The E-mail Game Phenomenon*

Yi-Chun Chen† Siyang Xiong‡

January 3, 2012

Abstract

Rubinstein (1989) constructs a simple coordination game showing that mutual knowledge of the payoffs up to any finite order can still result in strategic behaviors substantially different from those in the common knowledge scenario. In this paper, we study the notion of strategic discontinuity that naturally generalizes the idea in Rubinstein's example. Specifically, a (Harsanyi) type t is said to display strategic discontinuity in a game G, if there exists a sequence of types whose beliefs match those of t up to any finite order, but their strategic behaviors are substantially different from those of t in G. We first show that almost all types that are commonly used in economic analysis display strategic discontinuity in simple coordination games. In other words, the e-mail game phenomenon is pervasive among these types. We then consider the space of all types and demonstrate that some types display strategic discontinuity only in complicated finite games. Nevertheless, every type displays strategic discontinuity in some infinite-action game.

*We are grateful to Eddie Dekel and Jeff Ely for their constant support and encouragements. We thank Eduardo Faingold, Yossi Feinberg, Songying Fang, Simon Grant, Atsushi Kajii, Stephen Morris, Marcin Pęski, and Wolfgang Pesendorfer, an editor and anonymous referees for helpful comments and especially Satoru Takahashi for pointing out a mistake in an earlier draft. We also thank seminar/conference participants at NUS, Rice, Southern Methodist, Texas A&M, UT-Austin, the AEI-Four Joint Workshop, and the 10th ESWC. We gratefully acknowledge financial support from NSF (SES 0820333) and Northwestern University Economic Theory Center. All remaining errors are our own.

†Department of Economics, National University of Singapore, Singapore 117570, ecsycc@nus.edu.sg

‡Department of Economics, Rice University, Houston, TX 77005, xiong@rice.edu

1 Introduction

Virtually all game-theoretic models are built upon common knowledge assumptions. In particular, a game itself is typically assumed to be commonly known among all players. While such assumptions enhance tractability of the models, they are at best approximately satisfied in reality. It is thus important to understand how a "small departure" from a common knowledge scenario may alter our prediction of strategic behaviors.

The answer to this question clearly hinges on the notion of "small departure." It is by now well known that a natural notion—to assume that players mutually know the game up to any finite level—can drastically change strategic behaviors. A prominent example is the e-mail game proposed by Rubinstein (1989). Rubinstein constructs a simple 2×2 coordination game with two strict Nash equilibria, and considers incomplete information scenarios in which players mutually know the payoffs up to any arbitrarily high but finite level k.[1] When k is large, the scenario with mutual knowledge up to level k seems to "approximate" the scenario with common knowledge of the payoffs. However, as long as k is finite, only one of the equilibria is (ε-)rationalizable in the scenario with mutual knowledge.

The e-mail game provides "a useful, if extreme, illustration of the logic by which higher-order beliefs and knowledge might influence outcomes in strategic settings (Morris, 2002a, p.434)." Furthermore, it has by now inspired a large literature trying to explore its different aspects (see Section 2 for the detail). Since it is difficult in practice, if possible, to sort out the precise specification of players' higher-order beliefs, the e-mail game highlights a fundamental caveat on the way we analyze and predict strategic behaviors.

However, many games involved in economic analysis are neither 2×2 games nor with complete information. If the e-mail game phenomenon occurs only in such environments, whatsoever it suggests would be less of a concern. For a modeler, an important issue is to understand to what extent should she be worried about the phenomenon. Namely, it would be useful to learn which sort of higher-order beliefs will lead to the e-mail game phenomenon; in which kind of games the e-mail game phenomenon occurs; and above all, whether the e-mail game represents a general phenomenon or is only a special case. These are the questions we attempt to answer in the paper.

To answer these questions, we first use the notion of (Harsanyi) types to formu-

[1] Throughout the paper, we consider two-player games. An $m \times n$ game is a game in which player 1 has m actions and player 2 has n actions.

late incomplete information scenarios. Each type of a player identifies the player's belief about the payoff-relevant states (i.e., the first-order belief), the player's belief about the other player's beliefs about the payoff relevant states (i.e., the second-order belief), and so on. Mertens and Zamir (1985) construct a *universal type space* which "contains" all types.[2] We consider a natural generalization of the e-mail game approximation. Say a sequence of types $\{t_m\}_{m=1}^{\infty}$ *approximates* a type t if for any m, t_m and t have the same beliefs up to order m. This notion of proximity strengthens proximity in *product topology* which only requires that $\{t_m\}_{m=1}^{\infty}$ and t have *(weak*-)similar* beliefs up to any finite order, and both notions are formulated in terms of higher-order beliefs and thus easy to check.[3] The approximation is plausible because "common sense suggests that the players themselves will have their beliefs only partially articulated in their own minds (Weinstein and Yildiz, 2007, p.371)." For example, in the e-mail game, the common knowledge scenario corresponds to a type with complete information about a payoff relevant state, while the mutual knowledge scenarios correspond to a sequence of types which approximates the complete-information type.[4]

We then define a notion called *strategic discontinuity* which generalizes the e-mail game phenomenon. We say that a type t displays strategic discontinuity in a game G if there exists a sequence of types $\{t_m\}_{m=1}^{\infty}$ which approximates t, but for some fixed positive ε, some action in G is rationalizable for t but not $\varepsilon-$rationalizable for any t_m. For instance, the common knowledge type displays strategic discontinuity in Rubinstein's e-mail game. Strategic discontinuity is a property associated with both types and games; once we fix a class of games $\mathcal{G}$ and require that G be found in $\mathcal{G}$, it becomes a property of

[2]Types are usually defined by a type space in which each type specifies a belief about the payoff relevant states and the other player's types. More precisely, Mertens and Zamir (1985) show that every type space can be embedded into the universal type space in a way that preserves higher-order beliefs. In Harsanyi's original paper, a type specifies not only an implicit representation of belief hierarchy but also private information about payoffs, whereas in our paper, which follows Dekel, Fudenberg, and Morris (2006) and Ely and Pęski (2011), a type only specifies the former (see (Dekel, Fudenberg, and Morris, 2006, p.276) for discussion).

[3]To address the problem exhibited in the e-mail game, Dekel, Fudenberg, and Morris (2006) define the *strategic topology* on types. However, the strategic topology is defined using the rationalizable actions of types in games rather the primitives, i.e., the higher-order beliefs of types. Hence, it is often harder to check whether two types are close in the strategic topology than in product topology. As a result, product topology is still commonly adopted.

[4]These types assign probability 1 to the payoffs, assign probability 1 that the other player assigns probability 1 on the payoffs, and so on only up to some finite order. Throughout the paper, we do not distinguish between "knowledge" and "belief with probability 1."

types.

We follow Dekel, Fudenberg, and Morris (2006) to consider the natural classification of games according to the number of actions available to the players. Using similar terminology as in Ely and Pęski (2011) (see also Ely and Pęski (2007)), we say a type t is $n-$*critical* if t displays strategic discontinuity in an $n \times n$ game. A type is $n-$regular if it is not $n-$critical. For instance, the common-knowledge type in Rubinstein's e-mail game is $2-$critical. Intuitively, a game with more actions is more complicated and harder to solve. Indeed, an $n-$critical type is n'-critical for any $n' > n$. We have in mind a modeler who is analyzing an $n \times n$ game and cannot rule out any beliefs beyond a finite order. If a type in her model is $n-$critical, her prediction of the type's behaviors may be far away from the truth being approximated.

We start by analyzing finite types, i.e., types in a finite type space. Such types appear frequently in both applied and theoretical work. Our first result shows that every finite type displays strategic discontinuity in some 2×3 game and hence is $3-$critical. We then extend the argument to a class of types that have *common*$-p$ *belief in a proper first-order event* (see Definition 3).[5] This class includes all finite types and almost all types in any type space which admits a common prior. We show that every type in this class exhibits strategic discontinuity in some 2×4 game. Moreover, this class of types is "strategically" generic in the sense that it contains an open and dense set in the universal type space in the strategic topology defined in Dekel, Fudenberg, and Morris (2006). The simple games that we use to demonstrate strategic discontinuity resemble the e-mail game. Consequently, the e-mail game phenomenon arises around essentially all "economic types." In other words, even when the modeler can confine herself to the set of simple games, she cannot be sure that the "true type," which these "economic types" are meant to approximate, behaves similarly.

These results raise the question of whether strategic discontinuity in simple games persists around *any* type. To answer this question, we construct types which display strategic discontinuity *only* in complicated (albeit finite) games. Specifically, for every n, we construct an $n'-$critical type (with $n' > n$) which is $n-$regular. Finally, we show that if we allow for games with infinite actions, then every type displays strategic discontinuity. One key step in proving this result is to show that around every type t, there is a sequence of types which approximates t but fails to converge to t in the *uniform strategic topology*

[5]The notion of common$-p$ belief is first introduced by Monderer and Samet (1989). See Section 4.4 for the formal definition.

defined in Dekel, Fudenberg, and Morris (2006).[6]

Our main results can be summarized in two points:

- "Commonly seen types", such as finite types and types admitting common priors, display strategic discontinuity in simple games (Theorems 1 and 2), and moreover, these types are strategically generic (Theorem 3).
- Some types display strategic discontinuity only in complicated games (Theorem 4). Moreover, every type displays strategic discontinuity in some infinite-action game (Theorem 5).

The rest of the paper proceeds as follows. Section 2 surveys the related literature. Section 3 presents a non-technical illustration that the e-mail game phenomenon occurs around any finite type. Section 4 provides formal notations and definitions. Section 5 presents the results for strategic discontinuity in simple games. Section 6 constructs the $n'-$critical$-n-$regular types. Section 7 analyzes strategic discontinuity in infinite games. Section 8 discusses some related issues.

2 Related literature

Rubinstein (1989) mentioned that the e-mail game is closely related to the coordinated attack problem in the distributed system literature. The whole literature on the global game initiated by Carlsson and Van Damme (1993) exploits a similar idea to argue that introducing a vanishingly small noisy signal (similar to the approximation of types we study) in a game with multiple equilibria enables selecting a unique outcome. The selection can be interpreted as an alternative form of strategic discontinuity. In a recent paper, Weinstein and Yildiz (2007) substantially generalize the previous selection arguments by showing that when possible payoffs are sufficiently rich, every rationalizable outcome can be selected as the unique rationalizable outcome in a suitably perturbed game.

[6] Convergence in the uniform strategic topology requires that the speed of convergence of strategic behaviors be uniform over all finite games with a uniform payoff bound and thus is relevant for environments in which the game—both payoffs and action sets—is not a priori fixed. We note that whether this intermediate step (Proposition 5) is true is raised independently by Professor Drew Fudenberg on his class website.

Morris (2002b) is the first to study strategic discontinuity displayed by commonly seen types. In particular, Morris shows that a class of types, which includes all finite types, exhibits discontinuity in rationalizable behaviors in a specific infinite "higher-order expectations" game. Ely and Pęski (2011) advance this idea by defining and studying the notion of critical types, i.e., types which display strategic discontinuity in some finite-action game. Ely and Peski obtain a surprisingly concise characterization of critical types: a type is critical iff it has common $p-$belief in some closed proper subset in the universal type space. Consequently, all types in finite type spaces are critical; all common priors assign probability 1 to critical types; and moreover, regular types are generic in product topology.

In proving their characterization, for each critical type, Ely and Pęski (2011) construct a finite-action game to demonstrate its discontinuity. The game they construct is complicated and involves a large number of actions. This is in sharp contrast to the simple 2×2 game employed by Rubinstein (1989). On the one hand, our results for simple games show that if one wants to focus on "economic types," the notion of critical types goes too far by allowing all finite games of which many are complex and rarely-seen. Moreover, their arguments with complicated games make it difficult to see how critical types relate to the e-mail game. On the other hand, our construction of the $n'-$critical$-n-$regular types demonstrates that it is necessary for them to consider all finite games so as to fully characterize *all* critical types. Our results also clarify that even regular types display strategic discontinuity in infinite games and thus can alternatively be viewed as $\infty-$critical$-n-$regular types for any finite n.

Another line of research inspired by the e-mail game phenomenon seeks for alternative notion of proximity in beliefs that guarantees proximity of strategic behaviors. For different solution concepts and environments, different notions have been considered by, for instance, Monderer and Samet (1989, 1996); Kajii and Morris (1998); Dekel, Fudenberg, and Morris (2006); Chen, Di Tillio, Faingold, and Xiong (2010, 2011). In particular, Dekel, Fudenberg, and Morris (2006) define the strategic topology in which types are close iff they have similar rationalizable behaviors and show that the strategic topology is finer than product topology. Critical types can be viewed as types around which product topology is strictly coarser than the strategic topology. Contrary to the result in Ely and Pęski (2011) that regular types are generic in product topology, Chen, Di Tillio, Faingold, and Xiong (2008) show that the set of critical types is generic in the strategic topology — it contains an open and dense set in the strategic topology. Here our result shows that even strategic discontinuity in simple games is a generic phenomenon and thereby provide another motivation to study strategic proximity of types.

Finally, we remark that what we mention above is far from a comprehensive list of applications or extensions of the e-mail game phenomenon. For example, there have been numerous works in experimental and behavioral economics using the e-mail game phenomenon to motivate and study new models of reasoning in games (see, e.g., the recent paper by Strzalecki (2009) and references therein).

3 The "e-mail game" for finite types

In this section, we analyze the e-mail game phenomenon around finite types. It will be clear that the idea in fact resembles the one in Rubinstein (1989). To ease the comparison, we first recall Rubinstein's idea by considering the following two-player game.

	a_2	b_2
a_1	$1,1$	$0,0$
b_1	$0,0$	$1,1$

θ_0

	a_2	b_2
a_1	$1,1$	$-1,2$
b_1	$2,-1$	$0,0$

θ_1

There are two states, θ_0 and θ_1. At state θ_0, the game is "meeting in New York" and both (a_1, a_2) and (b_1, b_2) are Nash equilibria and hence all actions are rationalizable. At state θ_1, the game is "prisoner's dilemma" in which b_1 and b_2 are the unique $\frac{1}{2}-$rationalizable actions for player 1 (the row player) and player 2 (the column player), respectively. That is, for $i = 1, 2$, playing b_i is better than playing a_i by a payoff difference of at least $1/2$.

Suppose that all players either know the state is θ_0 or know the state is θ_1. First, Figure 1 represents a type space of the common knowledge scenario. In this type space, player 1 has only one type t^{θ_0} and player 2 has only one type s^{θ_0}. Both players (types) know that the state is θ_0, and know that they know the state is θ_0 (as they assign probability one to the only opponent's type which knows that the state is θ_0), and so on. In this case, both a_1 and b_1 are clearly rationalizable for t^{θ_0}.

Second, Figure 2 represents a type space of the mutual knowledge scenarios. For instance, $s^{\theta_0}_{2n+2}$ assigns probability 1 to θ_0 and $t^{\theta_0}_{2n+1}$; $t^{\theta_0}_{2n+1}$ assigns probability 1 to θ_0 and $s^{\theta_0}_{2n}$; and so on. The set of types for players 1 and 2 are $\{t^{\theta_1}\} \cup \left\{t^{\theta_0}_{2n-1}\right\}_{n=1}^{\infty}$ and $\{s^{\theta_1}\} \cup \left\{s^{\theta_0}_{2n}\right\}_{n=1}^{\infty}$ respectively. Like above, t^{θ_1} and s^{θ_1} represent the scenario of "common knowledge of θ_1" and thus only b_2 is the unique $\frac{1}{2}-$rationalizable for s^{θ_1}. However, all of

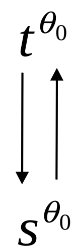

Figure 1: the type space of the common knowledge scenario (an arrow means "assigns probability 1 to")

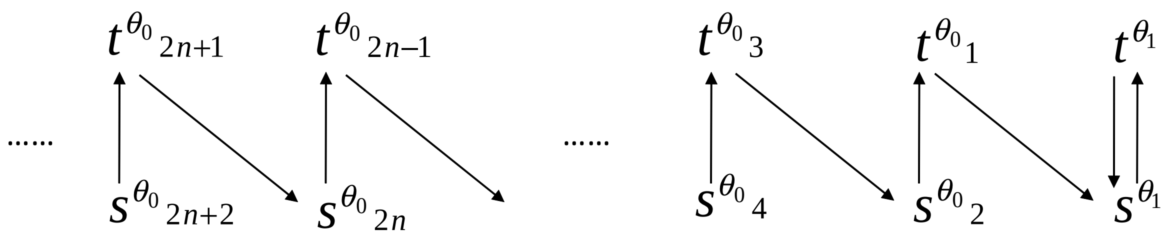

Figure 2: strategic discontinuity for the common-knowledge type t^{θ_0}

the types $t_{2n-1}^{\theta_0}$ and $s_{2n}^{\theta_0}$ know that the state is θ_0. Hence, $t_{2n-1}^{\theta_0}$ represents mutual knowledge of θ_0 up to level $2n-1$, i.e., it has the same beliefs as t^{θ_0} up to order $2n-1$. Thus, $\left\{t_{2n-1}^{\theta_0}\right\}_{n=1}^{\infty}$ approximates t^{θ_0}. Nevertheless, each $t_{2n-1}^{\theta_0}$ has the unique $\frac{1}{2}-$rationalizable action b_1: $t_1^{\theta_0}$ has a unique $\frac{1}{2}-$rationalizable action b_1 because his opponent's type s^{θ_1} has a unique $\frac{1}{2}-$rationalizable action b_2; $s_2^{\theta_0}$ has a unique $\frac{1}{2}-$rationalizable action b_2 because his opponent $t_1^{\theta_0}$ has a unique $\frac{1}{2}-$rationalizable action b_1, and so on. To sum up, the common-knowledge type t^{θ_0} displays strategic discontinuity in this 2×2 game.

We now show that a similar idea proves that all finite types display strategic discontinuity in simple games. Let Θ be a set of possible payoff parameters in a 2×3 game. Pick any finite type $\widehat{t}\in T$ in a type space $T\times S$ where T is the set of player 1's types and S is the set of player 2's types. Each $t\in T$ (resp. $s\in S$) is associated with a distribution $\mu_t\in\triangle(\Theta\times S)$ (resp. $\mu_s\in\triangle(\Theta\times T)$) which specifies all higher-order beliefs of t (resp. s). In particular, all types in T know their opponent's types are in S and vice versa, which can be illustrated in Figure 3.

Since S is finite, there exist some number $y\in(0,1)$ and some $\theta_0\in\Theta$ such that

$$\text{for any } s\in S, \text{ either } \mu_s[\theta_0]\geq y \text{ or } \mu_s[\theta_0]=0. \tag{1}$$

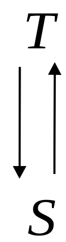

Figure 3: the finite type space which contains $\widehat{t}$

Namely, for any $s \in S$, the probability that s attaches to θ_0 never falls into the interval $(0, y)$. Choose $x_1 > 0$ such that $\frac{x_1}{1+x_1} = y$. Define $y' = \frac{x_1}{2+x_1}$. Thus, $0 < y' < y < 1$.

Consider the following 2×3 game G:

	a_2	b_2	c_2
a_1	$0, -1$	$0, -\frac{2}{x_1}$	$1, 0$
b_1	$1, 1$	$1, -\frac{2}{x_1}$	$0, 0$

$\theta = \theta_0$

	a_2	b_2	c_2
a_1	$0, -x_1$	$0, -1$	$1, 0$
b_1	$1, -x_1$	$1, 1$	$0, 0$

$\theta \neq \theta_0$

Observe that

$$\underbrace{1 \cdot \mu_s[\theta_0] + (-x_1) \cdot (1 - \mu_s[\theta_0])}_{\text{the payoff of } s \text{ by choosing } a_2 \text{ when player 1 chooses } b_1} \geq \underbrace{0}_{\text{the payoff of } s \text{ by choosing } c_2} \Leftrightarrow \mu_s[\theta_0] \geq \frac{x_1}{1+x_1} = y; \tag{2}$$

$$\underbrace{\left(-\frac{2}{x_1}\right) \cdot \mu_s[\theta_0] + 1 \cdot (1 - \mu_s[\theta_0])}_{\text{the payoff of } s \text{ by choosing } b_2 \text{ when player 1 chooses } b_1} \geq \underbrace{0}_{\text{the payoff of } s \text{ by choosing } c_2} \Leftrightarrow \mu_s[\theta_0] \leq \frac{x_1}{2+x_1} = y'. \tag{3}$$

By (2) and (3), when player 1 chooses b_1, player 2's best reply only depends on his belief on θ which can be illustrated by Figure 4. Furthermore, by (1), for any $s \in S$, $\mu_s[\theta_0]$ must be located on the red segment in Figure 4. Then, b_1 is rationalizable for all $t \in T$ because the following strategy profile constitutes a Bayesian Nash Equilibrium.

$$\left[\begin{array}{c} \text{every } t \in T \text{ plays } b_1; \\ \text{every } s \in S \text{ plays } b_2 \text{ if } \mu_s[\theta_0] = 0 \text{ and plays } a_2 \text{ if } \mu_s[\theta_0] \geq y. \end{array}\right]$$

Given this strategy profile, each type $t \in T$ gets the maximal payoff of this game (=1) by playing b_1; it is also optimal for each $s \in S$ to play b_2 if $\mu_s[\theta_0] = 0$ and to play a_2 if $\mu_s[\theta_0] \geq y$ (see Figure 4).

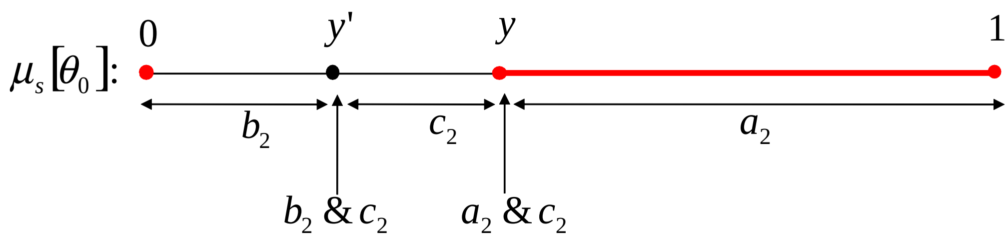

Figure 4: player 2's best reply when player 1 chooses b_1

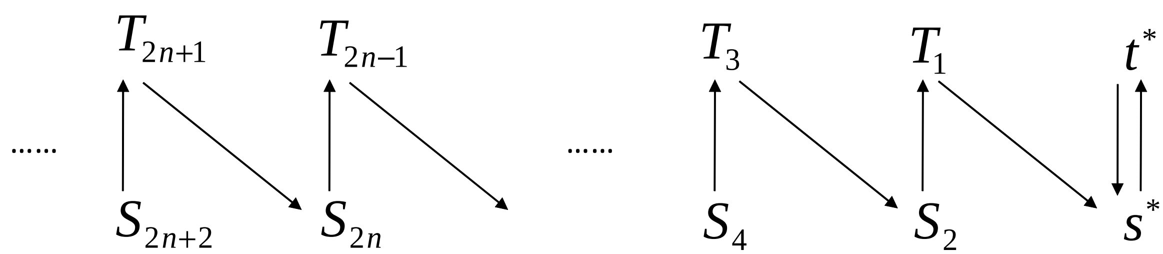

Figure 5: strategic discontinuity for the finite type $\widehat{t}$.

We are now ready to show that $\widehat{t}$ displays strategic discontinuity in this 2×3 game. First, consider the types t^* and s^* which commonly know $\Pr(\theta = \theta_0) = \frac{y+y'}{2}$.[7] By (2) and (3), there is some $\gamma \in \left(0, \min\left\{1, \frac{2}{x_1}, x_1\right\}\right)$ such that c_2 is the unique $\gamma-$rationalizable action for s^*.[8]

Then, consider countable copies T and countable copies of S. Specifically, let t_n (resp. s_n) denote the nth copy of $t \in T$ (resp. $s \in S$). Consider $T_n = \{t_n : t \in T\}$ and $S_n = \{s_n : s \in S\}$. For any $n \geq 1$, all types in T_{2n+1} (resp. S_{2n}) know his opponent's types are in S_{2n} (resp. T_{2n-1}), and moreover, t_{2n+1} (resp. s_{2n}) is endowed with the "same" beliefs as t (resp. s). Formally, $\mu_{t_{2n+1}}[(\theta, s_{2n})] = \mu_t[(\theta, s)]$ for any (θ, s) and $\mu_{s_{2n}}[(\theta, t_{2n-1})] = \mu_s[(\theta, t)]$ for any (θ, t). Furthermore, every t_1 in T_1 is endowed with the same first-order belief as t, but assigns probability 1 to s^*. That is, $\mu_{t_1}[(\theta, s^*)] = \mu_t[\theta]$ for every θ. We illustrate this type space using Figure 5.

[7] For exmaple, consider the types t^* and s^* defined by $\mu_{t^*}[(\theta_0, s^*)] = \mu_{s^*}[(\theta_0, t^*)] = \frac{y+y'}{2}$ and $\mu_{t^*}[(\theta_1, s^*)] = \mu_{s^*}[(\theta_1, t^*)] = 1 - \frac{y+y'}{2}$ for some $\theta_1 \neq \theta_0$.

[8] For example, set $\gamma = \frac{1}{4}\min\left\{1, \frac{2}{x_1}, x_1, \left|1 \cdot \frac{y+y'}{2} + (-x_1) \cdot \left[1 - \frac{y+y'}{2}\right]\right|, \left|\left(-\frac{2}{x_1}\right) \cdot \frac{y+y'}{2} + 1 \cdot \left[1 - \frac{y+y'}{2}\right]\right|\right\}$. Then, by choosing c_2, s^* always gets 0; by choosing a_2, s^* gets at most $1 \cdot \frac{y+y'}{2} + (-x_1) \cdot \left[1 - \frac{y+y'}{2}\right] < -\gamma < 0$; by choosing b_2, s^* gets at most $\left(-\frac{2}{x_1}\right) \cdot \frac{y+y'}{2} + 1 \cdot \left[1 - \frac{y+y'}{2}\right] < -\gamma < 0$.

For any $t \in T$, $s \in S$ and $n \geq 1$, t_1 and t have the same belief about θ, s_2 and s have the same belief about θ and the same belief about player 1's beliefs about θ,..., and thus inductively, t_{2n-1} has the same beliefs as that of t up to order $(2n-1)$. Hence, $\{\widehat{t}_{2n-1}\}_{n=1}^{\infty}$ approximates $\widehat{t}$.

However, for any n, all types in T_{2n-1} (including $\widehat{t}_{2n-1}$) have the unique $\gamma-$rationalizable action a_1, i.e., b_1 is not $\gamma-$rationalizable: all types in T_1 has the unique $\gamma-$rationalizable action a_1 because $\gamma < 1$ and she knows that player 2 has the type s^* which plays the unique $\gamma-$rationalizable action c_2; all types in S_2 has a unique $\gamma-$rationalizable action c_2 because $\gamma < \min\left\{1, \frac{2}{x_1}, x_1\right\}$ and she knows that player 1 has the types in T_1 which all play the unique $\gamma-$rationalizable action a_1, and so on. Recall that b_1 is rationalizable for all $t \in T$ (including $\widehat{t}$). Therefore, $\widehat{t}$ displays strategic discontinuity in this 2×3 game.

4 Preliminaries

Throughout this paper, we fix a two-player set I.[9] We also fix a finite set of payoff relevant states Θ which contains at least two distinct elements. Given a player $i \in I$, we write $-i$ to denote the other player in I. For any metric space Y, let $\Delta(Y)$ be the space of all probability measures on the Borel σ-algebra of Y endowed with the weak*-topology. Let $\text{supp}\mu$ be the support of a probability measure μ, i.e., the smallest closed set with measure 1 under μ. For any $y \in Y$, let δ_y be the Dirac measure on y. Unless otherwise mentioned, all product spaces are endowed with product topology and subspaces are endowed with relative topology. Every finite or countable set is endowed with discrete topology and $|E|$ denotes the cardinality of a finite set E.

4.1 Belief hierarchies and proximity of beliefs

Our formulation of incomplete information follows Mertens and Zamir (1985).[10] Let $Y^0 = \Theta$ and $Y^1 = Y^0 \times \Delta(Y^0)$. Then, for $k \geq 2$ define recursively

$$Y^k = \left\{ \left(\theta, \mu^1, ..., \mu^k\right) \in Y^0 \times \Delta\left(Y^0\right) \times \cdots \times \Delta\left(Y^{k-1}\right) : \text{marg}_{Y^{l-2}}\mu^l = \mu^{l-1}, \forall l = 2, ..., k \right\}.$$

[9]For simplicity, we restrict our attention to two-player games. All of our results can be easily extended to n-player games with $\infty > n \geq 3$.

[10]An alternative and equivalent formulation can be found in Brandenburger and Dekel (1993).

Then, define

$$\mathcal{T} = \left\{ \left(\mu^1, \mu^2, ...\right) \in \times_{k=0}^{\infty} \Delta \left(Y^k\right) : \text{marg}_{Y^{l-2}} \mu^l = \mu^{l-1}, \forall l \geq 2 \right\}.$$

For every player i, let $\mathcal{T}_i$ denote a copy of $\mathcal{T}$. An element $t_i \in \mathcal{T}_i$ is a type of player i. For any $t_i = (\mu^1, \mu^2, ...)$ and $k = 1, 2, ...,$ let $t_i^k = \mu^k$ be the k^{th}-order belief of t_i. Clearly, if two types have the same k^{th}-order belief, they must have the same k'^{th}-order belief for any $k' \leq k$. We say a sequence of types $\{t_{i,m}\}_{m=1}^{\infty}$ approximates a type t_i iff $t_{i,m}^m = t_i^m$ for every m. It follows that $\{t_{i,m}\}_{m=1}^{\infty}$ converges to t_i in product topology if it approximates t_i.[11] Mertens and Zamir (1985) show that $\mathcal{T}_i$ endowed with product topology is a compact metric space, and moreover, is homeomorphic to $\Delta(\Theta \times \mathcal{T}_{-i})$ under a homeomorphism π_i^*. The tuple $(\mathcal{T}_i, \pi_i^*)_{i \in I}$ is called the (Mertens-Zamir) universal type space.

In general, by a type space we mean a tuple $(T_i, \pi_i)_{i \in I}$ where T_i is a metric space and π_i is a continuous mapping which associates each $t_i \in T_i$ with a belief $\pi_i(t_i) \in \Delta(\Theta \times T_{-i})$. Say $(T_i, \pi_i)_{i \in I}$ is a finite type space if $T_i \cup T_{-i}$ is a finite set. Given any type space $(T_i, \pi_i)_{i \in I}$ and $t_i \in T_i$, we can compute the belief of t_i on Θ by setting $t_i^1 = \text{marg}_{\Theta} \pi_i(t_i)$. We can also compute the second-order belief of t_i (i.e., his belief about (θ, t_{-i}^1)) by setting

$$t_i^2(F) = \pi_i(t_i)\left(\left\{(\theta, t_{-i}) : \left(\theta, t_{-i}^1\right) \in F\right\}\right) \text{ for each measurable set } F \subseteq \Theta \times \Delta(\Theta).$$

We can similarly compute the entire hierarchy of beliefs $(t_i^1, t_i^2, ..., t_i^k, ...)$ by proceeding in this way. For any type space $(T_i, \pi_i)_{i \in I}$ and any $t_i \in T_i$, there is some $s_i \in \mathcal{T}_i$ such that $s_i^k = t_i^k$ for all k. For the solution concept that we adopt and will be introduced below, only the hierarchy of beliefs of a type matters and hence we often identify a type with its belief hierarchy. We call $s_i \in \mathcal{T}_i$ a finite type if there exits some type t_i in a finite type space such that $s_i^k = t_i^k$ for all k.

For expositional ease, we simplify our notations as follows.

1. For any $\mu \in \Delta(\Theta)$, we use t^{μ} to denote the type which has common knowledge that both players have the first-order belief μ, i.e. t^{μ} is contained in the following type space: $T_1 = T_2 = \{t^{\mu}\}$ and for all $i \in \{1, 2\}$ and all θ, $\pi_i(t^{\mu})[(\theta, t^{\mu})] = \mu[\theta]$.

2. We write t^{θ} instead of $t^{\delta_{\{\theta\}}}$ for the type under which a payoff parameter θ is commonly known.

[11] $\{t_{i,m}\}_{m=1}^{\infty}$ converges t_i in product topology iff for every positive integer k, $\left\{t_{i,m}^k\right\}_{m=1}^{\infty}$ converges to t_i^k in the weak*-topology. Since $t_{i,m}^k = t_i^k$ implies $t_{i,m}^{k'} = t_i^{k'}$ for any $k' \leq k$, $\{t_{i,m}\}_{m=1}^{\infty}$ converges to t_i in product topology if it approximates t_i.

3. We identify $t_i \in \mathcal{T}_i$ with its image under the homeomorphism π_i^* and write $t_i[E]$ instead of $\pi_i^*(t_i)[E]$ for the probability t_i assigns to some $E \subseteq \Theta \times \mathcal{T}_{-i}$.

4. We write $t_i[\theta_0]$ instead of $t_i[\{\theta_0\} \times \mathcal{T}_{-i}]$ for the probability a type t_i assigns to a payoff parameter θ_0.

5. We write $t_i[E_{-i}]$ instead of $t_i[\Theta \times E_{-i}]$ for the probability a type t_i assigns to a set of types E_{-i}.

4.2 Games and the solution concept

Let $G = \langle A_i, g_i \rangle_{i \in I}$ be a game where A_i is a finite or countably infinite set of actions for player i and $g_i : A_i \times A_{-i} \times \Theta \to \Re$ is player i's payoff function. Let $A = A_i \times A_{-i}$ and denote a typical element in A as $a = (a_i, a_{-i})$. We only deal with infinite games in Section 7. Following Dekel, Fudenberg, and Morris (2006), we define the set of $\gamma-$interim correlated rationalizable ($\gamma-$ICR) actions of type t_i as follows.

Given a type space $(T_i, \pi_i)_{i \in I}$ and a real number $\gamma \geq 0$, we say that $(\overline{R}_i)_{i \in I}$ with $\overline{R}_i : T_i \to 2^{A_i} \setminus \{\varnothing\}$ has the $\gamma-$*best-reply property* iff for every $i \in I$, every $t_i \in T_i$ and every $a_i \in \overline{R}_i(t_i)$, there exists a measurable function $\sigma_{-i} : \Theta \times T_{-i} \to \Delta(A_{-i})$ such that

$$\text{supp}\sigma_{-i}(\theta, t_{-i}) \subseteq \overline{R}_{-i}(t_{-i}) \text{ for } \pi_i(t_i) - \text{almost all } (\theta, t_{-i});$$
$$\int_{\Theta \times T_{-i}} \sum_{a_{-i} \in A_{-i}} \left[g_i(a_i, a_{-i}, \theta) - g_i(a_i', a_{-i}, \theta) \right] \sigma_{-i}(\theta, t_{-i})[a_{-i}] \, \pi_i(t_i)[(d\theta, dt_{-i})] \geq -\gamma, \forall a_i' \in A_i. \tag{4}$$

We define the set of $\gamma-$ICR actions of a type t_i in G by

$$R_i(t_i, G, \gamma) = \bigcup_{(\overline{R}_i, \overline{R}_{-i}) \text{ has the } \gamma-\text{best-reply property}} \overline{R}_i(t_i). \tag{5}$$

Clearly, $(R_i(\cdot, G, \gamma))_{i \in I}$ also has the $\gamma-$best-reply property.

When G is a finite game, $R_i(t_i, G, \gamma)$ has an equivalent recursive definition: Let $R_i^0(t_i, G, \gamma) = A_i$. For any integer $k \geq 1$, $a_i \in R_i^k(t_i, G, \gamma)$ iff there exists a measurable function $\sigma_{-i} : \Theta \times T_{-i} \to \Delta(A_{-i})$ such that

$$\text{supp}\sigma_{-i}(\theta, t_{-i}) \subseteq R_{-i}^{k-1}(t_i, G, \gamma) \text{ for } \pi_i(t_i) - \text{almost all } (\theta, t_{-i});$$
$$\int_{\Theta \times T_{-i}} \sum_{a_{-i} \in A_{-i}} \left[g_i(a_i, a_{-i}, \theta) - g_i(a_i', a_{-i}, \theta) \right] \sigma_{-i}(\theta, t_{-i})[a_{-i}] \, \pi_i(t_i)[(d\theta, dt_{-i})] \geq -\gamma, \forall a_i' \in A_i.$$

Dekel, Fudenberg, and Morris (2006, 2007) show that $R_i(t_i, G, \gamma) = \cap_{k=1}^{\infty} R_i^k(t_i, G, \gamma)$, and moreover, $R_i^k(t_i, G, \gamma) = R_i^k(s_i, G, \gamma)$ if t_i and s_i have the same beliefs up to order k. Consequently, $R_i(t_i, G, \gamma) = R_i(s_i, G, \gamma)$ if t_i and s_i have the same hierarchy of beliefs. Recall that for any type space $(T_i, \pi_i)_{i \in I}$ and any $t_i \in T_i$, there is one and only one $s_i \in \mathcal{T}_i$ such that $s_i^k = t_i^k$ for all k. We will identify s_i and t_i since they have the same rationalizable behaviors. When G is infinite, although it is still true that $R_i(t_i, G, \gamma) \subseteq \cap_{k=1}^{\infty} R_i^k(t_i, G, \gamma)$, these two sets are not necessarily equal.[12]

The strategic topology on $\mathcal{T}_i$ is the coarsest topology such that for each finite game G and $\gamma \geq 0$, the $\gamma-$rationalizable correspondence $R_i(\cdot, G, \gamma)$ is upper hemicontinuous and the *strict* $\gamma-$rationalizable correspondence $\cup_{\gamma' < \gamma} R_i(\cdot, G, \gamma')$ is lower hemicontinuous.[13] We conclude the section by recalling a result in Dekel, Fudenberg, and Morris (2006) that will be used later.

Lemma 1 (Dekel, Fudenberg, Morris (2006, Theorem 2)) *For any finite game G and real number $\gamma \geq 0$, $R_i(\cdot, G, \gamma)$ is an upper hemicontinuous correspondence on $\mathcal{T}_i$ when $\mathcal{T}_i$ is endowed with product topology. That is, for any $A_i' \subseteq A_i$, $\left\{t_i \in \mathcal{T}_i : R_i(t_i, G, \gamma) \subseteq A_i'\right\}$ is a product-open set.*

4.3 Strategic discontinuity and $n-$critical types

By Lemma 1, in a finite game, a product convergent (and in particular an approximating) sequence of types can only exhibit a failure of lower hemicontinuity in the strict $\gamma-$rationalizable correspondence. Ely and Pęski (2011) make use of this observation to define the notion of critical types. Here we follow their idea and further partition the set of critical types according to the "size" of the games in which they exhibit strategic discontinuity.

[12] The definition of rationalizability for infinite games studied here is also adopted by Bergemann, Morris, and Tercieux (2011) when they deal with "integer games" in their study of rationalizable implementation. Furthermore, when G is infinite, a recursive definition of $R_i(t_i, G, \varepsilon)$ may involve transfinite induction (see Lipman (1994)).

[13] The definition of strategic topology is different from the original definition in (Dekel, Fudenberg, and Morris, 2006, p.287). The original definition is given by a metric on types which compares their $\gamma-$rationalizable bahaviors in all finite games with a uniform payoff bound. (Ely and Pęski, 2007, Proposition 2 and footnote 15) show that the definition we adopt (which involve neither a metric nor a uniform bound) is equivalent to the original one (see also Chen, Di Tillio, Faingold, and Xiong (2010)).

Definition 1 *A type t_i displays strategic discontinuity in a game G if there exist $\varepsilon > 0$ and a sequence of types $\{t_{i,m}\}_{m=1}^{\infty}$ which approximates t_i such that for every m, there is some $a_i \in R_i(t_i, G, 0) \diagdown R_i(t_{i,m}, G, \varepsilon)$.*

Definition 2 *For $n = 1, 2, ..., \infty$, a type is $n-$critical if it displays strategic discontinuity in an $n \times n$ game. A type is $n-$regular if it is not $n-$critical. A type is critical if it is $n-$critical for some $n < \infty$. A type is regular if it is not critical.*

We make two remarks on this definition. First, it is clear that an $n-$critical type must be an $n'-$critical type for any $n' > n$. Namely, types displaying strategic discontinuity in simple games always display strategic discontinuity in complicated games. Second, we define strategic discontinuity in terms of $0-$rationalizability instead of $\gamma-$rationalizability. We may alternatively define a type t_i to be $n-$critical if there exist $\varepsilon > 0$, $\gamma \geq 0$ and a sequence of types $\{t_{i,m}\}_{m=1}^{\infty}$ which approximates t_i such that for every m, there is some $a_i \in R_i(t_i, G, \gamma) \diagdown R_i(t_{i,m}, G, \gamma + \varepsilon)$. All of our results remain true if we adopt the notion of strategic discontinuity in terms of $\gamma-$rationalizability.[14]

4.4 $p-$beliefs and common $p-$beliefs

We follow Ely and Pęski (2011) to define the notions of $p-$belief and common $p-$belief as follows. For any measurable set $E_{-i} \subseteq \mathcal{T}_{-i}$ and $p \in (0, 1]$, let $B_i^p(E_{-i})$ be the event that player i assigns at least probability p to (i.e., $p-$believes) E_{-i}. That is,

$$B_i^p(E_{-i}) = \{t_i \in \mathcal{T}_i : t_i[E_{-i}] \geq p\}.$$

Say $E \subseteq \mathcal{T}_i \times \mathcal{T}_{-i}$ is an *event* if $E = E_i \times E_{-i}$ for some measurable sets $E_i \subseteq \mathcal{T}_i$ and $E_{-i} \subseteq \mathcal{T}_{-i}$. For any event $E = E_i \times E_{-i}$, let $B_i^p(E) = E_i \cap B_i^p(E_{-i})$ and $B^p(E) = B_i^p(E) \times B_{-i}^p(E)$. Consequently, $B^p(E) \subseteq E$.

We say that the players have common $p-$belief in E when they $p-$believe in E, and they $p-$believe in $B^p(E)$, and so on. This concept was first formulated by Monderer and

[14] This difference in the definition of strategic discontinuity does not occur in Ely and Pęski (2011) because they consider *all* finite games, and Lemma 4 in Ely and Pęski (2007) (see Appendix A.1) shows that a critical type which exbibits strategic discontinuity in terms of $\gamma-$rationalizability in some $n \times n$ game must exhibit strategic discontinuity in terms of $0-$rationalizability in an $n^2 \times n^2$ game.

Samet (1989). Formally, denote by $[B^p]^k$ the iteration of the operator B^p for k times and define

$$C^p(E) = \bigcap_{k\geq 1} [B^p]^k (E).$$

For any event E, (Ely and Pęski, 2011, Lemma 2) proves that $C^p(E)$ is an event that can be written as

$$C^p(E) = C_i^p(E) \times C_{-i}^p(E), \tag{6}$$

and moreover,

$$C_i^p(E) = B_i^p C^p(E). \tag{7}$$

4.5 Patching types

In this section, we describe a useful technique of constructing types, which we call "patching types." For simplicity, here we only illustrate the idea of patching finite types, but the idea can be generalized to define patching infinite types (see Appendix A.2).

Let t_i be a finite type. Define the set of $1-$step reachable types from type t_i as

$$r(t_i) = \text{supp } \text{marg}_{\mathcal{T}_{-i}} \pi_i(t_i).$$

That is, $r(t_i)$ is the set of types on the support of $\pi_i(t_i)$. For any $E_i \subseteq \mathcal{T}_i$, define $r(E_i) = \cup_{t_i\in E_i} r(t_i)$. For any integer $k \geq 2$, let $r^k(t_i)$ denote the iteration of $r(\cdot)$ for k times. That is, $r^k(t_i) = r\left[r^{k-1}(t_i)\right]$ and call $r^k(t_i)$ the $k-$step reachable types from t_i. Furthermore, for notational ease, define $r^0(t_i) \equiv \{t_i\}$.[15]

Let $t_i \leftrightharpoons^m t_j$ denote the type s_i such that (a) $s_i^k = t_i^k$ for all $k = 1, ..., m$ and (b) $r^m(s_i) = \{t_j\}$. In words, $t_i \leftrightharpoons^m t_j$ is the type whose beliefs match those of t_i up to order m and the only $m-$step reachable type from the type $t_i \leftrightharpoons^m t_j$ is t_j. By definition, the sequence of types $\{t_i \leftrightharpoons^m t_j\}_{m=1}^{\infty}$ approximates t_i.

For example, for $\theta', \theta'' \in \Theta$, we can define $t^{\theta'} \leftrightharpoons^2 t^{\theta''}$ as follows. Let $T_1 = \{t_1', t_1''\}$ and $T_2 = \{t_2', t_2''\}$. Define the beliefs of types as follows: $\pi_1(t_1')\left[(\theta', t_2')\right] = 1$, $\pi_2(t_2')\left[(\theta', t_1'')\right] = 1$, and $\pi_i(t_i'')\left[(\theta'', t_{-i}'')\right] = 1$ for $i = 1, 2$. Then, t_1' is the type $t^{\theta'} \leftrightharpoons^2 t^{\theta''}$.

The previous idea can be easily generalized to define patching a countable set of finite types. For any three finite types $t_{i,1}$, $t_{i,2}$ and $t_{i,3}$ and positive integers k_1 and k_2

[15] If t_i is an infinite type, we still use $r^k(t_i)$ to denote the set of $k-$step reachable types from t_i. See Appendix A.2 for a formal definition.

given, we can patch $t_{i,1}$, $t_{i,2}$ and $t_{i,3}$ as follows. First, we construct a new type $t_{i,1} \leftrightharpoons^{k_1} t_{i,2}$ by patching $t_{i,1}$ with $t_{i,2}$ at order k_1. Second, we construct $t_{i,1} \leftrightharpoons^{k_1} t_{i,2} \leftrightharpoons^{k_2} t_{i,3}$ by patching $t_{i,1} \leftrightharpoons^{k_1} t_{i,2}$ with $t_{i,3}$ at order $k_1 + k_2$. Similarly, given a sequence of finite types $\{t_l\}_{l=1}^{\infty}$ and positive integers $\{k_l\}_{l=1}^{\infty}$, we can patch them altogether and get a type

$$t(1) \equiv t_1 \leftrightharpoons^{k_1} t_2 \leftrightharpoons^{k_2} \cdots t_l \leftrightharpoons^{k_l} \cdots.$$

The proof of following lemma is similar to the proof of Lemma 3 in Appendix A.2 and will also be sketched in Appendix A.2.

Lemma 2 *Let $\{t_l\}_{l=1}^{\infty}$ be a sequence of finite types, $k_0 = 0$, and $\{k_l\}_{l=1}^{\infty}$ be a sequence of positive integers. Then, there is a type $t(1)$ such that for any integers $l \geq 1$ and $k(l) = \sum_{m=0}^{l-1} k_m$, $r^{k(l)}(t(1)) = \{t(l)\}$ where for any $l \geq 1$, $t(l)$ is a type whose beliefs agrees with those of t_l up to order k_l, i.e., $t(l) = t_l \leftrightharpoons^{k_l} t_{l+1} \leftrightharpoons^{k_{l+1}} t_{l+2} \leftrightharpoons^{k_{l+2}} \leftrightharpoons \cdots$.*

5 Strategic discontinuity in simple games

In Section 3, we have proved the following result.

Theorem 1 *Every finite type is $3-$critical.*

We now extend the argument in Section 3 to a larger class of types.

Definition 3 *An event E is a proper first-order event if there is some i, some $\theta_0 \in \Theta$, and some $0 \leq y < z \leq 1$ such that $t_i[\theta_0] \notin (y, z)$ for all $t_i \in E_i$.*

Proposition 1 below provides a useful sufficient condition for strategic discontinuity in simple games. In the proof of Proposition 1 (see Appendix A.3), we show that every type which has common$-p$ belief in some proper first-order event E must display strategic discontinuity in a 2×4 game. The idea is similar to the argument in Section 3.

Proposition 1 *A type t_i is $4-$critical if $t_i \in C_i^p(E)$ for some $p > 0$ and some proper first-order event E.*

For instance, in a finite type space $(T_i, \pi_i)_{i\in I}$, $T = T_i \times T_{-i}$ is clearly a proper first-order event, and moreover, every $t_i \in C_i^1(T)$ is thus $4-$critical (while we have shown they are in fact $3-$critical in Section 3). In the next two subsections, we will use Proposition 1 to show that every common prior assigns probability 1 to $4-$critical types and $4-$critical types are generic in the strategic topology.

5.1 Common-prior types

Let $(T_i, \pi_i)_{i\in I}$ be a type space. Following Ely and Pęski (2011), we say that $(T_i, \pi_i)_{i\in I}$ is a common-prior type space if there exists a prior $\psi \in \Delta(\Theta \times T_i \times T_{-i})$ such that for any bounded measurable function $f : \Theta \times T_i \times T_{-i} \to \Re$ and any player i,

$$\int_{\Theta\times T_i\times T_{-i}} f(\theta, t_i, t_{-i})\, d\psi = \int_{T_i}\int_{\Theta\times T_{-i}} f(\theta, t_i, t_{-i})\, \pi_i(t_i)[(d\theta, dt_{-i})]\, \psi_i[dt_i],$$

where $\psi_i = \text{marg}_{T_i}\psi$.

The following theorem shows that every common prior assigns probability 1 to $4-$critical types.

Theorem 2 *Suppose that ψ is a common prior on a type space $(T_i, \pi_i)_{i\in I}$. Then, ψ_i assigns probability 1 to $4-$critical types.*

Proof. Suppose that ψ is a common prior on a type space $(T_i, \pi_i)_{i\in I}$. Let ψ^* be the common prior on the universal type space induced by ψ. Pick any $\theta_0 \in \Theta$ and any $\varepsilon > 0$. Since $\{t_i \in \mathcal{T}_i : t_i[\theta_0] \in (0, z)\} \to \varnothing$ as $z \to 0$, there is some $z^* > 0$ such that $\psi^*(E) \geq 1 - \varepsilon$ where $E = E_i \times E_{-i}$ with $E_j = \{t_j \in \mathcal{T}_j : t_j[\theta_0] \notin (0, z^*)\}$ for $j = i, -i$. Then, (Ely and Pęski, 2011, Lemma 6) (see Appendix A.1) implies that $\psi^*\left(C^{1/4}(E)\right) \geq 1 - \frac{3}{2}\varepsilon$. Furthermore, by Proposition 1, every type in $C_i^{1/4}(E)$ is $4-$critical. Hence, ψ_i^* assigns at least probability $1 - \frac{3}{2}\varepsilon$ to $4-$critical types. Since the latter is true for any $\varepsilon > 0$, ψ_i^* assigns probability 1 to $4-$critical types in $\mathcal{T}_i$. It follows that ψ_i assigns probability 1 to $4-$critical types.■

5.2 Genericity of $4-$critical types

By Theorem 2 in Ely and Pęski (2011), we know that regular types and hence $4-$regular types exist and contain a residual set in product topology. In sharp contrast to this result, we show that 4−critical types contain a set which is open and dense in the strategic topology.[16]

We now formally state and prove this genericity result.

Theorem 3 *The set of* 4−*critical types contains a set of types which is open and dense in the strategic topology.*

To prove this result, we need the following proposition in Chen, Di Tillio, Faingold, and Xiong (2008). Recall that $\mathcal{T}_i$ under product topology is a metric space and for any event E, denote by E_i^{ε} the $\varepsilon-$open ball of E_i and $E^{\varepsilon} = E_i^{\varepsilon} \times E_{-i}^{\varepsilon}$.

Proposition 2 (Chen, Di Tillio, Faingold, and Xiong (2008)) *If a sequence of types* $\{t_{i,m}\}_{m=1}^{\infty}$ *converges to* t_i *in the strategic topology, then for any* $\varepsilon > 0$, *any* $p \in (0,1]$ *and any product-closed* $E \subset \mathcal{T}_i \times \mathcal{T}_{-i}$ *such that* $t_i \in C_i^p(E)$, $t_{i,m} \in C_i^{p-\varepsilon}(E^{\varepsilon})$ *for any sufficiently large* m.

Proof of Theorem 3. We first prove that the closure of 4−regular types in the strategic topology contains no finite types. Then, since finite types are dense in the strategic topology (Theorem 3 in Dekel, Fudenberg, and Morris (2006)), the set of 4−critical types contains an open and dense set in the strategic topology.

Take any finite type t_i and any sequence of types $\{t_{i,m}\}_{m=1}^{\infty}$ converging to t_i in the strategic topology. Since t_i is a finite type, $t_i \in C_i^1(T)$ where $T = T_i \times T_{-i}$ is the finite type space which contains t_i. Hence, T_i is a closed set and there exist some $\theta_0 \in \Theta$ and some $y \in (0,1)$ such that $t_i[\theta_0] \notin (0,y)$ for all $t_i \in T_i$ and all $i \in I$,. Thus, for sufficiently small $\varepsilon > 0$, $t_i[\theta_0] \notin \left(\frac{y}{4}, \frac{y}{2}\right)$ for all $t_i \in T_i^{\varepsilon}$ and all $i \in I$, i.e., T^{ε} is a proper first-order event. Moreover, since $t_{i,m} \to t_i$ in the strategic topology, by Proposition 2, $t_{i,m} \in C_i^{1-\varepsilon}(T^{\varepsilon})$ for any sufficiently large m, and it is a 4−critical type by Proposition 1. Therefore, t_i is not in the closure of 4−regular types under the strategic topology.■

[16] Of course different topologies can be used to measure proximity of types for different purposes and we do not take a position on which is the "right" topology on the universal type space. The strategic topology is relevant when we are concerned about proximity of types being translated into proximity of strategic behaviors.

6 Critical types which are $n-$regular

Our results in the previous section suggest that almost all types commonly used in the economics literature are $4-$critical and the set of $4-$critical types is "large" in the strategic topology. This raises the question whether every critical type is in fact $4-$critical, or more generally, whether there is some integer n, such that every critical type is $n-$critical. In this section, we prove that for every integer $n \geq 2$, there exists a critical type which is $n-$regular. That is, some critical types exhibit strategic discontinuity *only* in complicated games and become irrelevant to a modeler who knows the number of actions in the game she analyzes.

Theorem 4 *For every integer $n \geq 2$, there is a critical type which is $n-$regular.*

The proof of Theorem 4 is involved and is relegated to Appendix A.4. We provide a sketch here. Our proof crucially relies on a notion called *minimal rationalizable types* studied in Ely and Pęski (2011). A type t_i is a minimal rationalizable type in a game G if there is no type whose $0-$rationalizable set in G is a proper subset of the $0-$rationalizable set of t_i.

If t_i is a minimal rationalizable type in G, then t_i does not display strategic discontinuity in G: Suppose that $\{t_{i,m}\}_{m=1}^{\infty}$ approximates t_i and hence converges to t_i in product topology. First, recall that product convergence implies upper hemicontinuity of the $0-$rationalizable correspondence (Lemma 1). Second, if t_i is a minimal rationalizable type, lower hemicontinuity of the $0-$rationalizable correspondence cannot fail either.[17] More generally, Ely and Pęski (2011) provide the insight that a type t_i does not display strategic discontinuity in G as long as there is some k such that the only $k-$step reachable type from t_i is a minimal rationalizable type in G (see Lemma 5 of Ely and Pęski (2011) in Appendix A.1).

Our key observation are the following two propositions (see Appendices A.4.1 and A.4.2 for the proofs).

Proposition 3 *For any $n \times n$ game G, there exists a finite type t_i^* which is a minimal rationalizable type in G, and $R_i^{2^{n+1}}\left(t_i^*, G, 0\right) = R_i\left(t_i^*, G, 0\right)$.*

[17] If t_i is a minimal rationalizable type, $R_i\left(t_{i,m}, G, 0\right) \subseteq R_i\left(t_i, G, 0\right)$ implies $R_i\left(t_{i,m}, G, 0\right) = R_i\left(t_i, G, 0\right)$. Lemma 1 implies that $\left\{t_i' : R_i\left(t', G, 0\right) \subseteq R_i\left(t_i, G, 0\right)\right\}$ is (product-)open and thus t_i exhibits strategic continuity in G.

Proposition 4 *There exists a countable set of $n^2 \times n^2$ games $\{G_l\}_{l=1}^{\infty}$ such that every $n-$critical type displays strategic discontinuity in some G_l.*

Proposition 3 says that in an $n \times n$ game, some finite minimal rationalizable type can achieve its rationalizable set in 2^{n+1} steps of iterated elimination of strictly dominated strategies. Proposition 4 says that to show a type is $n-$regular, we can restrict our attention to a countable set of $n^2 \times n^2$ games — if a type does not display strategic discontinuity in any game from this countable set of games, this type is $n-$regular.

Consider the countable set of $n^2 \times n^2$ games $\{G_l\}_{l=1}^{\infty}$ in Proposition 4. For each l by Proposition 3, we can find a finite type $t_{i,l}$ whose $0-$rationalizable set in G_l is not only minimal but is also fully determined by the k^{*th}-order belief where $k^* = 2^{n^2+1}$. Then, for any integer $l \geq 1$, define

$$t(l) \equiv t_{i,l} \rightleftharpoons^{k^*} t^{\theta_0} \rightleftharpoons^{1} t_{i,l+1} \rightleftharpoons^{k^*} t^{\theta_0} \rightleftharpoons^{1} t_{i,l+2} \rightleftharpoons^{k^*} t^{\theta_0} \rightleftharpoons^{1} t_{i,l+3} \rightleftharpoons^{k^*} t^{\theta_0} \rightleftharpoons^{1} \cdots\cdots.$$

Then, the beliefs of $t(l)$ match those of $t_{i,l}$ up to order k^*. We claim that $t(1)$ is the desired $n-$regular critical type. Since each $t_{i,l}$ is a minimal rationalizable type in G_l and achieves the minimal $0-$rationalizable set in k^* steps, $t(l)$ is also a minimal rationalizable type and hence does not display strategic discontinuity in G_l. Furthermore, by Lemma 5 in Ely and Pęski (2011), since $r^{(k^*+1)\times(l-1)}(t(1)) = \{t(l)\}$, $t(1)$ does not display strategic discontinuity in any G_l, which implies that $t(1)$ is $n-$regular by Proposition 4. Finally, it is common $1-$believed within the type space containing $t(1)$ that every type reaches t^{θ_0} in at most k^* steps, which is a closed proper event in the universal type space. Hence, $t(1)$ is critical by Theorem 1 in Ely and Pęski (2011).

7 $\infty-$critical types

In this section we show that every type is $\infty-$critical. As an intermediate step for the proof, Proposition 5 below shows that every type t_i is "uniform critical" in the sense that there exists a sequence of types $\{t_{i,m}\}_{m=1}^{\infty}$ which approximates t_i but does not converge to t_i in the uniform strategic topology defined in Dekel, Fudenberg, and Morris (2006).

To define the uniform strategic topology, let $\mathcal{G}_{-1,1}$ be the collection of all finite games which assume payoffs in the interval $[-1,1]$, i.e., $g_i(a,\theta) \in [-1,1]$ for any i and any (a,θ) in any game $G \in \mathcal{G}_{-1,1}$. For any $G \in \mathcal{G}_{-1,1}$, let $h_i(t_i, G, a_i) = \min\{\varepsilon : a_i \in R_i(t_i, G, \varepsilon)\}$.[18]

[18]Lemma 1 in Dekel, Fudenberg, and Morris (2006) shows that the minimum exists.

The uniform strategic topology is defined to be the topology induced by the following metric

$$d_i^{us}(t_i, s_i) = \sup_{a_i \text{ in } G,\, G \in \mathcal{G}_{-1,1}} |h_i(s_i, G, a_i) - h_i(t_i, G, a_i)|, \forall t_i, s_i \in \mathcal{T}_i.$$

The uniform strategic topology requires the degree of similarity of strategic behaviors to be uniform over all finite games and thus is finer than the strategic topology.[19] We now formally state the proposition that every type is "uniform critical" (see Appendix A.5 for the proof).

Proposition 5 *For every type t_i, there is a sequence of types $\{t_{i,m}\}_{m=1}^{\infty}$ which approximates t_i but does not converge to t_i in the uniform strategic topology.*

The proposition shows that for every type t_i, there is some sequence of types $\{t_{i,m}\}_{m=1}^{\infty}$ which approximates t_i, and moreover, there is some positive number ε such that for each m, we can find a finite game $G^m \in \mathcal{G}_{-1,1}$ in which some action is rationalizable for t_i but not $\varepsilon-$rationalizable for $t_{i,m}$. This result is trivial when t_i is a critical type: if t_i displays strategic discontinuity in a finite game G, we can normalize the payoffs to make $G \in \mathcal{G}_{-1,1}$ and set $G^m = G$ for every m. However, this result further shows that even if t_i is a regular type, we can always find some sequence of types which approximates t_i, but their strategic behaviors do not converge to those of t_i uniformly over all finite games. We note that uniform strategic convergence may be relevant for environments where the game—both payoffs and action sets—is not *a priori* fixed.

The following theorem is the main result in this section.

Theorem 5 *Every type is $\infty-$critical.*

In Appendix A.6, we will use Proposition 5 to prove Theorem 5. We briefly describe the intuition here. Recall that Proposition 5 implies that for some sequence of types $\{t_{i,m}\}_{m=1}^{\infty}$ which approximates t_i, there is some $\varepsilon > 0$ such that for each m, we can find a finite game G^m in which an action a_i^m is rationalizable for t_i but not $\varepsilon-$rationalizable for $t_{i,m}$. We will "combine" these countably many finite games $\{G^m\}$ to make an infinite-action game G in such a way that the $\gamma-$rationalizable behaviors are preserved (for all types and for all γ). Consequently, in the game G, a_i^m is still rationalizable for t_i but not $\varepsilon-$rationalizable for $t_{i,m}$. That is, t_i exhibits strategic discontinuity in G.

[19] See Chen, Di Tillio, Faingold, and Xiong (2010, 2011) for more discussion and results about the uniform strategic topology.

8 Discussion

8.1 Interim independent rationalizability

Throughout the paper, we only consider strategic discontinuity in terms of interim correlated rationalizability (ICR) . We note that Ely and Pęski (2007) study critical types using an alternative solution concept — interim independent rationalizability (IIR).[20] All of our main results hold if we consider IIR instead of ICR.

8.2 Strategic discontinuity in alternative classes of games

In games with incomplete information, players are not certain of their payoffs. Likewise, it might be difficult for a modeler to precisely specify a game. The modeler may only be sure that the game she analyzes comes from some particular family $\mathcal{G}$. Moreover, the modeler may be concerned about whether her predictions are robust to slight misspecifications of the higher-order beliefs for some game in $\mathcal{G}$. Even if the modeler can precisely specify a game, she may still wish to be able to generalize her predictions from a specific game to all games within a certain family $\mathcal{G}$ for robustness concerns.

The modeler would then be concerned about the notion of $\mathcal{G}-$critical types, i.e., types that display strategic discontinuity in some game in $\mathcal{G}$. As long as a type t is not $\mathcal{G}-$critical, we can approximate t by some simple (e.g. finite) type up to any finite but arbitrarily high order beliefs, and be confident that such an approximation will not change the strategic behavior of t in any game in $\mathcal{G}$. Ely and Pęski (2011) take $\mathcal{G}$ to be the set of all finite games to study critical types. However, in some situations, the modeler may know more than that the game she analyzes is finite and may be willing to impose further restrictions on $\mathcal{G}$.[21] In this paper, we take $\mathcal{G}$ to be the set of all $n \times n$ games. We can similarly study the problem by taking $\mathcal{G}$ to be some other interesting families, such as supermodular games. We leave this for future research.

[20] The notion of IIR also appears in previous works such as Morris and Skiadas (2000), Battigalli and Siniscalchi (2003) and Ely and Pęski (2006).

[21] Moreover, in some situations while a modeler indeed concerns all finite games, she may also want to guarantee strategic proximity of types to be uniform over all the finite games. Our Proposition 5 sheds some light on these situations.

8.3 Selections of rationalizability via perturbations of higher-order beliefs

In Rubinstein's e-mail game and the global game studied in Carlsson and Van Damme (1993), unique rationalizable outcomes are selected via perturbations of higher-order beliefs. In particular, with the perturbation on the common-knowledge type in the e-mail game, only one of the two Nash equilibria of the common-knowledge type remains rationalizable for the types with mutual knowledge. Weinstein and Yildiz (2007) prove that, in a fixed finite game with rich payoffs, for any rationalizable action a of any (Harsanyi) type t, we can slightly perturb (in product topology) the higher-order beliefs of t to make a new type t' for which a is uniquely rationalizable.

Weinstein and Yildiz's result is usually viewed as a generalization of the e-mail game argument. In Chen and Xiong (2011), we nonetheless point out a difference between the notion of selection highlighted in Carlsson and Van Damme (1993) and Rubinstein (1989) and that studied in Weinstein and Yildiz (2007). To see the difference, consider a rationalizable action a of a type t which has multiple rationalizable actions, and a sequence of types $\{t_n\}$ which converges to t in product topology. We call the selection in Weinstein and Yildiz (2007) a *WY selection*: an action a is WY-selected for t if a is the only $(0-)$rationalizable action for any t_n. However, the selection considered in Carlsson and Van Damme (1993) and Rubinstein (1989) goes beyond this: a is a *robust selection* for t if there is some $\varepsilon > 0$ such that a is the only $\varepsilon-$rationalizable action for any t_n. Therefore, strategic discontinuity must be exhibited in a robust selection, but not necessarily in a WY selection. In this vein, Weinstein and Yildiz (2007) is a generalization of the e-mail game regarding WY selections, while this paper complements Weinstein and Yildiz (2007) by generalizing the e-mail game on robust selections.

A Appendix

A.1 Some results in Ely and Peski (2007, 2010)

The following results are proved in Ely and Pęski (2007, 2011) and will be used in some of our proofs.

(Ely and Pęski, 2007, Lemma 4) *For each $n \times n$ game $G = \langle A_i, g_i \rangle$ and each $\gamma \geq 0$, there is an $n^2 \times n^2$ game $G' = \langle A'_i, g'_i \rangle$ such that $A'_i = A_i \times A_{-i}$ and for any t_i and $\varepsilon \geq 0$, $a_i \in R_i(t_i, G, \gamma + \varepsilon)$ iff $(a_i, a_{-i}) \in R_i(t_i, G', \varepsilon)$ for any $a_{-i} \in A_{-i}$.*

(Ely and Pęski, 2011, Lemma 5) *Let A_i^* be a minimal rationalizable set in a finite game G and $E_i = \{t_i \in \mathcal{T}_i : R_i(t_i, G, 0) \neq A_i^*\}$. Let $E = E_1 \times E_2$. Then, for any player j and any $k \geq 0$, if $t_j \notin C_j^p(E)$ for any $p > 0$, then t_j does not display strategic discontinuity in G.*

(Ely and Pęski, 2011, Lemma 6) *Let ψ^* be a common prior on the universal type space. For any event E, $\psi^*\left(C^{1/4}(E)\right) \geq \frac{3}{2}\psi^*(E) - \frac{1}{2}$.*

A.2 Patching types

In this section, we formally define how we patch two types t_i and t_j to produce $t_i \leftrightharpoons^m t_j$. Say every type reaches itself in 0 step. For any types $\bar{t}_i$ and $\bar{s}_i$ and any even number $m \geq 1$, say $\bar{t}_i$ *reaches* $\bar{s}_i$ *in* m *step(s)* iff there are measurable sets of types T^0, T^1,..., T^m, T^{m+1} such that (a) $T^0 = \{\bar{t}_i\}$, $T^{m+1} = \{\bar{s}_i\}$; (b) $T^k \subseteq \mathcal{T}_{-i}$ if k is odd and $T^k \subseteq \mathcal{T}_i$ if k is even, and moreover,

$$t_j\left[T^k\right] = 1 \text{ for all } t_j \in T^{k-1} \text{ and } k = 1, ..., m+1.$$

Similarly, we can define that $\bar{t}_i$ reaches $\bar{s}_{-i}$ in m steps for any $\bar{t}_i$ and $\bar{s}_{-i}$ and any odd number $m \geq 1$. Clearly, $\bar{t}_i$ reaches $\bar{s}_j$ in m steps implies that there is a set T_{-i} such that $\bar{t}_i[T_{-i}] = 1$ and t_{-i} reaches $\bar{s}_j$ in $m-1$ steps for any $t_{-i} \in T_{-i}$.

Lemma 3 (patching two types) *For any types $\bar{t}_i \in \mathcal{T}_i$ and $\bar{s}_{-i} \in \mathcal{T}_{-i}$ (resp. $\bar{s}_i \in \mathcal{T}_i$) and any odd (resp. even) integer m, there is a type $\bar{t}_i \leftrightharpoons^m \bar{s}_{-i}$ (resp. $\bar{t}_i \leftrightharpoons^m \bar{s}_i$) such that (a) the beliefs of $\bar{t}_i \leftrightharpoons^m \bar{s}_{-i}$ (resp. $\bar{t}_i \leftrightharpoons^m \bar{s}_i$) agree with the beliefs of t_i up to order m; (b) $\bar{t}_i \leftrightharpoons^m \bar{s}_{-i}$ (resp. $\bar{t}_i \leftrightharpoons^m \bar{s}_i$) reaches $\bar{s}_{-i}$ (resp. $\bar{s}_i$) in m steps.*

Proof. We only construct $\bar{t}_i \leftrightharpoons^m \bar{s}_{-i}$ and the construction of $\bar{t}_i \leftrightharpoons^m \bar{s}_i$ is similar. Let $\left(T_j^1, \pi_j^1\right)$ be a type space containing $\bar{t}_i$, and $\left(T_j^2, \pi_j^2\right)$ be a type space containing $\bar{s}_{-i}$ (e.g. we may set $\left(T_j^1, \pi_j^1\right) = \left(T_j^2, \pi_j^2\right) = \left(\mathcal{T}_j, \pi_j^*\right)$). To define $\bar{t}_i \leftrightharpoons^m \bar{s}_{-i}$, we first define a type space as follows. If $m = 1$, let $T_{-i}^{1,-1} = \varnothing$. If $m \geq 3$, for any odd $k = 1, 3..., m-2$, let $T_{-i}^{1,k}$ be an identical copy of T_{-i}^1 indexed by k, and similarly, for even $k = 0, 2, ..., m-1$, let $T_i^{1,k}$ be an identical copy of T_i^1 indexed by k. Moreover, for any $j \in I$ and $k \leq m-1$, let I_Θ be the identity mapping on Θ and $\varphi_{j,k} : T_j^1 \to T_j^{1,k}$ be the identity embedding of T_j^1 on $T_j^{1,k}$. Let $\varphi_{-i,m} : T_{-i}^1 \to T_{-i}^2$ be defined as $\varphi_{-i,m}(t_{-i}) = \bar{s}_{-i}$ for all $t_{-i} \in T_{-i}^1$.

Let $T_i = \left[\cup_{k=0}^{m-1} T_i^{1,k}\right] \cup T_i^2$ and $T_{-i} = \left(\cup_{k=-1}^{m-2} T_{-i}^{1,k}\right) \cup T_{-i}^2$ and define for $j = i, -i$,

$$\pi_j(t_j) = \begin{cases} \pi_j^1\left[\varphi_{j,k}^{-1}(t_j)\right] \circ \left[I_\Theta \times \varphi_{-j,k+1}\right]^{-1}, & \text{if } t_j \in T_j^{1,k}, 0 \le k \le m-1; \\ \pi_j^2(t_j), & \text{if } t_j \in T_j^2. \end{cases}$$

Let $\bar{t}_i \leftrightharpoons^m \bar{s}_{-i}$ be the type $\bar{t}_i \in T_i^{1,0}$. Then, property (b) follows directly from our construction and the proof of property (a) is identical to the proof of Lemma 3 in Ely and Pęski (2011).■

We also sketch the proof of Lemma 2.

Lemma 2 (patching a sequence of finite types) *Let $\{t_l\}_{l=1}^\infty$ be a sequence of finite types, $k_0 = 0$, and $\{k_l\}_{l=1}^\infty$ be a sequence of positive integers. Then, there is a type $t(1)$ such that for any integers $l \ge 1$ and $k(l) = \sum_{m=0}^{l-1} k_m$, $r^{k(l)}(t(1)) = \{t(l)\}$ where for any $l \ge 1$, $t(l)$ is a type whose beliefs agrees with those of t_l up to order k_l, i.e., $t(l) = t_l \leftrightharpoons^{k_l} t_{l+1} \leftrightharpoons^{k_{l+1}} t_{l+2} \leftrightharpoons^{k_{l+2}} \leftrightharpoons \cdots$.*

Proof. We only construct $t(1)$ for the case that $\{t_{i,l}\}_{l=1}^\infty$ are types of player i and $\{k_l\}_{l=1}^\infty$ are positive even integers. Other cases are similar. Let $\left(T_j^l, \pi_j^l\right)$ be the finite type space containing $t_{i,l}$. We define a type space as follows. For any $j \in I$ and odd integer m, let $T_{-i}^{(m)}$ be an identical copy of T_{-i}^l indexed by m if $k(l) \le m \le k(l+1) - 1$, and similarly, for even integer $m \ge 0$, let $T_i^{(m)}$ be an identical copy of T_i^l indexed by m if $k(l) \le m \le k(l+1) - 1$. Moreover, for any $j \in I$, $l \ge 1$, and $k(l) \le m \le k(l+1) - 1$, let I_Θ be the identity mapping on Θ and $\varphi_{j,m} : T_j^l \to T_j^{(m)}$ be the identity embedding of T_j^l on $T_j^{(m)}$. Let $\varphi_{i,k(l+1)} : T_i^l \to T_i^{l+1}$ be defined as $\varphi_{i,k(l+1)}(t_i) = t_{i,l+1}$ for all $t_i \in T_i^l$.

Let $T_i = \cup_{m=0}^\infty T_i^{(2m)}$ and $T_{-i} = \cup_{m=0}^\infty T_{-i}^{(2m+1)}$ and define

$$\pi_j(t_j) \equiv \pi_j^1\left[\varphi_{j,m}^{-1}(t_j)\right] \circ \left[I_\Theta \times \varphi_{-j,m+1}\right]^{-1}, \forall j \in I, t_j \in T_j^{(m)}, k(l) \le m \le k(l+1), l \in \mathbb{N}.$$

Let $t(1)$ be the type $\bar{t}_i \in T_i^{(0)}$.■

A.3 Proof of Proposition 1

Proposition 1 *A type t_i is 4−critical if $t_i \in C_i^p(E)$ for some $p > 0$ and some proper first-order event E.*

Proof. It is without loss of generality to consider $i = 2$. Pick any player 2's type $t^* \in C_2^p(E)$. Since E is a proper first-order event, there is some some $\theta_0 \in \Theta$, some $0 \leq y < z \leq 1$, and some player, say player 2 (the case with player 1 is similar), such that $t_2[\theta_0] \notin (y, z)$ for all $t_2 \in E_2$. Consider the following 2×4 game G with two positive numbers, x_1 and x_2 to be determined later.

$\theta = \theta_0$	a_2	b_2	c_2	d_2
a_1	$0, -x_2$	$0, 1$	$\frac{1}{1-p}, 0$	$\frac{1}{1-p}, \frac{1}{1-p}$
b_1	$\frac{1}{p}, -x_2$	$\frac{1}{p}, 1$	$0, 0$	$0, -\frac{1}{p}$

$\theta = \theta_0$

$\theta \neq \theta_0$	a_2	b_2	c_2	d_2
a_1	$0, 1$	$0, -x_1$	$\frac{1}{1-p}, 0$	$\frac{1}{1-p}, \frac{1}{1-p}$
b_1	$\frac{1}{p}, 1$	$\frac{1}{p}, -x_1$	$0, 0$	$0, -\frac{1}{p}$

$\theta \neq \theta_0$

First, choose $x_1, x_2 > 0$ such that $\frac{1}{1+x_2} = y$ and $\frac{x_1}{x_1+1} = z$. By choosing a_2, type s of player 2 gets $s[\theta_0] \cdot (-x_2) + (1 - s[\theta_0]) \cdot 1$, regardless player 1's strategy, and moreover,

$$s[\theta_0] \cdot (-x_2) + (1 - s[\theta_0]) \cdot 1 < 0 \Leftrightarrow s[\theta_0] > \frac{1}{1+x_2} = y. \tag{8}$$

Similarly, by choosing b_2, type s of player 2 gets $s[\theta_0] \cdot 1 + (1 - s[\theta_0]) \cdot (-x_1)$, regardless player 1's strategy, and moreover,

$$s[\theta_0] \cdot 1 + (1 - s[\theta_0]) \cdot (-x_1) < 0 \Leftrightarrow s[\theta_0] < \frac{x_1}{x_1+1} = z. \tag{9}$$

Let $\mu \in \Delta(\Theta)$ be a first-order belief such that $\mu[\theta_0] \in (y, z)$. By (8) and (9), we have

$$\begin{aligned} \mu[\theta_0] \cdot (-x_2) + (1 - \mu[\theta_0]) \cdot 1 &< 0; \\ \mu[\theta_0] \cdot 1 + (1 - \mu[\theta_0]) \cdot (-x_1) &< 0. \end{aligned}$$

Define

$$\gamma = \frac{1}{2} \min \left\{ \frac{1}{1-p}, \ |\mu[\theta_0] \cdot (-x_2) + (1 - \mu[\theta_0])|, \ |\mu[\theta_0] + (1 - \mu[\theta_0]) \cdot (-x_1)| \right\} > 0.$$

Recall that t^μ is the type which has common knowledge that both players have the first-order belief μ. By choosing c_2, type t^μ always gets 0, while he gets at most $-\gamma$ by choosing either a_2 or b_2. Hence

$$a_2, b_2 \notin R_2(t^\mu, G, \gamma). \tag{10}$$

Define a sequence of types $t_{2,m} \equiv \left(t^* \leftrightharpoons^{2m} t^\mu\right)$. Clearly, $\{t_{2,m}\}$ approximates t^*. Claims 1 and 2 below complete the proof of Proposition 1.

Claim 1 *$a_2, b_2 \notin R_2\left(t_{2,m}, G, \gamma\right)$ for any m.*

Proof. Since $\gamma < \frac{1}{1-p}$, given that player 2 chooses c_2 or d_2, player 1 has the unique $\gamma-$best reply a_1, regardless her belief about θ; given that player 1 chooses a_1, player 2 has a unique $\gamma-$best reply d_2, regardless his belief about θ. Then, $R_2\left(t_{2,m}, G, \gamma\right) = \{d_2\}$ by the usual infection argument: since $r^{2m}\left(t_{2,m}\right) = \{t^\mu\}$ and $a_2, b_2 \notin R_2\left(t^\mu, G, \gamma\right)$ by (10), all types in $r^{2m-1}\left(t_{2,m}\right)$ have the unique $\gamma-$rationalizable action a_1, which then imply that all types in $r^{2m-2}\left(t_{2,m}\right)$ have the unique $\gamma-$rationalizable action d_2, and so on. Therefore, $R_2\left(t_{2,m}, G, \gamma\right) = \{d_2\}$.■

Claim 2 *$\{a_2, b_2\} \cap R_2\left(t^*, G, 0\right) \neq \varnothing$.*

Proof. First, observe that in any type space, it is a Bayesian Nash Equilibrium for player 1 to play a_1 and player 2 to play d_2. Hence, $a_1 \in R_1\left(t_1, G, 0\right)$ and $d_2 \in R_2\left(t_2, G, 0\right)$ for any $\left(t_1, t_2\right) \in \mathcal{T}_1 \times \mathcal{T}_2$. Note that $t_2 \in C_2^p\left(E\right)$ implies $t_2\left[\theta_0\right] \notin \left(y, z\right)$. Then, define $\left(\overline{R}_i\right)_{i \in I}$ with $\overline{R}_i : \mathcal{T}_i \longrightarrow 2^{A_i} \backslash \{\varnothing\}$ as follows.

$$\overline{R}_1\left(t_1\right) = \begin{cases} \{b_1\}, & \text{if } t_1 \in C_1^p\left(E\right); \\ \{a_1\}, & \text{if } t_1 \notin C_1^p\left(E\right); \end{cases}$$

$$\overline{R}_2\left(t_2\right) = \begin{cases} \{a_2\}, & \text{if } t_2 \in C_2^p\left(E\right) \text{ and } t_2\left[\theta_0\right] \leq y; \\ \{b_2\}, & \text{if } t_2 \in C_2^p\left(E\right) \text{ and } t_2\left[\theta_0\right] \geq z; \\ \{d_2\}, & \text{if } t_2 \notin C_2^p\left(E\right). \end{cases}$$

We establish our claim by showing that $\left(\overline{R}_i\right)_{i \in I}$ satisfies the $0-$best-reply property.

First, consider $t_1 \in C_1^p\left(E\right)$. Suppose that t_1 believes that player 2 chooses a_2 if $t_2 \in C_2^p\left(E\right)$ and $t_2\left[\theta_0\right] \leq y$; player 2 chooses b_2 if $t_2 \in C_2^p\left(E\right)$ and $t_2\left[\theta_0\right] \geq z$, and player 2 chooses d_2 otherwise. Since $t_1 \in C_1^p\left(E\right)$, we have $t_1\left[C_2^p\left(E\right)\right] \geq p$ by (7). Hence, t_1 believes that player 2 plays a_2 or b_2 with at least probability p. Then, by choosing a_1, t_1 gets at most $p \cdot 0 + (1-p) \cdot \frac{1}{1-p}$, while by choosing b_1, t_1 gets at least $p \cdot \frac{1}{p} + (1-p) \cdot 0$. Therefore, b_1 is a $0-$best reply for t_1.

Second, consider $t_2 \in C_2^p\left(E\right)$ with $t_2\left[\theta_0\right] \leq y$ (and recall that $y < z$). Suppose t_2 believes that player 1 chooses b_1 if $t_1 \in C_1^p\left(E\right)$ and chooses a_1 otherwise. By (8) and (9),

t_2 gets at least 0 by choosing a_2 and he gets at most 0 by choosing b_2. Further, t_2 always gets 0 by choose c_2. Finally, by (7), $t_2\left[C_1^p(E)\right] \geq p$, the payoff of choosing d_2 is at most $p \cdot \left(-\frac{1}{p}\right) + (1-p) \cdot \frac{1}{1-p} = 0$. Therefore, a_2 is a 0−best reply for t_2. Similarly, for any $t_2 \in C_2^p(E)$ with $t_2[\theta_0] \geq z$, b_2 is a 0−best reply for t_2 under the belief of t_2 specified above.■

A.4 Proof of Theorem 4

A.4.1 Proof of Proposition 3

Proposition 3 *For any* $n \times n$ game G, *there exists a finite type* t_i^* *which is a minimal rationalizable type in* G, *and* $R_i^{2^{n+1}}\left(t_i^*, G, 0\right) = R_i\left(t_i^*, G, 0\right)$.

Proof. In this proof, say a type t_i is *k-minimal for* $A_i'\,(\subseteq A_i)$ if $R_i^k(t_i, G, 0) = A_i'$ and there is no $t_i' \in \mathcal{T}_i$ such that $R_i^{k-1}\left(t_i', G, 0\right) \subsetneq A_i'$. We divide the proof into two steps.

Step 1 *For any finite type* t_i *and integer* $k > 1$, *if* t_i *is k-minimal for* $R_i^k(t_i, G, 0)$, *then there exists* $t_{-i} \in$supp $\text{marg}_{\mathcal{T}_{-i}} t_i$ *and* t_{-i} *is* $(k-1)$*-minimal for* $R_{-i}^{k-1}(t_{-i}, G, 0)$.

Suppose that every $t_{-i} \in$supp $\text{marg}_{\mathcal{T}_{-i}} t_i$ is not $(k-1)$-minimal for $R^{k-1}(t_{-i}, G, 0)$. Then, for every $t_{-i} \in$supp $\text{marg}_{\mathcal{T}_{-i}} t_i$, there exists a type $s_{-i}^{t_{-i}}$ such that $R_{-i}^{k-2}\left(s_{-i}^{t_{-i}}, G, 0\right) \subsetneq R_{-i}^{k-1}(t_{-i}, G, 0)$. Consider a new type t_i' defined as follows.

$$t_i'\left[\left(\theta, s_{-i}^{t_{-i}}\right)\right] = t_i\left[(\theta, t_{-i})\right] \text{ for all } \theta \in \Theta \text{ and } t_{-i} \in \text{supp marg}_{\mathcal{T}_{-i}} t_i.$$

Then, $R_i^{k-1}\left(t_i', G, 0\right) \subsetneq R_i^k(t_i, G, 0)$ and hence t_i is not k-minimal for $R^k(t_i, G, 0)$, which is a contradiction. Hence, some $t_{-i} \in$supp $\text{marg}_{\mathcal{T}_{-i}} t_i$ is $(k-1)$-minimal for $R_{-i}^{k-1}(t_{-i}, G, 0)$.

Step 2 *There exists a finite type* t_i^* *which is a minimal rationalizable type in* G, *and* $R_i^{2^{n+1}}\left(t_i^*, G, 0\right) = R_i\left(t_i^*, G, 0\right)$.

Let A_i^* be a minimal 0−rationalizable set, i.e., there is no $t_i' \in \mathcal{T}_i$ such that $R_i\left(t_i', G, 0\right) \subsetneq A_i^*$. Then, there exists a finite type s_i such that $R_i(s_i, G, 0) = A_i^*$.[22] Consider the number

$$k^* = \min\left\{k \geq 0 : \exists \text{ a finite type } s_i \text{ s.t. } R_i^k(s_i, G, 0) = A_i^*\right\}. \quad (11)$$

[22] By Lemma 1, $\left\{t_i \in \mathcal{T}_i : R_i(t_i, G, 0) \subseteq A_i^*\right\}$ is a product-open set. Since finite types are dense in product topology (cf. Mertens and Zamir (1985)), $R_i(s_i, G, 0) \subseteq A_i^*$ for some finite type s_i, and moreover, since A_i^* is minimal, $R_i(s_i, G, 0) = A_i^*$.

Suppose that t_i^* is a finite type achieving the minimum in (11), i.e., $R_i^{k^*}(t_i^*, G, 0) = A_i^*$. Then, since A_i^* is minimal, t_i^* is a minimal rationalizable type in G, and t_i^* is k^*-minimal for $R_i^{k^*}(t_i^*, G, 0) = R_i(t_i^*, G, 0) = A_i^*$.

We now prove that $k^* \leq 2^{n+1}$, which implies $R_i^{2^{n+1}}(t_i^*, G, 0) = R_i(t_i^*, G, 0)$. Suppose instead $k^* > 2^{n+1}$. Without loss of generality suppose k^* is even.

Since t_i^* is k^*-minimal for $R_i^{k^*}(t_i^*, G, 0)$, we can apply step 1 $(k^* - 1)$ times and construct a finite sequence of types t^{k^*} $(= t_i^*)$, t^{k^*-1},..., t^1 such that for every k, t^{k-1} $\in$supp $\mathrm{marg}_{\mathcal{T}_{-i}} t^k$ and

$$\begin{cases} t^k \text{ is } k\text{-minimal for } R_i^k\left(t^k, G, 0\right), & \text{if } k \text{ is even;} \\ t^k \text{ is } k\text{-minimal for } R_{-i}^k\left(t^k, G, 0\right), & \text{if } k \text{ is odd.} \end{cases} \tag{12}$$

Since G is an $n \times n$ game, A_i only has $2^n - 1$ distinct nonempty subsets. Since $k^* > 2^{n+1}$, there are at least 2^n types of player i in the finite sequence $t^{k^*}, t^{k^*-2}, ..., t^2$. Hence, there are two even integers k, k' with $k > k'$ such that

$$R_i^k\left(t^k, G, 0\right) = R_i^{k'}\left(t^{k'}, G, 0\right).$$

Thus, t^k cannot be k-minimal for $R_i^k(t^k, G, 0)$, which is a contradiction to (12). Therefore, $k^* \leq 2^{n+1}$ and $R_i^{2^{n+1}}(t_i^*, G, 0) = R_i(t_i^*, G, 0)$.■

A.4.2 Proof of Proposition 4

Proposition 4 *There exists a countable set of $n^2 \times n^2$ games $\{G_l\}_{l=1}^{\infty}$ such that every $n-$critical type displays strategic discontinuity in some G_l.*

Proof. For any m, we use $\mathcal{G}^m$ to denote the set of all $m \times m$ games. First, take a countable dense set $\{G^k\}_{k=1}^{\infty}$ with $G^k = \langle A_i^k, g_i^k \rangle$ in $\mathcal{G}^n$ under the supmetric d_g where

$$d_g\left(G, G'\right) \equiv \sup_{j \in I,\, (a,\theta) \in A \times \Theta} \left| g_j(a, \theta) - g_j'(a, \theta) \right|. \tag{13}$$

Second, let $\mathbb{Q}_+$ be that set of all nonnegative rational numbers. By (Ely and Pęski, 2007, Lemma 4), for each positive integer k and $q \in \mathbb{Q}_+$, we can find $G^{k,q} = \left\langle A_i^{k,q}, g_i^{k,q} \right\rangle \in \mathcal{G}^{n^2}$ such that $A_i^{k,q} = A_i^k \times A_{-i}^k$ and for any t_i and $\varepsilon \geq 0$,

$$a_i \in R_i\left(t_i, G^k, q + \varepsilon\right) \text{ iff } (a_i, a_{-i}) \in R_i\left(t_i, G^{k,q}, \varepsilon\right) \text{ for any } a_{-i} \in A_{-i}^k. \tag{14}$$

Since $\{G^{k,q} : k \in \mathbb{N}, q \in \mathbb{Q}_+\}$ is also a countable set, we can enumerate $\{G^{k,q}\}$ as $\{G_l\}_{l=1}^{\infty}$.

We now show that $\{G_l\}_{l=1}^{\infty}$ is the desired countable set of $n^2 \times n^2$ games in this proposition. Suppose that t_i is $n-$critical. That is, t_i displays strategic discontinuity in some $n \times n$ game G, i.e., there exist some action a_i in G, some $\varepsilon \in (0,1)$ and a sequence of types $\{t_{i,m}\}_{m=1}^{\infty}$ which approximates t_i such that $a_i \in R_i(t_i, G, 0)$ but $a_i \notin R_i(t_{i,m}, G, \varepsilon)$ for all m. Since $\{G^k\}_{k=1}^{\infty}$ is dense in $\mathcal{G}^n$ under the supmetric d_g defined in (13), we can pick some G^k such that $d_g(G, G^k) < \frac{q}{2} < \frac{\varepsilon}{4}$ for some $q \in \mathbb{Q}_+$. By (Dekel, Fudenberg, and Morris, 2006, Lemma 10), $a_i \in R_i(t_i, G^k, q)$ but $a_i \notin R_i(t_{i,m}, G^k, \varepsilon - q)$ for all m. Finally, by (14), for any m, there is some $a_{-i} \in A_{-i}^k$, $(a_i, a_{-i}) \in R_i(t_i, G^{k,q}, 0)$ but $(a_i, a_{-i}) \notin R_i(t_{i,m}, G^{k,q}, \varepsilon - 2q)$. That is, t_i displays strategic discontinuity in $G^{k,q} \in \{G^{k,q} : k \in \mathbb{N}, q \in \mathbb{Q}_+\} = \{G_l\}_{l=1}^{\infty}$.■

A.4.3 Proof of Theorem 4

Theorem 4 *For every integer $n \geq 2$, there is a critical type which is $n-$regular.*

Proof. We divide the proof into three steps.

Step 1 *Construction of the $n-$regular type.*

Consider the countable set of $n^2 \times n^2$ game $\{G_l\}_{l=1}^{\infty}$ in Proposition 4. By Proposition 3, for each l, there is a finite minimal rationalizable type $t_{i,l}$ in G^l such that $R_i^{k^*}(t_{i,l}, G_l, 0) = R_i(t_{i,l}, G_l, 0)$ where $k^* = 2^{n^2+1}$. Now define

$$t(l) \equiv t_{i,l} \xrightleftharpoons{k^*} t^{\theta_0} \xrightleftharpoons{1} t_{i,l+1} \xrightleftharpoons{k^*} t^{\theta_0} \xrightleftharpoons{1} \cdots, \forall l \geq 1.$$

Step 2 *$t(1)$ is $n-$regular.*

We prove the step in the following two substeps:

(2.1) For any l, $t(l)$ is a minimal rationalizable type in G_l.

By Lemma 2, the beliefs of $t(l)$ agree with those of type $t_{i,l}$ up to order k^*. Hence, $R_i^{k^*}(t(l), G, 0) = R_i^{k^*}(t_{i,l}, G, 0)$. Since $R_i^{k^*}(t_{i,l}, G_l, 0) = R_i(t_{i,l}, G_l, 0)$, we have $R_i(t(l), G_l, 0) \subseteq R_i(t_{i,l}, G_l, 0)$. Moreover, since $t_{i,l}$ is a minimal rationalizable type in G_l, we have $R_i(t(l), G_l, 0) = R_i(t_{i,l}, G_l, 0)$, i.e., the type $t(l)$ is also a minimal rationalizable type in G_l.

(2.2) For any l, $t(1)$ does not display strategic discontinuity in G_l.

By Lemma 2, $r^{(l-1)(k^*+1)}(t(1)) = \{t(l)\}$. Since the type $t(l)$ is a minimal rationalizable type in G^l, by (Ely and Pęski, 2011, Lemma 5) in Appendix A.1, $t(1)$ does not display strategic discontinuity in any G_l. Therefore, $t(1)$ is $n-$regular by Proposition 4.

Step 3 *The $n-$regular type $t(1)$ is critical.*

Let $\left(T_j^*, \pi_j^*\right)$ be the type space defined in the proof of Lemma 2 which contains $t(1)$. First, define

$$E_i = \left\{t_i \in \mathcal{T}_i : \text{there exists some } k \leq k^* + 1 \text{ such that } r^k(t_i) = \left\{t^{\theta_0}\right\}\right\}.$$

Clearly, E_i is a product-closed set. Second, $T_i^* \subset E_i$ and hence $t(1) \in C_i^1(E_i \times \mathcal{T}_{-i})$. Third, $E_i \times \mathcal{T}_{-i}$ is a proper subset of $\mathcal{T}_i \times \mathcal{T}_{-i}$ because for any $\theta \neq \theta_0$, $t^\theta \notin E_i$. Therefore, $t(1)$ is critical by (Ely and Pęski, 2011, Theorem 1).■

A.5 Proof of Proposition 5

Proposition 5 *For every type t_i, there is a sequence of types $\{t_{i,m}\}_{m=1}^{\infty}$ which approximates t_i but does not converge to t_i in the uniform strategic topology.*

The proof of Proposition 5 relies on the following lemma.

Lemma 4 *Let θ_0 and θ_1 be two distinct parameters in Θ. For every $m \geq 0$ and $i \in I$, there exists a finite game $G = \langle A_j, g_j \rangle_{j\in I} \in \mathcal{G}_{-1,1}$ with some a_i^* and $\left(\bar{a}_j\right)_{j\in I}$ such that*

(1) $g_j\left(\bar{a}_j, a_{-j}, \theta\right) = 1$ for any $j \in I$, any θ and any a_{-j};

(2) $a_i^* \in R_i(t_i, G, 0)$ and $a_i^* \notin R_i(s_i, G, 1/2)$ for any types t_i and s_i such that $r^m(t_i) = \{t^{\theta_0}\}$ and $r^m(s_i) = \{t^{\theta_1}\}$.

Proof. We prove the proposition by induction on m. For $m = 0$, define $G = \langle A_j, g_j \rangle_{j\in I}$ by the matrix:

	$\bar{a}_{-i}$
a_i^*	1, 1
$\bar{a}_i$	1, 1

$\theta = \theta_0$

	$\bar{a}_{-i}$
a_i^*	0, 1
$\bar{a}_i$	1, 1

$\theta \neq \theta_0$

Clearly, for every player j, $g_j\left(\bar{a}_j, a_{-j}, \theta\right) = 1$ for any θ and a_{-j}. Moreover, by choosing a_i^*, player i with type t^{θ_0} gets the payoff $t^{\theta_0}[\theta_0] = 1$ and player i with type t^{θ_1} gets 0. Since player i always gets the payoff 1 by choosing $\overline{a}_i$, we conclude that $a_i^* \in R_i\left(t^{\theta_0}, G, 0\right)$ and $a_i^* \notin R_i\left(t^{\theta_1}, G, 1/2\right)$.

Now assume that our claim holds for some integer $m-1 \geq 0$ and we prove the case for m. By the induction hypothesis, there is a game $G' = \left\langle A'_j, g'_j \right\rangle_{j\in I}$ with actions $a^{*\prime}_{-i}$ and $\left(\overline{a}'_j\right)_{j\in I}$ such that properties (1) and (2) hold for $m-1$. Define $G = \langle A_j, g_j \rangle_{j\in I}$ as follows. Let $A_i = \left\{b_i^*, \overline{b}_i\right\} \times A'_i$ and $A_{-i} = A'_{-i}$ and for any $j \in I$ and any $\left(\left(b, a'_i\right), a'_{-i}, \theta\right) \in A_i \times A_{-i} \times \Theta$

$$g_j\left(\left(b, a'_i\right), a'_{-i}, \theta\right) = \min\left\{g''_j\left(b, a'_{-i}\right), g'_j\left(a'_i, a'_{-i}, \theta\right)\right\}$$

where $g''_j : \left\{\overline{b}_i, b_i^*\right\} \times A'_{-i}$ is defined using the following matrix:

	$a'_{-i} = a^{*\prime}_{-i}$	$a'_{-i} \neq a^{*\prime}_{-i}$
b_i^*	1, 1	0, 1
$\overline{b}_i$	1, 1	1, 1

Claim 3 *$R_{-i}\left(t_{-i}, G, \gamma\right) = R_{-i}\left(t_{-i}, G', \gamma\right)$ for any $\gamma \geq 0$ and any $t_{-i} \in \mathcal{T}_{-i}$.*

Proof. First, we prove that $R_{-i}\left(t_{-i}, G', \gamma\right) \subseteq R_{-i}\left(t_{-i}, G, \gamma\right)$. Define $\overline{R}_{-i}\left(t_{-i}\right) = R_{-i}\left(t_{-i}, G', \gamma\right)$ and $\overline{R}_i\left(t_i\right) = \left\{\overline{b}_i\right\} \times R_i\left(t_i, G', \gamma\right)$ for all $\left(t_i, t_{-i}\right) \in \mathcal{T}_i \times \mathcal{T}_{-i}$. Observe that

$$\min\left\{g''_j\left(\overline{b}, a'_{-i}\right), g'_j\left(a'_i, a'_{-i}, \theta\right)\right\} = g'_j\left(a'_i, a'_{-i}, \theta\right), \forall\left(a'_i, a'_{-i}, \theta\right).$$

Thus, $\left(\overline{R}_j\right)_{j\in I}$ has the $\gamma-$ best-reply property in G. Hence, $R_{-i}\left(t_{-i}, G', \gamma\right) \subseteq R_{-i}\left(t_{-i}, G, \gamma\right)$.

Second, we prove that $R_{-i}\left(t_{-i}, G', \gamma\right) \supseteq R_{-i}\left(t_{-i}, G, \gamma\right)$. Define $\widehat{R}_{-i}\left(t_{-i}\right) = R_{-i}\left(t_{-i}, G, \gamma\right)$ and $\widehat{R}_i\left(t_i\right) = \left\{a'_i \in A'_i : \left(\overline{b}_i, a'_i\right) \in R_i\left(t_i, G, \gamma\right)\right\}$. Then, $\left(\widehat{R}_j\right)_{j\in I}$ has the $\gamma-$ best-reply property in G'. Hence, $R_{-i}\left(t_{-i}, G, \gamma\right) \subseteq R_{-i}\left(t_{-i}, G', \gamma\right)$.■

G satisfies property (1) by setting $\overline{a}_i = \left(\overline{b}_i, \overline{a}'_i\right)$ and $\overline{a}_{-i} = \overline{a}'_{-i}$. We now show that G satisfies property (2). Since $r^m\left(t_i\right) = \left\{t^{\theta_0}\right\}$, there is a set $T^{t_i}_{-i}$ such that $t_i\left[T^{t_i}_{-i}\right] = 1$ and $r^{m-1}\left(t_{-i}\right) = \left\{t^{\theta_0}\right\}$ for every $t_{-i} \in T^{t_i}_{-i}$. Similarly, since $r^m\left(s_i\right) = \left\{t^{\theta_1}\right\}$, there is a set $T^{s_i}_{-i}$ such that $s_i\left[T^{s_i}_{-i}\right] = 1$ and $r^{m-1}\left(s_{-i}\right) = \left\{t^{\theta_1}\right\}$ for every $s_{-i} \in T^{s_i}_{-i}$. By the induction hypothesis, $a^{*\prime}_{-i} \in R_i\left(t_{-i}, G', 0\right)$ for every $t_{-i} \in T^{t_i}_{-i}$ and $a^{*\prime}_{-i} \notin R_{-i}\left(t_{-i}, G', 1/2\right)$ for every $t_{-i} \in T^{s_{-i}}_{-i}$. Moreover, by Claim 3, $a^{*\prime}_{-i} \in R_i\left(t_{-i}, G, 0\right)$ for every $t_{-i} \in T^{t_i}_{-i}$ and $a^{*\prime}_{-i} \notin R_{-i}\left(s_{-i}, G, 1/2\right)$ for every $s_{-i} \in T^{s_i}_{-i}$.

Let $a_i^* = \left(b_i^*, \bar{a}_i'\right)$. First, since $a_{-i}^{*\prime} \in R_i\left(t_{-i}, G, 0\right)$ for every $t_{-i} \in T_{-i}^{t_i}$ and $t_i\left[T_{-i}^{t_i}\right] = 1$, we define the following conjecture: $\sigma_{-i}\left(\theta, t_{-i}\right) = \delta_{a_{-i}^*}$ for any θ and any type $t_{-i} \in T_{-i}^{t_i}$. Moreover, $a_i^* = \left(b_i^*, \bar{a}_i'\right)$ is a $0-$best reply to σ_{-i} for t_i. Therefore, $a_i^* \in R_i\left(t_i, G, 0\right)$. Second, since $a_{-i}^* \notin R_{-i}\left(s_{-i}, G, 1/2\right)$ for every $s_{-i} \in T_{-i}^{s_i}$ and $s_i\left[T_{-i}^{s_i}\right] = 1$, $a_i^* = \left(b_i^*, \bar{a}_i'\right)$ is not an $1/2-$best reply to any conjecture σ_{-i} with supp$\sigma_{-i}\left(\theta, t_{-i}\right) \subseteq R_{-i}\left(t_{-i}, G, 0\right)$ for s_i-almost all $\left(\theta, t_{-i}\right)$. Therefore, $a_i^* = \left(b_i^*, \bar{a}_i'\right) \notin R_i\left(s_i, G, 1/2\right)$.■

Proof of Proposition 5. First, $\left\{t_i \leftrightharpoons^m t^{\theta_0}\right\}$ and $\left\{t_i \leftrightharpoons^m t^{\theta_1}\right\}$ both approximate t_i. Then, by Lemma 4 and the triangle inequality, we have

$$1/2 \leq d^{us}\left(t_i \leftrightharpoons^m t^{\theta_0}, t_i \leftrightharpoons^m t^{\theta_1}\right) \leq d^{us}\left(t_i \leftrightharpoons^m t^{\theta_0}, t_i\right) + d^{us}\left(t_i, t_i \leftrightharpoons^m t^{\theta_1}\right), \forall m.$$

Hence, there is either a subsequence of $\left\{t_i \leftrightharpoons^m t^{\theta_0}\right\}_m$ or a subsequence of $\left\{t_i \leftrightharpoons^m t^{\theta_1}\right\}_m$ such that our claim holds.■

A.6 Proof of Theorem 5

Theorem 5 *Every type is $\infty-$critical.*

Proof. Without loss of generality, consider some type t_1 of player 1. By Proposition 5 and Lemma 4, we can find some $\varepsilon \in (0,1)$ and a sequence of types $\{t_{1,m}\}$ which approximates t_1, and furthermore, for every m, there is a finite game $G^m = \langle A_i^m, g_i^m \rangle_{i \in I} \in \mathcal{G}_{-1,1}$, an action a_1^m in A_1^m, and action profile $(\bar{a}_i^m)_{i \in I}$ such that (1) a_1^m is $0-$rationalizable for t_1 but a_1^m is not $\varepsilon-$rationalizable for $t_{1,m}$ in G^m; (2) $g_i\left(\bar{a}_i^m, a_{-i}, \theta\right) = 1$ for any $i \in I$, any θ and any a_{-i}. Note that for any m, 1 is the maximal payoff in G^m and hence $\bar{a}_i^m$ is $0-$rationalizable in G^m for any type of player i.

We now define a new game $G = \langle A_i, g_i \rangle_{i \in I}$ such that

$$A_i = \bigcup_{m=1}^{\infty} A_i^m \text{ and } g_i\left(a_i, a_{-i}, \theta\right) = \begin{cases} g_i^m\left(a_i, a_{-i}, \theta\right), & \text{if } a_i \in A_i^m, a_{-i} \in A_{-i}^m; \\ g_i^m\left(a_i, \bar{a}_{-i}^m, \theta\right) - 3, & \text{if } a_i \in A_i^m, a_{-i} \notin A_{-i}^m. \end{cases}$$

We divide the rest of the proof into two steps.

Step 1 $R_i\left(t_i, G^m, \gamma\right) = R_i\left(t_i, G, \gamma\right) \cap A_i^m, \forall m, \forall t_i \in \mathcal{T}_i, \forall \gamma \in [0,1)$.

Fix a positive integer m. First, since $G^m \in \mathcal{G}_{-1,1}$, deviation to any $a_i \notin A_i^m$ under a conjecture in G^m will deliver at most a payoff -2, while choosing any $a_i \in A_i^m$ guarantees some payoff at least -1. Since $\gamma \in [0,1)$, it follows that $\left(\widehat{R}_i\right)_{i \in I}$ with $\widehat{R}_i\left(t_i\right) =$

$R_i(t_i, G^m, \gamma)$ for every $t_i \in \mathcal{T}_i$ has the $\gamma-$best-reply property in G. Hence, $R_i(t_i, G^m, \gamma) \subseteq R_i(t_i, G, \gamma) \cap A_i^m$.

We now prove that $R_i(t_i, G, \gamma) \cap A_i^m \subseteq R_i(t_i, G^m, \gamma)$ by showing that $\left(\widetilde{R}_i\right)_{i \in I}$ with $\widetilde{R}_i(t_i) \equiv R_i(t_i, G, \gamma) \cap A_i^m$ satisfies $\gamma-$best-reply property in G^m. Note that $\overline{a}_i^m \in \widetilde{R}_i(t_i)$ because $\overline{a}_i^m \in R_i(t_i, G^m, \gamma) \subseteq R_i(t_i, G, \gamma) \cap A_i^m$ as showed above. Consider any $a_i \in \widetilde{R}_i(t_i)$. Then, there exists a conjecture σ_{-i} in G such that supp$\sigma_{-i}(\theta, t_{-i}) \subseteq R_{-i}(t_{-i}, G, \gamma)$ for t_i-almost all (θ, t_{-i}) and

$$\int \sum_{a_{-i} \in A_{-i}} \left[g_i(a_i, a_{-i}, \theta) - g_i\left(a_i', a_{-i}, \theta\right)\right] \sigma_{-i}(\theta, t_{-i})[a_{-i}]\, dt_i[(\theta, t_{-i})] \geq -\gamma, \forall a_i' \in A_i^m. \tag{15}$$

Define a new conjecture σ_{-i}^m in G^m as

$$\sigma_{-i}^m(\theta, t_{-i})[a_{-i}] \equiv \begin{cases} \sigma_{-i}(\theta, t_{-i})[a_{-i}], & \text{if } a_{-i} \in A_{-i}^m \setminus \left\{\overline{a}_{-i}^m\right\}; \\ 1 - \sum\limits_{a_{-i} \in A_{-i}^m \setminus \left\{\overline{a}_{-i}^m\right\}} \sigma_{-i}(\theta, t_{-i})[a_{-i}], & \text{if } a_{-i} = \overline{a}_{-i}^m. \end{cases}$$

I.e., for $a_{-i} \in A_{-i}^m \setminus \left\{\overline{a}_{-i}^m\right\}$, σ_{-i}^m assigns the same probability as σ_{-i}; σ_{-i}^m assigns the rest of the probability to $\overline{a}_{-i}^m$. Since $\overline{a}_{-i}^m \in \widetilde{R}_{-i}(t_{-i})$ and supp$\sigma_{-i}(\theta, t_{-i}) \subseteq R_{-i}(t_{-i}, G, \gamma)$ for t_i-almost all (θ, t_{-i}), supp$\sigma_{-i}^m(\theta, t_{-i}) \subseteq \widetilde{R}_{-i}(t_{-i})$ for t_i-almost all (θ, t_{-i}). Then, for any (θ, t_{-i}),

$$\begin{aligned} & \sum_{a_{-i} \in A_{-i}} \left[g_i(a_i, a_{-i}, \theta) - g_i\left(a_i', a_{-i}, \theta\right)\right] \sigma_{-i}(\theta, t_{-i})[a_{-i}] \\ = & \sum_{a_{-i} \in A_{-i}^m \setminus \left\{\overline{a}_{-i}^m\right\}} \left[g_i^m(a_i, a_{-i}, \theta) - g_i^m\left(a_i', a_{-i}, \theta\right)\right] \sigma_{-i}(\theta, t_{-i})[a_{-i}] \\ & + \left[g_i^m\left(a_i, \overline{a}_{-i}^m, \theta\right) - g_i^m\left(a_i', \overline{a}_{-i}^m, \theta\right)\right] \sigma_{-i}(\theta, t_{-i})\left[\overline{a}_{-i}^m\right] \\ & + \sum_{a_{-i} \notin A_{-i}^m} \left[\left(g_i^m\left(a_i, \overline{a}_{-i}^m, \theta\right) - 3\right) - \left(g_i^m\left(a_i', \overline{a}_{-i}^m, \theta\right) - 3\right)\right] \sigma_{-i}(\theta, t_{-i})[a_{-i}] \\ = & \sum_{a_{-i} \in A_{-i}^m} \left[g_i^m(a_i, a_{-i}, \theta) - g_i^m\left(a_i', a_{-i}, \theta\right)\right] \sigma_{-i}^m(\theta, t_{-i})[a_{-i}] \end{aligned} \tag{16}$$

Then, (15) and (16) imply that a_i is a $\gamma-$best reply to σ_{-i}^m for t_i. Thus, $\left(\widetilde{R}_i\right)_{i \in I}$ has the $\gamma-$best-reply property in G^m and $R_i(t_i, G, \gamma) \cap A_i^m \subseteq R_i(t_i, G^m, \gamma)$.

Step 2 *$a_1^m \in R_1(t_1, G, 0)$ and $a_1^m \notin R_1(t_{1,m}, G, \varepsilon)$.*

Recall that $a_1^m \in R_1(t_1, G^m, 0)$ and $a_1^m \notin R_1(t_{1,m}, G^m, \varepsilon)$. Then, by "$\subseteq$" in Step 1, $a_1^m \in R_1(t_1, G, 0)$; by "$\supseteq$" in Step 1, $a_1^m \notin R_1(t_{1,m}, G, \varepsilon)$ for every m. Therefore, t_1 is $\infty-$critical.■

References

BATTIGALLI, P., AND M. SINISCALCHI (2003): "Rationalizable Bidding in First-Price Auctions," *Games and Economic Behavior*, 45, 38–72.

BERGEMANN, D., S. MORRIS, AND O. TERCIEUX (2011): "Rationalizable Implementation," *Journal of Economic Theory*, 146, 1253–1274.

BRANDENBURGER, A., AND E. DEKEL (1993): "Hierarchies of Belief and Common Knowledge," *Journal of Economic Theory*, 59, 189–197.

CARLSSON, H., AND E. VAN DAMME (1993): "Glabal Games and Equilibrium Selection," *Econometrica*, 61, 989–1018.

CHEN, Y.-C., A. DI TILLIO, E. FAINGOLD, AND S. XIONG (2008): "Genericity of Critical Types," mimeo.

——— (2010): "Uniform Topologies on Types," *Theoretical Economics*, 5, 445–478.

——— (2011): "The Strategic Impact of Higher-Order Beliefs," mimeo.

CHEN, Y.-C., AND S. XIONG (2011): "Robust Selection of Rationalizability," mimeo.

DEKEL, E., D. FUDENBERG, AND S. MORRIS (2006): "Topologies on Types," *Theoretical Economics*, 1, 275–309.

——— (2007): "Interim Correlated Rationalizability," *Theoretical Economics*, 2, 15–40.

ELY, J. C., AND M. PĘSKI (2006): "Hierarchies of Belief and Interim Rationalizability," *Theoretical Economics*, 1, 19–65.

——— (2007): "Critical Types," mimeo.

——— (2011): "Critical Types," *Review of Economic Studies*, forthcoming.

KAJII, A., AND S. MORRIS (1998): "Payoff Continuity in Incomplete Information Games," *Journal of Economic Theory*, 82, 267–276.

LIPMAN, B. (1994): "A Note on the Implications of Common Knowledge of Rationality," *Games and Economic Behavior*, 6, 114–129.

MERTENS, J.-F., AND S. ZAMIR (1985): "Formulation of Bayesian Analysis for Games with Incomplete Information," *International Journal of Game Theory*, 14, 1–29.

MONDERER, D., AND D. SAMET (1989): "Approximating Common Knowledge with Common Beliefs," *Games and Economic Behavior*, 1, 170–190.

——— (1996): "Proximity of Information in Games with Common Beliefs," *Mathematics of Operations Research*, 21, 707–725.

MORRIS, S. (2002a): "Coordination, Communication and Common Knowledge: A Retrospective on the Electronic-Mail Game," *Oxford Review of Economic Policy*, 18, 433–445.

MORRIS, S. (2002b): "Typical Types," mimeo.

MORRIS, S., AND C. SKIADAS (2000): "Rationalizable Trade," *Games and Economic Behavior*, 31, 311–323.

RUBINSTEIN, A. (1989): "The Electronic Mail Game: the Strategic Behavior under Almost Common Knowledge," *American Economic Review*, 79, 385–391.

STRZALECKI, T. (2009): "Depth of Reasoning and Higher Order Beliefs," mimeo, Harvard University.

WEINSTEIN, J., AND M. YILDIZ (2007): "A Structure Theorem for Rationalizability with Application to Robust Predictions of Refinements," *Econometrica*, 75, 365–400.