive integer to be chosen later. For each message profile $m \in M$, the allocation is defined as follows:

$$g(m)=\frac{1}{I(I-1)}\sum_{i\in\mathcal{I}}\sum_{j\neq i}\left[e_{i,j}\left(m_i,m_j\right)\frac{1}{2}\sum_{k=i,j}y_k\left(m_k^1\right)\oplus\frac{1-e_{i,j}(m_i,m_j)}{H}\left[x\left(m_i^2,m_j^{H+2}\right)\oplus\sum_{h=3}^{H+1}\phi\left(m^h\right)\right]\right]$$

where $y_k:\Theta\rightarrow X$ is the dictator lottery[28] for agent k defined in Lemma 1, $\phi(\cdot)$ is an outcome function such that

$$\phi\left(m^h\right)=\begin{cases} f(\tilde{\theta}), & \text{if } m_i^h=\tilde{\theta}\in\Theta \text{ for at least } I-1 \text{ agents;}\\ b, & \text{otherwise, where } b \text{ is an arbitrary outcome in } A,\end{cases}$$

and

$$e_{i,j}(m_i,m_j)=\begin{cases} 0, & \text{if } m_i^2\in\Theta,\ m_i^2=m_j^2=m_i^h=m_j^h \text{ and } x\left(m_i^2,m_j^{H+2}\right)=f(m_i^2),\ \forall h\in\{3,\ldots,H+1\};\\ 1, & \text{if } m_i^2\notin\Theta;\\ \varepsilon, & \text{otherwise.}\end{cases}$$

We now define the transfer rule. For every message profile m and agent i, we specify the transfer to agent i as follows:

$$\tau_i(m)=\sum_{j\neq i}\left[\tau_{i,j}^{1,2}(m)+\tau_{i,j}^{2,2}(m)\right]+\sum_{h=3}^{H+1}\tau_i^h(m)+d_i(m^2,\ldots,m^{H+1})$$

where $\gamma,\kappa,\xi>0$ (their size are determined later)

$$\tau_{i,j}^{1,2}(m)=\begin{cases} 0, & \text{if } m_{i,j}^2=m_{j,j}^2;\\ -\gamma & \text{if } m_{i,j}^2\neq m_{j,j}^2 \text{ and } m_{i,j}^2\neq m_j^1;\\ \gamma & \text{if } m_{i,j}^2\neq m_{j,j}^2 \text{ and } m_{i,j}^2=m_j^1.\end{cases}$$

$$\tau_{i,j}^{2,2}(m)=\begin{cases} 0, & \text{if } m_{i,i}^2=m_{j,i}^2;\\ -\gamma, & \text{if } m_{i,i}^2\neq m_{j,i}^2;\end{cases}$$

moreover, for every $h\in\{3,\ldots,H+1\}$,

$$\tau_i^h(m)=\begin{cases} -\kappa, & \text{if there exists } \tilde{\theta} \text{ such that } m_i^h\neq\tilde{\theta} \text{ but } m_j^h=\tilde{\theta} \text{ for all } j\neq i;\\ 0, & \text{otherwise,}\end{cases}$$

[28] Although the dictator lotteries may contain transfers, we do not take into account the scale of transfers in it. To drop the transfers in the dictator lotteries, we can use arbitrarily small money to make all the outcomes from the best challenge schemes and social choice function generic.

and

$$d_i(m^2,\ldots,m^{H+1}) = \begin{cases} -\xi, & \text{if there exists } h\in\{3,\ldots,H+1\} \text{ such that } m_i^h \neq m_i^2 \text{ and } m_j^{h'} = m_j^2, \\ & \text{for all } h'\in\{2,\ldots,h-1\} \text{ and all } j\neq i; \\ 0, & \text{otherwise.} \end{cases}$$

Finally, we choose positive numbers γ, ξ, H, κ, and ε such that

$$\begin{aligned} \bar{\tau} &> \gamma + (H-1)\,\kappa + \xi \\ \gamma &> \xi + \varepsilon\eta \\ \kappa &> \varepsilon\eta \\ \xi &> \frac{1}{H}\eta + \kappa. \end{aligned}$$

More precisely, we first fix $\bar{\tau}$ and choose $\gamma < \frac{1}{3}\bar{\tau}$ and $\xi < \min\left\{\frac{1}{3}\bar{\tau},\gamma\right\}$. Second, we choose H large enough so that $\xi > \frac{1}{H}\eta$. Third, we choose κ small enough such that $(H-1)\,\kappa < \frac{1}{3}\bar{\tau}$ and $\xi > \frac{1}{H}\eta + \kappa$. Fourth, we choose ε small enough such that $\gamma > \xi + \varepsilon\eta$ and $\kappa > \varepsilon\eta$. We can now prove Theorem 3 following the three steps as in the proof of Theorem 1.

A.6.1 Contagion of Truth

First, note that Claims 1 and 2 hold. The proof of Claim 2 applies with only one minor difference: Here m_i^2 may affect agent i's payoff through $d_i(\cdot)$. However, a similar argument follows, since we have $\gamma > \xi + \varepsilon\eta$.[29] Let θ denote the true state.

Claim 9 *If every agent j reports the truth in his first report σ_j-probability one, then every agent j reports the truth in his 2nd,...,$(H+1)$th report. That is, $m_j^h = \theta$ for $h = 2,\ldots,H+1$.*

By Claims 1 and 2, every agent j reports the state truthfully in his 2nd report. Then, we can follow verbatim the argument on p. 12 of Abreu and Matsushima (1994) which shows that every agent j reports the state truthfully in his hth report for every $h = 2,...,H+1$.

A.6.2 Consistency

Claim 10 *There exists a state $\tilde{\theta}$ such that every agent announces $\tilde{\theta}$ in the second report all the way to the last/$(H+1)$th report with probability one.*

[29] This step corresponds to Property (b) in Abreu and Matsushima (1994).

Proof. We prove consistency by considering the two cases as in the proof of Claim 3. The proof for the first case remains the same. For the second case, suppose that one agent, say i, tells a lie in the first report. As agent i believes that all the other agents report the same state $\tilde{\theta}$ in their second all the way to the last report. By the same argument in the second case in the proof of Claim 3, we can show that agent i announces $\tilde{\theta}$ in the second report with probability one. In addition, for every $h = 2, \dots, H+2$, as agent i believes that all the other agent report the same state $\tilde{\theta}$, by the rule $\phi\left(m^h\right)$ and $\tau_i^h\left(m^h\right)$, we know $m_i^h = \tilde{\theta}$. ■

A.6.3 No Challenge

Claim 11 *No agent challenges with positive probability the common state $\tilde{\theta}$ announced in the second report.*

Proof. The argument is the same as the proof of Claim 4. ■

A.7 Proof of Theorem 4

Recall we assume that set of alternatives A is a finite set, and the state space Θ is a Polish (i.e., separable and complete metric) space. For every $\ell \in \Delta(A)$, we write $v_i(\ell, \theta_i) = \ell \cdot \bar{v}_i(\theta_i)$ where $\bar{v}_i(\theta_i) \in [0,1]^{|A|}$ is a vector of utilities over A induced by $v_i(\cdot, \theta_i)$. Assume that agents' utilities remain bounded, f is also bounded in money if f specifies every transfer in outcome and $\eta > 0$ still satisfies condition (2). Let $\bar{X} \equiv \Delta(A) \times [-2\eta, 2\eta]^I$ and identify $\bar{X}$ with a compact subset of $\mathbb{R}^{I+|A|}$ (endowed with the Euclidean topology). We identify Θ with a subset of the product set $\times_{j=1}^I \Theta_j$. Let d denote the metric on Θ, d_i the metric on Θ_i, and $\rho : \bar{X} \times \bar{X} \to \mathbb{R}$ a metric on the outcome space. We endow $\times_{j=1}^I \Theta_j$ with the product topology and $\times_{j=1}^I \Theta_j$ and $\bar{X}$ with the Borel σ-algebra. We say that the setting is *compact and continuous* if Θ is compact and $(v_i(a,\cdot))_{a\in A, i\in\mathcal{I}}$ and the SCFs are all continuous functions on Θ.

We introduce the following version of challenge scheme. First, for $(x, \theta_i) \in X \times \Theta_i$, we use $\mathcal{SL}_i(x, \theta_i)$ to denote the strict lower-contour set at allocation x for type θ_i, i.e.,

$$\mathcal{SL}_i(x, \theta_i) = \{x' \in X : u_i(x, \theta_i) > u_i(x', \theta_i)\}.$$

For agent i of type θ_i, an allocation $x \in \bar{X}$, and $\tilde{\theta} \in \Theta$, we construct the following compound lottery:

$$\ell(x, \tilde{\theta}) = \frac{\rho(x, f(\tilde{\theta}))}{1 + \rho(x, f(\tilde{\theta}))} x \oplus \frac{1}{1 + \rho(x, f(\tilde{\theta}))} f(\tilde{\theta}).$$

Define

$$x(\tilde{\theta}, x) = \begin{cases} \ell(x, \tilde{\theta}), & \text{if } x \in \mathcal{SL}_i(f(\tilde{\theta}), \tilde{\theta}_i); \\ f(\tilde{\theta}), & \text{otherwise.} \end{cases}$$

As we mentioned in the main text, in the infinite setting we know of no way to construct a challenge scheme by pre-selecting a test allocation (depending on type θ_i) in a continuous manner. As a result, we cannot have the agents report their type (let alone the true type) to cast a challenge to state $\tilde{\theta}$. Instead, we will restore continuity of the outcome function by asking them to report an allocation x directly (see Section A.7.1). The definition of strict lower-contour set can be found in Section A.5.

Claim 12 *$x(\tilde{\theta}, x)$ is a continuous function on $\bar{X} \times \Theta$.*

Proof. Since $\rho(\cdot, \cdot)$ is continuous, we have that $\ell(\cdot, \cdot)$ is continuous. We show that $x(\tilde{\theta}[n], x[n]) \to x(\tilde{\theta}, x)$ in each of the following two cases.

Case 1. $x \in \mathcal{SL}_i(f(\tilde{\theta}), \tilde{\theta}_i)$.

In this case, $x(\tilde{\theta}, x) = \ell(x, f(\tilde{\theta}))$. Since f and u_i are both continuous, it follows that $x[n] \in \mathcal{SL}_i(f(\tilde{\theta}[n]), \tilde{\theta}_i[n])$ for each n large enough. Thus, $x(\tilde{\theta}[n], x[n]) = \ell(x[n], \tilde{\theta}[n])$. Hence, $x(\tilde{\theta}[n], x[n]) \to \ell(x, f(\tilde{\theta}))$ as $(x[n], \tilde{\theta}[n]) \to (x, \tilde{\theta})$.

Case 2. $x \notin \mathcal{SL}_i(f(\tilde{\theta}), \tilde{\theta}_i)$.

In this case, $x(\tilde{\theta}, x) = f(\tilde{\theta})$. If there is some $\overline{n}$ such that $x[n] \notin \mathcal{SL}_i(f(\tilde{\theta}[n]), \tilde{\theta}_i[n])$ for every $n \geq \overline{n}$, then $x(\tilde{\theta}[n], x[n]) = f(\tilde{\theta}[n])$. Since f is continuous and $\tilde{\theta}[n] \to \tilde{\theta}$, it follows that $x(\tilde{\theta}[n], x[n]) \to f(\tilde{\theta})$. Now suppose that there is a subsequence of $x[n], \tilde{\theta}[n]$, say itself, such that $x[n] \in \mathcal{SL}_i(f(\tilde{\theta}[n]), \tilde{\theta}_i[n])$ for every n. Then, we have $x(\tilde{\theta}[n], x[n]) = \ell(x[n], f(\tilde{\theta}[n]))$. Since ρ is jointly continuous, we must have $\rho(x[n], f(\tilde{\theta}[n])) \to \rho(x, f(\tilde{\theta}))$. Since $x \notin \mathcal{SL}_i(f(\tilde{\theta}), \tilde{\theta}_i)$, it follows that $\rho(x, f(\tilde{\theta})) = 0$. By construction of $\ell(x[n], f(\tilde{\theta}[n]))$, we must have $\ell(x[n], f(\tilde{\theta}[n])) \to f(\tilde{\theta})$. Hence, $x(\tilde{\theta}[n], x[n]) \to f(\tilde{\theta})$. ■

Lemma 4 *For each $i \in \mathcal{I}$, there exists a continuous function $y_i : \Theta_i \to X$ such that for all types θ_i and θ_i' of agent i with $\theta_i \neq \theta_i'$, we have*

$$u_i(y_i(\theta_i), \theta_i) > u_i(y_i(\theta_i'), \theta_i)\,; \tag{21}$$

and for each type θ_j' of agent $j \in \mathcal{I}$, we also have that for every $x \in f(\Theta)$

$$u_i(y_j(\theta_j'), \theta_i) < u_i(x, \theta_i). \tag{22}$$

Moreover, $y_i(\cdot)$ is continuous on Θ_i.

Proof. We construct the dictator lotteries in the infinite state space. We first construct $\ell_i(\theta_i) \in \Delta(A)$ for each $\theta_i \in \Theta_i$ and let

$$y_i(\theta_i) = (\ell_i(\theta_i), -2\eta, ..., -2\eta).$$

Hence, we obtain $u_i(a, \theta_i) > u_i(y_k(\theta'_k), \theta_i)$ for all type θ_i and type θ'_k of agents i and k. Let ℓ^* be the uniform lottery over A, i.e., $\ell^*[a] = 1/|A|$. Pick $r < 1/|A|$. Consider the maximization problem as follows:

$$\max_{l \in \Delta(A)} l \cdot \bar{v}_i(\theta_i)$$
$$\text{s.t. } \|\ell - \ell^*\| \leq r$$

The Kuhn-Tucker condition for $l_i(\theta_i)$ to be the solution is

$$\bar{v}_i(\theta_i) - 2\lambda_i(\theta_i)(\ell_i(\theta_i) - \ell^*) = 0$$

We know that $\bar{v}_i(\theta_i)$ is not a zero vector. Hence, $\lambda_i(\theta_i) > 0$ and $l_i(\theta_i)$ is equal to the normalization of $\frac{1}{2}\left(\frac{\bar{v}_i(\theta_i)}{\lambda_i(\theta_i)} + \ell^*\right)$ as a lottery. For every $\theta_i \neq \theta'_i$, since $\bar{v}_i(\theta_i)$ is not an affine transform of $\bar{v}_i(\theta'_i)$, it follows that $\ell_i(\theta_i) \neq \ell_i(\theta'_i)$. Moreover, by Theorem of the maximum, $\ell_i(\cdot)$ is a continuous function on Θ_i. ■

A.7.1 The Mechanism

A generic message of agent i is described as follows:

$$m_i = (m_i^1, m_i^2, m_i^3) \in M_i = M_i^1 \times M_i^2 \times M_i^3 = \Theta_i \times \left[\times_{j=1}^I \Theta_j\right] \times \bar{X}.$$

That is, agent i is asked to make (1) one announcement of agent i's type (i.e., m_i^1); and (2) one announcement of a type profile (i.e., m_i^2); and (3) one announcement of an allocation (i.e., m_i^3). As in the main text, we write $m_{i,j}^2 = \tilde{\theta}_j$ if agent i reports in m_i^2 that agent j's type is $\tilde{\theta}_j$.

A.7.1.1 Allocation Rule

For each message profile $m \in M$, the allocation is defined as follows:

$$g(m) = \frac{1}{I(I-1)} \sum_{i \in \mathcal{I}} \sum_{j \neq i} \left[e_{i,j}(m_i, m_j) \frac{1}{2} \sum_{k=i,j} y_k(m_k^1) \oplus (1 - e_{i,j}(m_i, m_j)) x(m_i^2, m_j^3) \right]$$

where $y_k(\theta_k) = (l_k(\theta_k), t_1(\theta_k), ..., t_I(\theta_k))$ is the dictator lottery for agent k with type θ_k defined in Lemma 4 and we define

$$e_{i,j}(m_i, m_j) \equiv \min\left\{\max\left\{\tilde{d}\left(m_i^2, m_j^2\right), d\left(m_i^2, m_j^2\right), \rho(m_j^3, f(m_i^2))^3\right\}, 1\right\},$$

where[30]

$$\tilde{d}\left(m_i^2, m_j^2\right) = \inf_{\theta\in\Theta} d\left(m_i^2, \theta\right) + \inf_{\theta\in\Theta} d\left(m_j^2, \theta\right). \tag{23}$$

For each message profile (m_i, m_j) of agents i and j, let

$$C_{i,j}(m_i.m_j) \equiv e_{i,j}(m_i, m_j) \frac{1}{2} \sum_{k=i,j} y_k\left(m_k^1\right) \oplus (1 - e_{i,j}(m_i, m_j))\, x(m_i^2, m_j^3).$$

To sum, with probability $\frac{1}{I(I-1)}$ an ordered pair (i, j) is chosen, then $C_{i,j}(m_i.m_j)$ is implemented.

Claim 13 *The outcome function g is continuous.*

Proof. It follows from Claim 12 and Lemma 4. ■

A.7.1.2 Transfer Rule

We now define the transfer rule. For every message profile $m \in M$ and agent $i \in \mathcal{I}$, we specify the transfer to agent i as:

$$\tau_i(m) = \sum_{j\neq i} \left[\tau_{i,j}^1(m) + \tau_{i,j}^2(m)\right],$$

where $\tau_{i,j}^1$ and $\tau_{i,j}^2$ will be defined as follows: Given a message profile m and agent j, let $\tilde{m}_i^m = \left(m_i^1, \left(m_j^1, m_{i,-j}^2\right), m_i^3\right)$ (which replaces $m_{i,j}^2$ in m_i by m_j^1), $\hat{m}_i^m = \left(m_i^1, \left(m_{j,j}^2, m_{i,-j}^2\right), m_i^3\right)$ (which replaces $m_{i,j}^2$ in m_i by $m_{j,j}^2$), and $\bar{m}_i^m = \left(m_i^1, \left(m_{j,i}^2, m_{i,-i}^2\right), m_i^3\right)$ (which replaces $m_{i,i}^2$ as $m_{j,i}^2$). We define $\tau_{i,j}^1$ as follows:

$$\begin{aligned}\tau_{i,j}^1(m) &= -\sup_{\theta_i'} |u_i(g(m), \theta_i') - u_i(g(\tilde{m}_i^m, m_{-i}), \theta_i')| \\ &\quad + \sup_{\theta_i'} |u_i(g(\hat{m}_i^m, m_{-i}), \theta_i') - u_i(g(\tilde{m}_i^m, m_{-i}), \theta_i')| \\ &\quad + d_j\left(m_{j,j}^2, m_j^1\right) - d_j(m_{i,j}^2, m_j^1).\end{aligned} \tag{24}$$

[30] In comparison with the $e_{i,j}(\cdot)$ defined in the proof of Theorem 1, here the terms of $\tilde{d}$ and d correspond to the consistency check and the term ρ corresponds to the no challenge check.

Observe that $\tau^1_{i,j}$ satisfies two important properties: (1) neither $u_i(g(\hat{m}^m_i, m_{-i}), \theta'_i) - u_i(g(\tilde{m}^m_i, m_{-i}), \theta'_i)$ nor $d_j\left(m^2_{j,j}, m^1_j\right)$ depends on agent i's choice of $m^2_{i,j}$, since $m^2_{i,j}$ has been replaced by agent j's announcements in both $\hat{m}^m_i$ and $\tilde{m}^m_i$; (2) $\tau^1_{i,j}(m) = 0$ if $m^2_{i,j} = m^2_{j,j}$. We next define $\tau^2_{i,j}$ as follows:

$$\tau^2_{i,j}(m) = -\sup_{\theta'_i} |u_i(g(m), \theta'_i) - u_i(g(\bar{m}^m_i, m_{-i}), \theta'_i)| - d_i(m^2_{i,i}, m^2_{j,i}) \tag{25}$$

Say a function $\alpha(\cdot)$ between two metric spaces S and Y, both endowed with the Borel σ-algebra, is *analytic* if its pre-image of every open set on Y is an analytic set. Since every analytic set is universally measurable, an analytic function is "almost" a measurable function (see pp. 498-499 of Stinchcombe and White (1992)). We show below that the mechanism which we are about to construct has analytic outcome function and transfer rule. Hence, whenever we fix a mixed-strategy Nash equilibrium which is a Borel probability measure on M, we can work with the σ-completion of the Borel σ-algebra on M to make all the expected payoffs well defined.[31]

Claim 14 *The transfer rule $\tau_i : M \to \mathbb{R}$ is an analytic function. Moreover, if the setting is compact and continuous, then $\tau_i(\cdot)$ is a continuous function.*

Proof. It follows from Theorem 2.17 of Stinchcombe and White (1992) that τ_i is an analytic function. Suppose that the setting is compact and continuous. Then, by Claim 13, g is also continuous on M. Moreover, by the theorem of maximum, $\tau^1_{ij}(\cdot)$ and $\tau^2_{ij}(\cdot)$ are continuous on M. Hence $\tau^1_{ij}(\cdot)$ and $\tau^2_{ij}(\cdot)$ are both continuous. ■

With the claims above, we have defined the implementing mechanism $\mathcal{M} = ((M_i), g, (\tau_i))_{i\in\mathcal{I}}$; moreover, when the setting is compact and continuous, $\mathcal{M}$ is a mechanism with compact sets of message and continuous outcome function and transfer rule. To show that implementation is achieved by the constructed mechanism, we only emphasize the differences from the argument in finite state space. Before we provide the main argument, we establish two lemmas which play an important role in the proof of Theorem 4.

Throughout the proof, we denote by θ the true state. First, we show that it is strictly worse for every agent to challenge the truth.

[31] As will be clear in the argument later, the outcome function and transfer rule which we construct is a value function of some optimization problem. Such value functions are not necessarily continuous when Θ is not compact (which is the case when we apply Theorem 4 to prove Theorem 5 later).

Lemma 5 *Let (m_i, m_j) be a message profile of agents i and j with $x(m_i^2, m_j^3) \neq f(m_i^2)$. Then, $u_j(C_{i,j}(m_i, m_j), m_{i,j}^2) < u_j(f(m_i^2), m_{i,j}^2)$.*

Proof. Since $x(m_i^2, m_j^3) \neq f(m_i^2)$, we have $u_j(x(m_i^2, m_j^3), m_{i,j}^2) < u_j(f(m_i^2), m_{i,j}^2)$. Moreover, by (22) for every $x \in f(\Theta)$ we have $u_i(x, \theta_i) > u_i(y_k(\theta_k'), \theta_i)$ for each type θ_i and type θ_k' of agents i and k, we conclude that

$$u_j(C_{i,j}(m_i, m_j), m_{i,j}^2) < u_j(f(m_i^2), m_{i,j}^2). \tag{26}$$

This completes the proof. ■

Second, whenever $\mathcal{SL}_j(f(\tilde{\theta}), \tilde{\theta}_i) \cap \mathcal{SU}_j(f(\tilde{\theta}), \theta_i) \neq \varnothing$, we show that it is strictly better for agent j to challenge.

Lemma 6 *Let (m_i, m_j) be a message profile of agents i and j with $m_i^2 = m_j^2 = \tilde{\theta}$ with $\tilde{\theta} \in \Theta$ and $\mathcal{SL}_j(f(\tilde{\theta}), \tilde{\theta}_j) \cap \mathcal{SU}_j(f(\tilde{\theta}), m_j^1) \neq \varnothing$. Then, we can choose $m_j^3 \in \bar{X}$ such that $u_j(C_{i,j}(m_i.m_j), m_j^1) > u_j(f(m_i^2), m_j^1)$.*

Proof. First, we fix an arbitrary $x \in \mathcal{SL}_j(f(\tilde{\theta}), \tilde{\theta}_j) \cap \mathcal{SU}_j(f(\tilde{\theta}), m_j^1)$. Let

$$m_j^3 = \alpha x \oplus (1-\alpha) f(\tilde{\theta}),$$

where $\alpha \in (0, 1)$. Note that $m_j^3 \in \mathcal{SL}_j(f(\tilde{\theta}), \tilde{\theta}_j) \cap \mathcal{SU}_j(f(\tilde{\theta}), m_j^1)$ for every $\alpha \in (0, 1)$. As $\alpha \to 0$, we have

$$\rho(m_j^3, f(\tilde{\theta})) \to 0 \text{ and } u_j(m_j^3, m_j^1) \to u_j(f(\tilde{\theta}), m_j^1).$$

Observe that $\rho(m_j^3, f(\tilde{\theta})) = \alpha \left\| x - f(\tilde{\theta}) \right\|$ (recall that $\bar{X}$ is a compact subset of $\mathbb{R}^{I+|A|}$). Hence, we can choose $\alpha > 0$ small enough such that

$$\rho(m_j^3, f(\tilde{\theta})) < 1; \tag{27}$$

$$\rho(m_j^3, f(\tilde{\theta}))^3 (-3\eta) + \alpha \rho(m_j^3, f(\tilde{\theta}))(u_j(x, m_j^1) - u_j(f(\tilde{\theta}), m_j^1)) > 0. \tag{28}$$

Now, we have

$$
\begin{aligned}
& u_j(C_{i,j}(m_i.m_j), m_j^1) - u_j(f(\tilde{\theta}), m_j^1) \\
= \; & e_{i,j}\left(m_i, m_j\right) u_j(\frac{1}{2}\sum_{k=i,j} y_k(m_k^1), m_j^1) + \left(1 - e_{i,j}\left(m_i, m_j\right)\right) u_j\left(x(m_i^2, m_j^3), m_j^1\right) - u_j(f(\tilde{\theta}), m_j^1) \\
= \; & \rho(m_j^3, f(\tilde{\theta}))^3 u_j(\frac{1}{2}\sum_{k=i,j} y_k(m_k^1), m_j^1) + (1 - \rho(m_j^3, f(\tilde{\theta}))^3) u_j(\ell(m_j^3, f(\tilde{\theta})), m_j^1) - u_j(f(\tilde{\theta}), m_j^1) \\
= \; & \rho(m_j^3, f(\tilde{\theta}))^3 \left[u_j(\frac{1}{2}\sum_{k=i,j} y_k(m_k^1), m_j^1) - u_j(f(\tilde{\theta}), m_j^1)\right] \\
& + (1 - \rho(m_j^3, f(\tilde{\theta}))^3) \left[u_j\left(\frac{\rho(m_j^3, f(\tilde{\theta}))}{1 + \rho(m_j^3, f(\tilde{\theta}))} m_j^3 \oplus \frac{1}{1 + \rho(m_j^3, f(\tilde{\theta}))} f(\tilde{\theta}), m_j^1\right) - u_j(f(\tilde{\theta}), m_j^1)\right] \\
> \; & \rho(m_j^3, f(\tilde{\theta}))^3 \left(-3\eta\right) + (1 - \rho(m_j^3, f(\tilde{\theta}))^3) \left[\frac{\rho(m_j^3, f(\tilde{\theta}))}{1 + \rho(m_j^3, f(\tilde{\theta}))}\left(u_j\left(m_j^3, m_j^1\right) - u_j(f(\tilde{\theta}), m_j^1)\right)\right] \\
= \; & \rho(m_j^3, f(\tilde{\theta}))^3 \left(-3\eta\right) + (1 - \rho(m_j^3, f(\tilde{\theta}))^3) \left[\frac{\alpha\rho(m_j^3, f(\tilde{\theta}))}{1 + \rho(m_j^3, f(\tilde{\theta}))}\left(u_j\left(x, m_j^1\right) - u_j(f(\tilde{\theta}), m_j^1)\right)\right] \\
> \; & \rho(m_j^3, f(\tilde{\theta}))^3 \left(-3\eta\right) + \frac{1}{2}\alpha\rho(m_j^3, f(\tilde{\theta}))\left(u_j\left(x, m_j^1\right) - u_j(f(\tilde{\theta}), m_j^1)\right)
\end{aligned}
$$

where second equality follows because $m_i^2 = m_j^2 = \tilde{\theta}$ with $\tilde{\theta} \in \Theta$ and $\rho(m_j^3, f(\tilde{\theta}))^3 < 1$ (by (27)); the third equality follows from the definition of $\ell(m_j^3, f(\tilde{\theta}))$; the fourth equality follows from linearity of u_j in allocation; the first inequality follows because $u_j\left(\frac{1}{2}\sum_{k=i,j} y_k(m_k^1), m_j^1\right) - u_j(f(\tilde{\theta}), m_j^1) > -3\eta$; the last inequality follows from (27). Hence, it follows from (28) that $u_j(C_{i,j}(m_i.m_j), m_j^1) - u_j(f(\tilde{\theta}), m_j^1) > 0$. ■

A.7.2 Existence of Good Equilibrium

Consider an arbitrary true state θ. The proof consists of two parts. In the first part, we argue that truth-telling m where $m_i = (\theta_i, \theta, x)$ for each $i \in \mathcal{I}$ constitutes a pure-strategy equilibrium, where $x\left(\theta, x\right) = f\left(\theta\right)$. Under this message profile m, $e_{i,j}(m_i, m_j) = 0$. Firstly, reporting $\tilde{m}_i$ with either $\tilde{m}_{i,i}^2 \neq \theta_i$ or $\tilde{m}_{i,j}^2 \neq \theta_j$ suffers the penalty of $\tau_{i,j}^2\left(m\right)$ or $\tau_{i,j}^1\left(m\right)$ and hence cannot be a profitable deviation by Claim 15. Secondly, reporting $\tilde{m}_i$ with $\tilde{m}_i^2 = \theta$ and $\tilde{m}_i^3 = x' \neq x$ either leads to $x\left(\theta, x'\right) = f(\theta)$ and results in no change in payoff or $x\left(\theta, x'\right) \neq f(\theta)$ which is strictly worse than $f(\theta)$. By Lemma 5, this is not a profitable deviation. Finally, reporting $\tilde{m}_i$ with $\tilde{m}_i^2 = \theta$, $\tilde{m}_i^3 = \theta_i$, and $\tilde{m}_i^1 \neq \theta_i$ does not affect the allocation or transfer, since we still have $\tau_i\left(m\right) = 0$ and $e_{j,k}(m_j, m_k) = 0$ for every j and k.

In the second part, we show that for every Nash equilibrium σ of the game $\Gamma(\mathcal{M},\theta)$ and every $m \in$supp(σ), we have that $g(m) = f(\theta)$ and $\tau_i(m) = 0$ for every $i \in \mathcal{I}$. The proof of the second part is divided into three steps: (Step 1) *contagion of truth*: if agent j announces his type truthfully in his first report, then every agent must also report agent j's type truthfully in their second report; (Step 2) *consistency*: every agent reports the same state $\tilde{\theta}$ in the second report; and (Step 3) *no challenge*: no agent challenges the common reported state $\tilde{\theta}$, i.e., $x(\tilde{\theta}, m_j^3) = f(\tilde{\theta})$ for every $j \in \mathcal{I}$. Then, consistency implies that $\tau_i(m) = 0$ for every $i \in \mathcal{I}$, whereas no challenge is invoked so that monotonicity of f together with Lemma 6 implies that $g(m) = f(\tilde{\theta}) = f(\theta)$.

As we do not make use of the notion of best challenge scheme in the infinite setting, the proof of Claim 1 is more straightforward. Here m_i^1 only affects agent i's own payoff through controlling the dictator lottery y_i; in particular, both $e_{i,j}(m_i, m_j)$ and $e_{j,i}(m_j, m_i)$ are not determined by m_i^1 or m_j^1. Hence, if $e_{i,j}(m_i, m_j) = \varepsilon$ or $e_{j,i}(m_j, m_i) = \varepsilon$, then $m_i^1 = \theta_i$ by (21). We now establish Steps 1-3.

A.7.3 Contagion of Truth

Claim 15 *We establish two results:*
(a) If agent j sends a truthful first report with σ_j-probability one, then every agent $i \neq j$ must report agent j's type truthfully in his second report with σ_i-probability one.
(b) If every agent $i \neq j$ reports a type $\tilde{\theta}_j$ of agent j in his second report with σ_i-probability one, then agent j must also report $\tilde{\theta}_j$ in his second report with σ_j-probability one.

Proof. First, we prove part (a). Suppose instead that there exists some message m_i played with σ_i-positive probability such that agent i misreports agent j's type in his second report, i.e., $m_{i,j}^2 \neq \theta_j$. Let $\tilde{m}_i$ be a message which differs from m_i only in reporting j's type truthfully $\tilde{m}_{i,j}^2 = \theta_j$. Then, for each m_{-i} played with σ_{-i}-positive probability, since agent j reports his type truthfully ($m_j^1 = \theta_j$), we have $\tilde{m}_i = \tilde{m}_i^m$ where $m = (m_i.m_{-i})$. Recall the definition of $\tilde{m}_i^m$ in Section A.7.1.2. Hence, we reach a contradiction if we show $\tilde{m}_i$ is a profitable deviation, i.e., for each m_{-i} played with σ_{-i}-positive probability, we have

$$u_i(g(\tilde{m}_i^m, m_{-i}), \theta_i) + \tau_i(\tilde{m}_i^m, m_{-i}) > u_i(g(m), \theta_i) + \tau_i(m).$$

First, observe that

$$\begin{aligned}\tau_i\left(\tilde{m}_i^m, m_{-i}\right) - \tau_i\left(m\right) &= \tau_{i,j}^1\left(\tilde{m}_i^m, m_{-i}\right) - \tau_{i,j}^1\left(m\right) \\ &= \sup_{\theta_i'}\left|u_i\left(g\left(m\right), \theta_i'\right) - u_i\left(g\left(\tilde{m}_i^m, m_{-i}\right), \theta_i'\right)\right| + d_j\left(m_{i,j}^2, m_j^1\right)\end{aligned}$$

where the first equality follows because $\tilde{m}_i^m$ only differs from m_i in agent i's second report of agent j's type. Thus we have

$$\begin{aligned}& \left[u_i\left(g\left(\tilde{m}_i^m, m_{-i}\right), \theta_i\right) + \tau_i\left(\tilde{m}_i^m, m_{-i}\right)\right] - \left[u_i\left(g\left(m\right), \theta_i\right) + \tau_i\left(m\right)\right] \\ = \;& u_i\left(g\left(\tilde{m}_i^m, m_{-i}\right), \theta_i\right) - u_i\left(g\left(m\right), \theta_i\right) \\ & + \sup_{\theta_i'}\left|u_i\left(g\left(m\right), \theta_i'\right) - u_i\left(g\left(\tilde{m}_i^m, m_{-i}\right), \theta_i'\right)\right| + d_j\left(m_{i,j}^2, m_j^1\right) \\ > \;& 0.\end{aligned}$$

where the last inequality follows because $m_{i,j}^2 \neq \theta_j = m_j^1$.

Second, we prove part (b). Suppose, on the contrary, that there exists some message m_j played with σ_j-positive probability such that agent j misreports agent i's type in his second report, i.e., $m_{j,j}^2 \neq \tilde{\theta}_j$. Let $\bar{m}_j$ be a message that is identical to m_j except that $\bar{m}_{j,j}^2 = \tilde{\theta}_j$. Then, for each m_{-j} played with σ_{-j}-positive probability, since every agent $i \neq j$ reports $\tilde{\theta}_j$ in agent i's second report, we have $\bar{m}_j = \bar{m}_j^m$ where $m = (m_j, m_{-j})$. Recall the definition of $\bar{m}_i^m$ in Section A.7.1.2. Hence, we reach a contradiction if we show that $\bar{m}_j$ is a profitable deviation, i.e., for each m_{-j} played with σ_{-j}-positive probability, we have

$$u_j\left(g\left(\bar{m}_j^m, m_{-j}\right), \theta_j\right) + \tau_j\left(\bar{m}_j^m, m_{-j}\right) > u_j\left(g\left(m\right), \theta_j\right) + \tau_j\left(m\right).$$

Notice that $\tau_{j,i}^1(\bar{m}_j^m, m_{-j}) = \tau_{j,i}^1(m)$.

$$\begin{aligned}\tau_j\left(\bar{m}_j^m, m_{-j}\right) - \tau_j\left(m\right) &= \sum_{i\neq j}\left\{\tau_{j,i}^2\left(\bar{m}_j^m, m_{-j}\right) - \tau_{j,i}^2\left(m\right)\right\} \\ &= \sum_{i\neq j}\left\{\begin{array}{c}\sup_{\theta_j'}\left|u_j\left(g\left(m\right), \theta_j'\right) - u_j\left(g\left(\bar{m}_j^m, m_{-j}\right), \theta_j'\right)\right| \\ +d_j\left(m_{j,j}^2, m_{i,j}^2\right)\end{array}\right\}.\end{aligned}$$

Thus, we have

$$\begin{aligned}& \left[u_j\left(g\left(\bar{m}_j^m, m_{-j}\right), \theta_j\right) + \tau_j\left(\bar{m}_j^m, m_{-j}\right)\right] - \left[u_j\left(g\left(m\right), \theta_j\right) + \tau_j\left(m\right)\right] \\ = \;& u_j\left(g\left(\bar{m}_j^m, m_{-j}\right), \theta_j\right) - u_j\left(g\left(m\right), \theta_j\right) \\ & + \sum_{i\neq j}\left\{\begin{array}{c}\sup_{\theta_j'}\left|u_j\left(g\left(m\right), \theta_j'\right) - u_j\left(g\left(\bar{m}_j^m, m_{-j}\right), \theta_j'\right)\right| \\ +d_j\left(m_{j,j}^2, m_{i,j}^2\right)\end{array}\right\} \\ > \;& 0.\end{aligned}$$

where the last inequality follows because $m^2_{j,j} \neq \tilde{\theta}_j = m^2_{i,j}$. This completes the proof. ■

A.7.4 Consistency

The argument for consistency follows verbatim the proof of Claim 3 in the proof of Theorem 1.

A.7.5 No Challenge

Claim 16 *No agent challenges with positive probability the common state $\tilde{\theta}$ announced in the second report.*

Proof. Given a consistent report of $\tilde{\theta}$ is achieved, it suffices to show that $\mathcal{SL}_j(f(\tilde{\theta}), \tilde{\theta}_j) \cap \mathcal{SU}_j(f(\tilde{\theta}), \theta_j) = \varnothing$ for every agent $j \in \mathcal{I}$. Suppose to the contrary that $\mathcal{SL}_j(f(\tilde{\theta}), \tilde{\theta}_j) \cap \mathcal{SU}_j(f(\tilde{\theta}), \theta_j) \neq \varnothing$ for some agent j. Then, we first show that $x(\tilde{\theta}, m^3_j) \neq f(\tilde{\theta})$ for every $m_j \in$supp(σ_j). By Lemma 6, there exists $x \in \mathcal{SL}_j(f(\tilde{\theta}), \tilde{\theta}_j) \cap \mathcal{SU}_j(f(\tilde{\theta}), \theta_j)$. If $x(\tilde{\theta}, m^3_j) = f(\tilde{\theta})$, then $\tilde{m}_j = \left(m^1_j, m^2_j, x\right)$ is a strictly profitable deviation from announcing m_j. This deviation results in a better allocation $x(\tilde{\theta}, x) \in \mathcal{SU}_j(f(\tilde{\theta}), \theta_j)$. Hence, we have $x(\tilde{\theta}, m^3_j) \neq f(\tilde{\theta})$ for every $m_j \in$supp(σ_j). It follows that the dictator lottery is triggered with positive probability. Thus, by (21), each agent i has a strict incentive to announce the true type in his first report (i.e., $m^1_i = \theta_i$) with σ_i-probability one. By Claim 15, we conclude that $\tilde{\theta} = \theta$ and hence $\mathcal{SL}_j(f(\theta), \theta_j) \cap \mathcal{SU}_j(f(\theta), \theta_j) \neq \varnothing$, which is impossible. ■

A.8 Proof of Theorem 5

First, we obtain a stronger version of Lemma 1, namely, there is a set of dictator lotteries which can used to elicit the agents' true types, regardless of their cardinal representation. This follows from the same proof of Lemma in Abreu and Matsushima (1992); moreover, the dictator lottery constructed remains valid (in the sense of (29)) as long as the preferences exhibit monotonicity with respect to first-order stochastic dominance.

Lemma 7 *For each $i \in \mathcal{I}$, there exists a function $y_i : \Theta_i \to \Delta(A)$ such that for every $\theta_i, \theta'_i \in \Theta_i$ with $\theta_i \neq \theta'_i$ and every cardinal representation $v_i(\cdot)$ of $(\succeq^{\theta}_i)_{\theta\in\Theta}$,*

$$v_i(y_i(\theta_i), \theta_i) > v_i(y_i(\theta'_i), \theta_i); \tag{29}$$

moreover, for each type θ'_j of agent $j \in \mathcal{I}$, we also have that for every $x \in f(\Theta)$

$$v_i(y_j(\theta'_j), \theta_i) < v_i(x, \theta_i). \tag{30}$$

To satisfy condition (30), we simply add a penalty of η to each outcome of the lotteries $\{y'_i(\theta_i)\}$ where η is chosen in the same fashion as in (2) given each cardinal representation $v_i(\cdot)$. Moreover, since each $v_i(\cdot)$ takes values in $[0, 1]$, we also save the notation and take η to be independent of $v_i(\cdot)$.

We first introduce the following definitions of contour set under ordinal preferences. For $(a, \theta_i) \in A \times \Theta_i$, under ordinal preference $\succeq_i^{\theta}$, we denote the upper-contour set, the lower-contour set, the strict upper-contour set, and the strict lower-contour set as follows:

$$\begin{aligned} U_i(a, \theta_i) &= \left\{a' \in A : a' \succeq_i^{\theta} a\right\}; \\ L_i(a, \theta_i) &= \left\{a' \in A : a \succeq_i^{\theta} a'\right\}; \\ SU_i(a, \theta_i) &= \left\{a' \in A : a' \succ_i^{\theta} a\right\}; \\ SL_i(a, \theta_i) &= \left\{a' \in A : a \succ_i^{\theta} a'\right\}; \end{aligned}$$

where $\succ_i^{\theta}$ denotes the strict preference induced by $\succeq_i^{\theta}$. We now introduce the notion of ordinal almost monotonicity proposed by Sanver (2006) as the key condition for Theorem 5.

Definition 9 *An SCF f satisfies* ***ordinal almost monotonicity*** *if, for every pair of states θ and $\tilde{\theta}$, with $f(\tilde{\theta}) \neq f(\theta)$, there is some agent $i \in \mathcal{I}$ such that either*

$$L_i(f(\tilde{\theta}), \tilde{\theta}_i) \cap SU_i(f(\tilde{\theta}), \theta_i) \neq \varnothing,$$

or

$$SL_i(f(\tilde{\theta}), \tilde{\theta}_i) \cap U_i(f(\tilde{\theta}), \theta_i) \neq \varnothing.$$

A.8.1 Proof of the Only-If Part

In the proof, we make use of Claim C from Mezzetti and Renou (2012) which is reproduced as follows:

Claim 17 *Suppose that $L_i(f(\theta), \theta_i) \subseteq L_i(f(\theta), \tilde{\theta}_i)$ and $SL_i(f(\theta), \theta_i) \subseteq SL_i(f(\theta), \tilde{\theta}_i)$. Then, given every cardinal representation $v_i(\cdot, \theta)$ of $\succcurlyeq_i^{\theta}$, there exists a cardinal representation $v_i(\cdot, \tilde{\theta})$ of $\succcurlyeq_i^{\tilde{\theta}}$ such that $v_i(a, \tilde{\theta}) \leq v_i(a, \theta)$ for all $a \in A$ and $v_i(f(\theta), \tilde{\theta}) = v_i(f(\theta), \theta)$*

Suppose f is ordinally Nash implementable in a mechanism $\mathcal{M} = ((M_i), g, (\tau_i))_{i\in\mathcal{I}}$ but not almost monotonic. That is, there exists $\theta, \tilde{\theta} \in \Theta$ such that for each agent $i \in \mathcal{I}$, we have $L_i(f(\theta), \theta_i) \subseteq L_i(f(\theta), \tilde{\theta}_i)$ and $SL_i(f(\theta), \theta_i) \subseteq SL_i(f(\theta), \tilde{\theta}_i)$, but $f(\theta) \neq f(\tilde{\theta})$. Since $\mathcal{M}$ ordinally Nash implements f, we know that for every cardinal representation $(v_i)_{i\in\mathcal{I}}$, there exists a pure-strategy Nash equilibrium m^* in the game $\Gamma(\mathcal{M}, \theta, v)$ such that $g(m^*) = f(\theta)$. Since $f(\theta) \neq f(\tilde{\theta})$, the message profile m^* cannot be a Nash equilibrium at state $\tilde{\theta}$ for every cardinal representation v_i. Then, there exists an agent i, and a message $m_i \in M_i$ such that

$$\begin{aligned} v_i\left(g\left(m_i^*, m_{-i}^*\right), \theta\right) + \tau_i\left(m_i^*, m_{-i}^*\right) &\geq v_i\left(g\left(m_i, m_{-i}^*\right), \theta\right) + \tau_i\left(m_i, m_{-i}^*\right); \\ v_i(g\left(m_i^*, m_{-i}^*\right), \tilde{\theta}) + \tau_i\left(m_i^*, m_{-i}^*\right) &< v_i(g\left(m_i, m_{-i}^*\right), \tilde{\theta}) + \tau_i\left(m_i, m_{-i}^*\right). \end{aligned}$$

Summing up the two inequalities, we obtain that for every cardinal representation v_i

$$v_i\left(g\left(m_i^*, m_{-i}^*\right), \theta\right) - v_i(g\left(m_i^*, m_{-i}^*\right), \tilde{\theta}) > v_i\left(g\left(m_i, m_{-i}^*\right), \theta\right) - v_i(g\left(m_i, m_{-i}^*\right), \tilde{\theta}). \quad (31)$$

Note that $g\left(m_i^*, m_{-i}^*\right) = f(\theta)$. By Claim 17, however, we can construct a cardinal utility representation $v_i(\cdot, \tilde{\theta})$ of $\succsim_i^{\tilde{\theta}}$ such that $v_i(a, \tilde{\theta}) \leq v_i(a, \theta)$ for all $a \in A$ and $v_i(f(\theta), \tilde{\theta}) = v_i(f(\theta), \theta)$. Therefore, the left-hand side of (31) is zero, while the right-hand side is non-negative. This is a contradiction.

A.8.2 Proof of the If Part

Let f be an SCF which is ordinally almost monotonic on Θ. Recall that V_i^θ denotes the set of all cardinal representations $v_i(\cdot, \theta_i)$ of $\succeq_i^\theta$. Define $V^\theta = \times_{i\in\mathcal{I}} V_i^\theta$ with a generic element v^θ. By no redundancy, $\theta \neq \theta'$ implies that $\succeq_i^\theta \neq \succeq_i^{\theta'}$ for some agent $i \in \mathcal{I}$. Hence, $\{V^\theta : \theta \in \Theta\}$ forms a partition of $\Theta^* \equiv \cup_{\theta\in\Theta} V^\theta$ which is the set of all cardinal utility profiles of agent i induced by Θ. Observe that Θ^* is a Polish space.[32] For the sake of notational simplicity, we write θ_i^* as a generic element in Θ_i^* and $\theta^* = (\theta_i^*)_{i\in\mathcal{I}}$. Let $f^* : \Theta^* \to A$ be the SCF on Θ^* induced by $f : \Theta \to A$ such that $f^*(\theta^*) = f(\theta)$ if and only if $\theta^* \in V^\theta$.

[32] Since any product or disjoint union of a countable family of Polish spaces remains a Polish space (see Proposition A.1(b) in p. 550 of Treves (2016)), it suffices to argue that V_i^θ is a Polish space. Let $V = [0,1]^{|A|}$ be the set of possible cardinalizations. We may write $V_i^\theta = \bigcap_{a\in A} V_{i,a}^\theta$ where for each $a \in A$, we set

$$V_{i,a}^\theta \equiv \bigcap_{\left\{b\in A: a\succ_i^\theta b\right\}} \{v \in V : v(a) > v(b)\} \bigcap \bigcap_{\left\{b\in A: a\sim_i^\theta b\right\}} \{v \in V : v(a) = v(b)\}.$$

It follows that V_i^θ is a finite intersection of open subsets and closed subsets of the Polish space V and hence remains a Polish space (see Proposition A.1(a)(c)(e) in p. 550 of Treves (2016)).

We prove the if-part by establishing two claims: First, we show that f^* is Maskin-monotonic in Claim 18. Hence, Theorem 4 implies that f^* is implementable in Nash equilibria on Θ^*. Second, it follows from Claim 19 that f is ordinally Nash implementable on Θ.

Claim 18 *If f is ordinally almost monotonic on Θ, then f^* is strictly Maskin-monotonic on Θ^*.*

Proof. Consider θ^* and $\tilde{\theta}^*$ in Θ^* such that $f^*(\theta^*) \neq f^*(\tilde{\theta}^*)$. Since $f^*(\theta^*) = f(\theta)$ if and only if $\theta^* \in V^\theta$, we must have two states θ and $\tilde{\theta} \in \Theta$ such that $\theta^* \in V^\theta$ and $\tilde{\theta}^* \in V^{\tilde{\theta}}$, and $f(\theta) \neq f(\tilde{\theta})$. Since f satisfies ordinal almost monotonicity, there exists agent $i \in \mathcal{I}$ and outcomes a, $a' \in A$ such that either $a \in L_i(f(\tilde{\theta}), \tilde{\theta}_i) \cap SU_i(f(\tilde{\theta}), \theta_i)$ or $a' \in SL_i(f(\tilde{\theta}), \tilde{\theta}_i) \cap U_i(f(\tilde{\theta}), \theta_i)$. Then, either choose $t_i < 0$ such that $(a, (t_i, \mathbf{0})) \in \mathcal{SL}_i(f^*(\tilde{\theta}^*), \tilde{\theta}_i^*) \cap \mathcal{SU}_i(f^*(\tilde{\theta}^*), \theta_i^*)$ or $t_i' > 0$ such that $(a', (t_i', \mathbf{0})) \in \mathcal{SL}_i(f^*(\tilde{\theta}^*), \tilde{\theta}_i^*) \cap \mathcal{SU}_i(f^*(\tilde{\theta}^*), \theta_i^*)$ where $\mathbf{0} \in \mathbb{R}^{I-1}$ means that every agent $j \neq i$ incurs no transfer. Hence, f^* is strictly Maskin-monotonic on Θ^*. ■

Claim 19 *If f^* is implementable in Nash equilibria, then f is ordinally Nash implementable.*

Proof. Suppose an SCF f^* is implementable in Nash equilibria on Θ^*. Then, there exists a mechanism $\mathcal{M} = ((M_i), g, (\tau_i))_{i \in \mathcal{I}}$ such that for every state $\theta^* \in \Theta^*$ and $m \in M$, (i) there exists a pure-strategy Nash equilibrium in the game $\Gamma(\mathcal{M}, \theta^*)$; and (ii) $m \in \text{supp}\ (NE(\Gamma(\mathcal{M}, \theta^*))) \Rightarrow g(m) = f^*(\theta^*)$ and $\tau_i(m) = 0$ for every $i \in \mathcal{I}$. Thus, for every state $\theta^* \in V^\theta$, we must have (i) there exists a pure-strategy Nash equilibrium in the game $\Gamma(\mathcal{M}, \theta, v^{\theta^*})$; and (ii) $m \in \text{supp}\ (NE(\Gamma(\mathcal{M}, \theta, v^{\theta^*}))) \Rightarrow g(m) = f(\theta)$ and $\tau_i(m) = 0$ for every $i \in \mathcal{I}$. Hence, f is ordinally Nash implementable on Θ. ■

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