Belief operators in infinite models[*]

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Abstract

We follow Morris (JET 69 (1996) 1-23) to study the decision-theoretic notion of belief and its logical properties in infinite models; in particular, we extend Morris's main results to the belief operators derived from regular preferences due to Epstein and Wang (Econometrica 64 (1996) 1343-1373). As an application, we formulate and show a general impossibility result of speculative trade in infinite state spaces. Our approach is applicable to the "probabilistic" notion of belief. *JEL Classification: D80, C70.*

Keywords: infinite model; belief operator; general preferences

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1 Introduction

Following Aumann [2], economists and game theorists usually define the notion of "belief" (or "knowledge") in terms of some *exogenous* information structure. This semantic approach, although very useful in game theory and economics, ignores behavioral and decision-theoretic aspects of belief. In an interesting paper, Morris [26] offered a framework for thinking about information and belief in the context of decision-making under uncertainty, in which the decision maker is assumed to be endowed with preferences, at each state of the world, over acts on the space of states of the world. Without referring to an exogenous information structure, the notion of belief is defined directly by more primitive preferences; accordingly, an event is believed if its complement is null in the sense of Savage [30].[1] In a *finite* state space, Morris [26] showed how some substantive properties of beliefs can be related to axioms on preferences; among others, Morris showed that (1) if preference relation is a complete ordering, then the belief operator elicited from preferences is "normal" – i.e. the belief operator can be generated by some information correspondence in the standard semantic model, and (2) if preferences are coherent, then the belief operator satisfies the axioms of "knowledge" and "positive introspection." The purpose of this short paper is to further extend these results to infinite models where individual(s) may exhibit general preferences.

The primary reason for pursuing the study of this paper is as follows. In complex informational and strategic environments, it is important to consider an infinite regress of a hierarchy of "beliefs about beliefs about beliefs about ...". The full description of a state pertaining to such a decision problem is required to represent this exhaustive subjective uncertainty facing each individual (cf. Harsanyi [17]). In doing so, the crucial prerequisite is whether there exists a well-defined state space so that a state contains a "limit-closure" description of general preferences over the state space. For example, Epstein and Wang [11] (hereafter EW) constructed such a well-defined space of types

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[1] Without using a probabilistic apparatus, this approach extends the notion of belief to models which accommodate a very broad class of nonexpected utility maximizing behaviors, including the prominent models of uncertain-aversion such as Choquet expected utility model due to Schmeidler [32] and multiple-prior expected utility model due to Gilboa and Schmeidler [14] among others. See Gilboa and Marinacci [15] for a recent survey of some of the decision-theoretic literature on general preferences.

where each type is an infinite regress of a hierarchy of "preferences over preferences over preferences over ..." and is complete in the sense that it is homeomorphic to preferences over acts on the product space of the states of nature and types. Such a state space is necessarily infinite.[2]

In this paper we advance the line of research advocated by Morris [26, 27] by working with infinite models. More specifically, in infinite models where the decision maker's preferences are regular in the sense of Epstein and Wang [11] (see Appendix I for the formal definition), we show that the belief operator defined by preferences associated with states satisfies the most basic property of "normality" (see Theorem 1). This result extends both Morris's [26] "normality" result to infinite models[3] and Zamir and Vassilakis's [33] result in terms of the "probabilistic" notion of belief to allow beliefs based upon general preferences, so it enables us to exploit the relatively familiar and simple semantic way of analysis whenever doing so is more convenient in more general situations. In addition, we study properties of the belief operator and the information structure elicited from the preferences associated with states (see Lemma 1).[4]

Our analysis is conducted in the framework of infinite models with regular preferences and hence is applicable to EW's complete state space. In this application, our result is

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[2] Infinite state spaces also naturally arise in some applications. For example, in financial economics, it is desired to represent an agent's information by the value of a nonatomically distributed random variable and to do this we must introduce infinite state spaces. Chatterjee and Krishna [6] demonstrated that a finite model may not capture the behavioral property of the sign of a state in an infinite model − i.e. the space of infinitely many subjective states.

[3] See also Lehrer and Samet [20] and Lipman [21] for discussions on the consistency and finite order implications of the common prior assumption in countable infinite models.

[4] In an infinite state space, the finite version of "distributivity" in Lemma 1**B4** is not enough to guarantee that the belief operator is normal, and hence Morris's proof is not applicable to this case. For example, consider a belief operator B as follows: for event $E \subseteq \Omega \equiv [0,1]$,

$$BE = \begin{cases} \Omega, & \text{if } E = [0,1] \\ \{0\}, & \text{if } [0,1] \supsetneq E \supseteq [0,r) \text{ where } 0 < r \leq 1 \\ \varnothing, & \text{otherwise} \end{cases} .$$

Clearly, B satisfies the finite version of "distributivity" − i.e. $B(E \cap F) = BE \cap BF$ for all events $E, F \subseteq \Omega$. For any information correspondence P which is represented by B, information set $P(0)$ at state 0 must be $\{0\}$ or $\varnothing$, contradicting $B\{0\} = \varnothing$. The belief operator defined in this paper, however, can be shown to be distributive over any arbitrary (possibly uncountable) declining family of events; see Lemmas 2 and 3 in Appendix II.

immunized from the "circularity" problem that arises when using Morris's [26] finite state space where preferences at a state are *exogenously* given.[5] Furthermore, our approach is also significant, because the assumption that the model is commonly known can be stated formally, whereas this sort of assumption must be understood informally in a meta-sense within the semantic formalism.

We also extend Morris's [26] Theorem 3 to infinite models (see Theorem 2). This result provides decision-theoretic insights into two assumptions on belief commonly used in the economics literature – i.e. the "knowledge" and "positive introspection" axioms. To demonstrate the usefulness of our approach, we apply the set-up used in this paper to show a general impossibility result of speculative trade in an infinite state space (see Theorem 3).

The rest of this paper is organized as follows. Section 2 presents the set-up. Section 3 studies properties of belief operator and information structure which are defined directly from the preferences associated with states in an infinite model. To facilitate reading, the definition of "regular preferences" is summarized in Appendix I, and all the proofs are relegated to Appendix II.

2 Set-up

Let Ω be an arbitrary compact Hausdorff state space endowed with the Borel σ-algebra. The decision maker makes a choice among different *acts*; i.e., Borel measurable functions $f : \Omega \to [0, 1]$. We denote by $\mathcal{F}(\Omega)$ the set of the decision maker's acts and by $\mathcal{P}(\Omega)$ the set of the *preferences over* $\mathcal{F}(\Omega)$. Throughout this paper, we restrict ourselves to EW's regular preferences that admit representations by utility functions – i.e., the preferences satisfy **U.1-4** in Appendix I. Thus, we identify a preference as a utility function and write $u \in \mathcal{P}(\Omega)$. Each $\omega \in \Omega$ is associated with a preference profile $(u_i^\omega)_{i \in I}$ where $u_i^\omega \in \mathcal{P}(\Omega)$

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[5] Morris [25] identified the "circularity" problem in his framework. Namely, a decision maker's belief is described in terms of preferences over acts which depend on the state of the world which (implicitly) contains a description of his belief. This problem is closely related to a well-known "self-referential" problem in the conventional semantic framework used in game theory; see, e.g., Aumann [3, p.264] and Fagin *et al.* [12, p.332] for more discussions.

and I is the set of decision makers.

Our model applies to EW's complete state space. EW consider a (nonempty) compact Hausdorff space X that represents the primitive uncertainty faced by a set of decision makers (or players); for example, X can be the space of all uncertain parameters in a game with incomplete information or the set of all strategy profiles in a game with complete information. In EW's complete state space, $\Omega \equiv X \times (\times_{i \in I} T_i)$ where T_i is the set of Harsanyi's types for decision maker i. Under EW's construction, $T_i \sim^{homeomorphic} \mathcal{P}(\Omega)$ where Ω is a compact Hausdorff space. That is, at each state, each decision maker's type identifies, through a homeomorphism, a utility function in $\mathcal{P}(\Omega)$.

We restrict attention to the case of single-agent decision making problems, i.e., to consider the case that I is a singleton.[6] We refer to an arbitrary subset $E \subseteq \Omega$ as an *event*. Given a closed event E, let $\mathcal{P}(\Omega|E)$ denote the set of the decision maker's preferences for which the complement of E is *Savage-null*; i.e. any two acts that agree on E are ranked as being indifferent. Following Epstein [10], we say the decision maker *believes an event E at ω* if there exists a closed subset $\overline{E} \subseteq E$ such that $u^\omega \in \mathcal{P}(\Omega|\overline{E})$. Let BE denote the set of all the states where the decision maker believes E; i.e.,

$$BE \equiv \left\{\omega \in \Omega : u^\omega \in \mathcal{P}(\Omega|\overline{E}) \text{ for some closed set } \overline{E} \subseteq E\right\}.$$

Clearly, for a closed set E, $BE = \{\omega \in \Omega : u^\omega \in \mathcal{P}(\Omega|E)\}$. The decision maker's *information structure* is defined as a correspondence P mapping each $\omega \in \Omega$ to some $P(\omega) \subseteq \Omega$. Within the conventional semantic framework, the set $P(\omega)$ is interpreted as: at ω the decision maker knows only that the state is in $P(\omega)$. The information structure P represents *all* information aspects of uncertainty on the part of the decision maker. It constitutes the standard model for "differential" information commonly used in economics. We say the decision maker's information structure P generated by the belief operator B if for all $\omega \in \Omega$,

$$P(\omega) = \bigcap_{\{E \subseteq \Omega:\ BE \ni \omega\}} E.$$

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[6] We will consider the multi-person setting when we prove the impossibility result of speculative trade in Section 3.

3 Results

We start with presenting some basic properties of the belief operator B and the information structure P defined directly from the preferences associated with states in the infinite state space Ω.

Lemma 1 ***(B1)*** *$B\varnothing = \varnothing$.* ***(B2)*** *$B\Omega = \Omega$.* ***(B3)*** *$E \subseteq F \Rightarrow BE \subseteq BF$.* ***(B4)*** *$BE \cap BF = B(E \cap F)$ for any events E and F.*

Remark 1. These properties are common and self-explanatory. The belief operator B may fail to satisfy the other three properties: *the axiom of knowledge*, *the axiom of positive introspection*, and *the axiom of negative introspection* – i.e., $BE \subseteq E$, $BE \subseteq B(BE)$, and $\Omega \backslash BE \subseteq B(\Omega \backslash BE)$. We would like to mention that within the framework of this paper the decision maker is possibly uncertain about his own type or own preferences; see also Heifetz and Samet's [18, p.330] Remark.

In accordance with the standard semantic framework of information and knowledge, a belief operator is expressed in terms of an information structure. We may wonder if the belief operator B defined directly from the preferences associated with states can be defined in a semantic fashion. The following Theorem 1 shows that the belief operator defined by preferences associated with states in the infinite state space Ω is indeed consistent with the one defined by an information structure. Formally, say a belief operator B is *normal* if, for some information structure $\widetilde{P}$, $BE = \left\{\omega \in \Omega : \ \widetilde{P}(\omega) \subseteq E\right\}$ for any event E.

Theorem 1 *B is normal; in particular, $BE = \{\omega : \ P(\omega) \subseteq E\}$ $\forall E$.*

Remark 2. Morris [26, 27] showed this result in a finite state space following the preference-based approach. In an infinite state space, Zamir and Vassilakis [33] among others referred to a similar type of formulation of the "probabilistic" notion of belief.[7] Our result can

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[7] The normality property of the "probabilistic" belief notion is somewhat trivial: that is, we may consider the only states in the support of a probability measure to be possible. But it is not obvious that there is a natural counterpart of "support" for general preferences.

therefore be viewed as both extending the former to an infinite state space and the latter to allow beliefs based upon general preferences.

We also offer an alternative characterization of the information structure P defined in this paper. The following corollary is an immediate consequence of Theorem 1.

Corollary 1 *For every ω, $P(\omega)$ is nonempty and closed, and moreover, $P(\omega) = \{\omega' : \omega \notin B(-\omega')\}$.*

We next extend Morris's [26] Theorem 3 to infinite models. This result offers a decision-theoretic rationale for two additional properties of belief operator – i.e. the axioms of "knowledge" and "positive introspection" – that are often used in economics; see also EW [11, pp.1351-1352] for an alternative approach to incorporate these axioms into the state space. In doing so, we assume that preferences satisfy the following two additional conditions:[8]

> **Sup-norm Continuity**: $\forall \varepsilon > 0$, $\exists \delta$ such that $|u(f) - u(f')| < \varepsilon$, whenever $\sup_{\omega \in \Omega} |f(\omega) - f'(\omega)| < \delta$.
>
> **Non-null Eventwise Monotonicity:** $u(f) > u(f')$ if, for non-null event E, $f(\omega) > f'(\omega)$ $\forall \omega \in E$ and $f(\omega) = f'(\omega)$ $\forall \omega \notin E$.

Following Morris [26], let D be a finite subset of acts in $\mathcal{F}(\Omega)$. Define

$$C_{\omega}[D] \equiv \{f \in D : u^{\omega}(f) \geq u^{\omega}(f') \ \ \forall f' \in D\},$$

i.e. the set of optimal choices over D at state ω. Define

$$C^{*}[D] \equiv \{f^{*} \in \mathcal{F}(\Omega) : \ \forall \omega \in \Omega, \ f^{*}(\omega) = f_{\omega}(\omega) \ \text{ for some } f_{\omega} \in C_{\omega}[D]\}.$$

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[8] Morris [26, **P4**(ii)] assumed conditions similar to "Sup-norm Continuity" and "Non-null Eventwise Monotonicity" in proving his result (see also [27, **P6**]). The models of countably additive SEU and CEU preferences, for example, satisfy these conditions. Moreover, EW's Theorems 6.1 and 4.3 can be used to derive various belief-closed and complete (infinite) models to which our results are applicable.

We say u is *full support* if $u \notin \mathcal{P}(\Omega|E)$ for closed $E \subsetneq \Omega$. Preference relations are said to be *coherent* if there is a full-support $u \in \mathcal{P}(\Omega)$ such that for each finite $D \subset \mathcal{F}(\Omega)$, there exists $f^* \in C^*[D]$ such that $u(f^*) \geq u(f)$ for all $f \in D$. The coherence condition requires that the choices made at different states of the world can be seen as reflecting an appropriate meta-preference ordering over acts. We are now ready to show the version of Morris's [26] Theorem 3 in an infinite state space.

Theorem 2 *In a model with regular preferences satisfying Sup-norm Continuity and Non-null Eventwise Monotonicity, if preference relations are coherent, then the belief operator B satisfies the axiom of knowledge and the axiom of positive introspection.*

Finally, to demonstrate the usefulness of our approach, we apply the set-up used in this paper to explore interactive epistemology.[9] In particular, we show a general impossibility result of speculative trade in an infinite state space.

For simplicity, we consider a state space Ω for two individuals $i = 1, 2$. For $u_i \in \mathcal{P}(\Omega)$ we denote by $u_i|_E$ the conditional preference on event E. Note that, at state ω, an individual i knows only that the state is in $P_i(\omega)$. For our purpose, we require u_i^ω to be consistent with i's *a prior* preference relation $u^{[i]} \in \mathcal{P}(\Omega)$, i.e. $u_i^\omega = u^{[i]}|_{P_i(\omega)}$. In many economic applications, it is natural that, at every state, an act would be preferred to another act, the former act must then be preferred to the latter act before learning which state occurs. Let $B_i E$ denote the set of all the states where i believes event E. For two acts $f, g \in \mathcal{F}(\Omega)$, we make the following mild "consistency" assumption of preferences for each individual $i = 1, 2$.

> **Consistency of Preferences:** Suppose $\emptyset \neq E \subseteq B_i E$. If $u_i^\omega(f) \geq u_i^\omega(g)$ $\forall \omega \in E$, then $u^{[i]}|_E(f) \geq u^{[i]}|_E(g)$; if $u_i^\omega(f) > u_i^\omega(g)$ $\forall \omega \in E$, then $u^{[i]}|_E(f) > u^{[i]}|_E(g)$.

Let us consider an event that is self-evident to i, i.e., whenever it occurs, i believes that it occurs. This assumption expresses the idea that, if i *interim* (strictly) prefers act

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[9] In a separate paper [7], we applied the set-up used in this paper to study the epistemic foundation for a "stable" pattern of strategic behavior.

f to act g across all the states, then i *ex ante* (strictly) prefers f to g. (It makes little sense to postulate this assumption for an event not being self-evident to i – i.e., i is aware that some other states might occur – since i's *interim* preferences at these other states might be quite different.) This assumption is related to Rubinstein and Wolinsky's [29] property of "being preserved under union."

Let $BE = B_1E \cap B_2E$ denote the event "E is mutually believed by the individuals." Let CBE denote the event "E is commonly believed by the individuals," i.e.,

$$CBE = BE \cap BBE \cap BBBE \cap ....$$

For acts $f, g \in \mathcal{F}(\Omega)$ let E^0 denote the event "i prefers f to g and j strictly prefers g to f (where $i, j = 1, 2$ and $i \neq j$)," i.e.,

$$E^0 = \left\{ \omega \in \Omega | \; u_i^\omega (f) \geq u_i^\omega (g) \text{ and } u_j^\omega (f) < u_j^\omega (g) \; \right\}.$$

The following result shows that, under the "common prior" assumption, it cannot be common knowledge that two individuals bet with each other. Formally,

Theorem 3 *Assume that there is a common prior preference relation* $u \in \mathcal{P}(\Omega)$ *for individuals* 1 *and* 2*, i.e.,* $u^{[1]} = u = u^{[2]}$*. Then,* $CBE^0 = \emptyset$*.*

Remark 3. The impossibility result of speculative trade in Theorem 3 is under general preferences and nonpartitional information structures. By restricting preferences to the subjective expected utility, we can derive the "agreeing to disagree" type results in the literature, e.g., Aumann's [2] "agreeing to disagree" result and Milgrom and Stokey's [24] "no trade theorem," as special cases of Theorem 3 (cf. Luo and Ma [22]). See also Green [16] for an alternative approach to the agreement theorem in uncountable partitional information structures and Samet [31] for related discussions of the "agreeing to disagree" in infinite nonpartitional information structures.[10]

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[10] In a standard Bayesian framework, Feinberg [13] characterized common priors by a disagreement in expectations. For characterization of agreeable bets and trades under general preferences (possibly with infinite spaces), see Billot et al. [4], Kajii and Ui [19], and Rigotti, Shannon, and Strzalecki [28].

4 Concluding remarks

We have studied the logical properties of belief operator in infinite models from a decision-theoretic point of view. In particular, we have extended Morris's [26] results to infinite state spaces with regular preferences. We would like to emphasize several major features in this paper: (1) The framework is perfectly flexible to accommodate general preferences which include the subjective expected utility as a special case. (2) The belief operator defined directly from preferences associated with states would not necessarily satisfy some other axioms of belief, e.g., the axiom of knowledge, the axiom of positive introspection, and the axiom of negative introspection. (3) The information structure defined directly from preferences associated with states may not be partitional. This general approach is suitable for a wide range of informational and strategic situations in economic applications. As an application, we have presented a general impossibility result of speculative trade in infinite state spaces.

As emphasized, this paper has focused on studying the logical properties of beliefs and information structure elicited from preferences associated with states in a infinite state space, especially on studying relations with those in the conventional semantic framework for the study of information and knowledge. For this purpose we have adopted Epstein's [10] definition of belief, which is consistent with the definition in a probabilistic setting by using the support of the relevant measure (see, e.g., Dekel and Gul [8] and Zamir and Vassilakis [33]) and can have merits in Choquet and multiple-priors models; see Epstein [10] for more discussions. In this respect Theorem 1 in this paper extends both Morris's [26] "normality" result to an infinite state space and Zamir and Vassilakis's [33] result in terms of the "probabilistic" notion of belief to allow beliefs based upon general preferences.[11]

Di Tillio [9] provided an alternative construction of type space where even weaker assumptions about preferences are maintained (for example, regularity is not assumed). It is certainly an interesting topic for future study how to conduct our analysis in this paper using his framework.

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[11] There are other possible definitions of belief; in particular, an alternative definition of belief is by using "measurable subsets" in place of "closed subsets." The definition of belief used in this paper enables us to show the "normality" result by using the key Lemma 3 in Appendix II that states a variant version of "knowledge continuity" property. Under the alternative definition by using "measurable sets," normality in general does not hold. For example, let $\Omega = [0, 1]$ and consider the expected utility preference at a state whose belief is the Lebesgue measure on Ω. At that state there is no "minimal (w.r.t. set-inclusion) measurable set with measure 1" because the Lebesgue measure is atomless.

Appendix I: Regular preferences

Let

$$\begin{aligned}\mathcal{F}^u(\Omega) &= \left\{f \in \mathcal{F}(\Omega) : f(\Omega) \text{ is finite; } f^{-1}\left([r,1]\right) \text{ is closed for any } r \in [0,1]\right\};\\ \mathcal{F}^l(\Omega) &= \left\{f \in \mathcal{F}(\Omega) : f(\Omega) \text{ is finite; } f^{-1}\left((r,1]\right) \text{ is open for any } r \in [0,1]\right\}.\end{aligned}$$

A preference is said to be *regular* if it has a numerical representation $u : \mathcal{F}(\Omega) \to [0,1]$ satisfying:

U.1. Certainty Equivalence: $u(r) = r$, $\forall r \in [0,1]$.

U.2. Weak Monotonicity: $f' \geq f \Rightarrow u(f') \geq u(f)$, $\forall f, f' \in \mathcal{F}(\Omega)$.

U.3. Inner Regularity: $u(f) = \sup\left\{u(g) : g \leq f,\ g \in \mathcal{F}^u(\Omega)\right\}$, $\forall f \in \mathcal{F}(\Omega)$.

U.4. Outer Regularity: $u(g) = \inf\left\{u(h) : h \geq g,\ h \in \mathcal{F}^l(\Omega)\right\}$, $\forall g \in \mathcal{F}^u(\Omega)$.

Examples of preference models satisfying **U.1-4** include: the subjective expected utility model [30], the ordinal expected utility model [5], the probabilistic sophistication model [23], the Choquet expected utility model [32], the multi-priors model [14], and so on.

Appendix II: Proofs

Proof of Lemma 1. Clearly, **B1** holds by **U.1**. **B2** holds because $\varnothing$ is Savage-null. **B3** holds because any subset of a Savage-null event is Savage-null. For **B4** we show first that $BE \cap BF \subseteq B(E \cap F)$ for any closed sets $E, F \subseteq \Omega$. Suppose $\omega \in BE \cap BF$. Since E and F are closed, $u^\omega \in \mathcal{P}(\Omega|E)$ and $u^\omega \in \mathcal{P}(\Omega|F)$. That is, $u^\omega(f) = u^\omega(g)$ for any f and g which agree on E; $u^\omega(f) = u^\omega(g)$ for any f and g which agree on F. Therefore, for any f and g which agree on $E \cap F$,

$$u^\omega(f) = u^\omega(\left[\begin{array}{ll} f & \text{on } E \\ g & \text{on } \Omega\backslash E \end{array}\right]) = u^\omega(\left[\begin{array}{ll} f & \text{on } E \cap F \\ g & \text{on } \Omega\backslash\left(E \cap F\right) \end{array}\right]) = u^\omega(g).$$

So $\omega \in B(E \cap F)$. Now suppose $\omega \in BE \cap BF$ for arbitrary events E and F. Then, there is some closed sets $\overline{E} \subseteq E$ and $\overline{F} \subseteq F$ such that $\omega \in B\overline{E} \cap B\overline{F}$. Thus, $\omega \in B(\overline{E} \cap \overline{F})$ by our previous step. Since $\overline{E} \cap \overline{F} \subseteq E \cap F$, it follows that $\omega \in B(E \cap F)$. Hence, $BE \cap BF \subseteq B(E \cap F)$. Finally, $B(E \cap F) \subseteq BE \cap BF$ by **B3**.■

To prove Theorem 1, we need the following Lemmas 2 and 3. For any $f \in \mathcal{F}(\Omega)$ and $E \subseteq \Omega$, let $f1_E$ denote the function such that

$$f1_E(\omega) = \begin{cases} f(\omega), & \text{if } \omega \in E \\ 0, & \text{otherwise} \end{cases}.$$

Lemma 2. *Let $E \subseteq \Omega$ be closed and $u \in \mathcal{P}(\Omega)$. Suppose that $u(g) = u(g1_E)$ for all $g \in \mathcal{F}^u(\Omega)$. Then, $u(f) = u(f1_E)$ $\forall f \in \mathcal{F}(\Omega)$.*
Proof. Let $f \in \mathcal{F}(\Omega)$. Note that, if $g \in \mathcal{F}^u(\Omega)$ and E is closed, $g1_E \in \mathcal{F}^u(\Omega)$ and, furthermore, $g1_E \leq f1_E$ whenever $g \leq f$. By **U.3** we have

$$\begin{aligned} u(f) &= \sup\{u(g) : g \leq f,\, g \in \mathcal{F}^u(\Omega)\} \\ &= \sup\{u(g1_E) : g \leq f,\, g \in \mathcal{F}^u(\Omega)\} \\ &\leq \sup\{u(g1_E) : g \leq f1_E,\, g \in \mathcal{F}^u(\Omega)\} \\ &\leq \sup\{u(g) : g \leq f1_E,\, g \in \mathcal{F}^u(\Omega)\} \\ &= u(f1_E), \end{aligned}$$

where the first and the last equalities are because of U.3, the second equality is due to our assumption that $u(g) = u(g1_E)$ for all $g \in \mathcal{F}^u(\Omega)$, the first inequality follows because $g1_E \in \mathcal{F}^u(\Omega)$ and $g1_E \leq f1_E$ whenever $g \in \mathcal{F}^u(\Omega)$, $g \leq f$, and E is closed, and the second inequality is due to U.2. Hence, $u(f) \leq u(f1_E)$. By U.2, $u(f) \geq u(f1_E)$. Hence, $u(f) = u(f1_E)$. ■

Lemma 3. *Suppose that $u \in \mathcal{P}(\Omega)$ and Λ is a directed set with a direction $\succeq$.*[12] *Let $\{E_\lambda\}_{\lambda\in\Lambda}$ be a family of closed subsets of Ω declining in λ, i.e., $E_\lambda \subseteq E_{\lambda'}$ if $\lambda \succeq \lambda'$. Then, for $g \in \mathcal{F}^u(\Omega)$, $\inf_\lambda u(g1_{E_\lambda}) = u(g1_{E^*})$ where $E^* = \cap_\lambda E_\lambda$.*
Proof. (This lemma is an extension of EW's Lemma D.1 to the situation of arbitrary declining family of closed subsets.) Clearly, E^* is closed and $g1_{E^*} \in \mathcal{F}^u(\Omega)$ for $g \in \mathcal{F}^u(\Omega)$. By **U.4**, for any $\varepsilon > 0$, there is a simple lsc function $h \geq g1_{E^*}$ such that $u(h) < u(g1_{E^*}) + \varepsilon$. We now show there is some $\overline{\lambda}$ such that $g1_{E_{\overline{\lambda}}} \leq h$. Suppose to the contrary that for each λ there is $\omega_\lambda \in \Omega$ such that

$$g1_{E_\lambda}(\omega_\lambda) > h(\omega_\lambda). \qquad (\bigstar)$$

Obviously, $\{\omega_\lambda\}_{\lambda\in\Lambda}$ is a net. For $g \in \mathcal{F}^u(\Omega)$ we can express $g = \sum_{m=1}^M \alpha_m 1_{F_m}$ where $\alpha_m \geq 0$ and $\Omega = F_1 \supset F_2 \supset \ldots \supset F_M$ are all closed (see EW [11, p.1366]). Therefore, $g1_{E_\lambda} = \sum_{m=1}^M \alpha_m 1_{F_m \cap E_\lambda}$. Since $g1_{E_\lambda}$ has only finitely many possible values, for simplicity we may assume that, for every λ, $g1_{E_\lambda}(\omega_\lambda) = \alpha_1 + \alpha_2$ and $\omega_\lambda \in (F_2 \cap E_\lambda)$ but $\omega_\lambda \notin (F_m \cap E_\lambda)$ for all $m > 2$.

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[12] For the definitions of a directed set and a net, see Aliprantis and Border [1, pp.27-28].

Likewise, since h has also only finitely many possible values, $\{\omega_\lambda\}_{\lambda\in\Lambda}$ has a subnet, without loss of generality say it is $\{\omega_\lambda\}_{\lambda\in\Lambda}$ itself, such that $h(\omega_\lambda) = \beta$ for some $\beta \in [0,1]$ for all λ. By ($\bigstar$), $\alpha_1 + \alpha_2 > \beta$. Since Ω is compact Hausdorff, $\{\omega_\lambda\}_{\lambda\in\Lambda}$ has a subnet, without loss of generality say it is $\{\omega_\lambda\}_{\lambda\in\Lambda}$ itself, such that $\omega_\lambda \to \omega^*$. Since $F_2 \cap E_\lambda$ is closed, it follows $\omega^* \in F_2 \cap E^*$, which implies $g1_{E^*}(\omega^*) \geq \alpha_1 + \alpha_2$. Since $h \in \mathcal{F}^l(\Omega)$, $\beta = \liminf_\lambda h(\omega_\lambda) \geq h(\omega^*)$. Since $\alpha_1 + \alpha_2 > \beta$, $g1_{E^*}(\omega^*) > h(\omega^*)$, contradicting the fact that $h \geq g1_{E^*}$. Therefore, there is some $\overline{\lambda}$ such that $g1_{E_{\overline{\lambda}}} \leq h$. Thus, by U.2, $u(g1_{E_\lambda}) \leq u(h) < u(g1_{E^*}) + \varepsilon$ for all $E_\lambda \subseteq E_{\overline{\lambda}}$. Since ε is arbitrary, by the monotonicity of u, we conclude $\inf_\lambda u(g1_{E_\lambda}) = u(g1_{E^*})$.$\blacksquare$

Proof of Theorem 1. Let $\omega \in \Omega$. We proceed to show that there is a closed set $\widetilde{P}(\omega)$ such that $\omega \in B\widetilde{P}(\omega)$ and $\widetilde{P}(\omega) \subseteq E$ for any closed set E satisfying $\omega \in BE$. Consider the following family of closed events

$$\mathcal{E} = \{E \subseteq \Omega : E \text{ is closed and } \omega \in BE\}.$$

By Lemma 1**B2**, $\Omega \in \mathcal{E}$ and hence $\mathcal{E}$ is a nonempty. Moreover, make $\mathcal{E}$ a directed set with the "inverse" inclusion $\subseteq$ as the direction. Note that, if $\widetilde{P}(\omega)$ is a "maximal" element in $\mathcal{E}$, then for any given $E \in \mathcal{E}$, by Lemma 1**B4,** $\left[\widetilde{P}(\omega) \cap E\right] \in \mathcal{E}$ and, hence, $\widetilde{P}(\omega) \subseteq E$. By Zorn's Lemma (see, e.g., Aliprantis and Border [1, p.14]), it remains to show for every chain $\{E_\lambda\}_{\lambda\in\Lambda}$ in $\mathcal{E}$, there is an upper bound E^* in $\mathcal{E}$ – i.e. $E^* \subseteq E_\lambda$ for all λ. To see that $E^* \equiv \cap_\lambda E_\lambda$ is the desired upper bound, it suffices to show that $u^\omega(f) = u^\omega(f1_{E^*})$ $\forall f \in \mathcal{F}(\Omega)$. By Lemma 2, we only need to verify that $u^\omega(g) = u^\omega(g1_{E^*})$ $\forall g \in \mathcal{F}^u(\Omega)$. However, since $\omega \in BE_\lambda$, $u^\omega(g) = u^\omega(g1_{E_\lambda})$ for all λ. So by Lemma 3, $u^\omega(g) = \inf_\lambda u^\omega(g1_{E_\lambda}) = u^\omega(g1_{E^*})$.

By the definition of the belief operator B, $P(\omega) = \bigcap_{\{E\subseteq\Omega:\ E \text{ is closed and } \omega\in BE\}} E$. Since $\widetilde{P}(\omega)$ is closed and $\omega \in B\widetilde{P}(\omega)$, $P(\omega) = \widetilde{P}(\omega)$.$\blacksquare$

Proof of Corollary 1. Nonemptiness and closedness of $P(\omega)$ follow from **B1** and the proof of Theorem 1. To see $P(\omega) = \{\omega' : \omega \notin B(-\omega')\}$, note that, by Theorem 1, $\omega \in B(-\omega')$ iff $P(\omega) \subseteq \Omega\backslash\{\omega'\}$ iff $\omega' \notin P(\omega)$.$\blacksquare$

Proof of Theorem 2. By Theorem 1, it suffices to show (1) $\omega \in P(\omega)$; and (2) $P(\omega) \supseteq P(\omega')$ $\forall \omega' \in P(\omega)$. To see (1), suppose on the contrary that $a \notin P(a)$ for state a. Since Ω is compact and Hausdorff, there exist open sets $G \supset P(a)$ and $G_a \ni a$ satisfying $G \cap G_a = \varnothing$ (cf. Figure 1.1). Define a continuous act $f \in \mathcal{F}(\Omega)$ such that

$$f(\omega) = \begin{cases} r_\omega, & \text{if } \omega \in G_a \\ 1/2, & \text{otherwise} \end{cases}$$

where $r_a = 1$, $r_\omega \in (1/2, 1]$, and G_a is an open set such that $a \in G_a$ and $1/2 > u^\omega (f - \epsilon)$ $\forall \omega \in G_a$ for sufficiently small $\epsilon > 0$. Now consider the decision problem $D \equiv \{f - \epsilon, 1/2\}$. Note that, by Measurable Maximum Theorem (see, e.g., Aliprantis and Border [1, pp.570-571]), $C^* [D] \neq \emptyset$. Clearly, $1/2 \geq f^*$ for all $f^* \in C^* [D]$. By **U.1-2**, $1/2 \geq u (f^*)$ for any $u \in \mathcal{P} (\Omega)$. By Non-null Eventwise Monotonicity, for any full-support $u \in \mathcal{P} (\Omega)$, $u (f) > 1/2$. By Sup-norm Continuity we can appropriately adjust ϵ to satisfy $u (f - \epsilon) > 1/2$. Consequently, $u (f - \epsilon) > u (f^*)$ for all $f^* \in C^* [D]$; coherence fails.

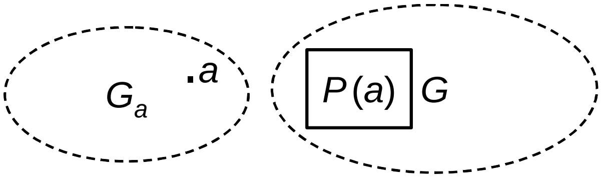

Figure 2.1

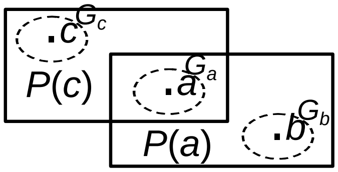

Figure 2.2

To see (2), suppose on the contrary that there exist $a, b, c \in \Omega$ such that $b \in P (a)$, $a \in P (c)$, but $b \notin P (c)$ (cf. Figure 1.2). By (1), $c \in P (c)$, so a, b, and c are all distinct states. Similarly to the proof of (1), define a continuous act $f \in \mathcal{F} (\Omega)$ such that

$$f (\omega) = \begin{cases} 1/2 + \epsilon_a r_\omega, & \text{if } \omega \in G_a \\ 1/2 - \frac{1}{2} r_\omega, & \text{if } \omega \in G_b \\ 1/2 - \epsilon_c r_\omega, & \text{if } \omega \in G_c \\ 1/2, & \text{otherwise} \end{cases}$$

where $r_a = r_b = r_c = 1$, $r_\omega \in (0, 1]$, and G_a, G_b and G_c are mutually disjoint open sets satisfying $a \in G_a$, $b \in G_b$, and $c \in G_c$. By Sup-norm Continuity we can choose $\epsilon_a, \epsilon_c > 0$ such that $1/2 > u^\omega (f)$ on G_a and $1/2 < u^\omega (f)$ on G_c. Consider the decision problem $D \equiv \{f, 1/2\}$. Therefore, $f^* \in C^* [D]$ satisfies $1/2 \geq f^*$ and $1/2 > f^* (\omega)$ for all $\omega \in G_c$. By **U.1** and Non-null Eventwise Monotonicity, for any full-support $u \in \mathcal{P} (\Omega)$, $1/2 > u (f^*)$ $\forall f^* \in C^* [D]$. So by $C^* [D] \neq \emptyset$, coherence fails.■

Proof of Theorem 3. Assume, in negation, that $CBE^0 \neq \emptyset$. By Theorem 1, B is normal and, thus, CBE^0 is self-evident to individuals 1 and 2 (see, e.g., Aumann [3]). By the Consistency of Preferences, we have $u^{[i]}|_{CBE^0} (f) \geq u^{[i]}|_{CBE^0} (g)$ and $u^{[j]}|_{CBE^0} (f) < u^{[j]}|_{CBE^0} (g)$. But, since there is a common prior preference relation $u \in \mathcal{P} (\Omega)$ for the individuals, $u^{[i]}|_{CBE^0} = u|_{CBE^0} = u^{[j]}|_{CBE^0}$. A contradiction.■

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