Online Appendix to "A Structural Theorem for Rationalizability in the Normal Form of Dynamic Games"

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Abstract

Omitted proofs for results in "A Structural Theorem for Rationalizability in the Normal Form of Dynamic Games" Chen (2011) are presented.

A Online Appendix

Lemmas 1 and 2 are the counterparts of Lemmas 6 and 7 of WY under RURA. Before we present their proofs, we provide some sketches for readers who are familiar with WY's arguments. WY prove their Lemma 7 by induction on k. Their Richness assumption guarantees that when $k = 0$, i.e., when it is vacuously true that $\widetilde{t}_i^{k'} = t_i^{k'}$ for all $k' \leq k$, choosing $\widetilde{t}_i$ with $\widetilde{t}_i [\theta^{s_i}] = 1$ proves the claim. Here when $k = 0$, we use RURA to set $\widetilde{t}_i$ to be a finite type with $S_i^\infty [t_i] = \{s_i\}$. This is possible because finite types are dense in T_i^* (see (Mertens and Zamir, 1985, Theorem 3.1)) and $S_i^\infty [\cdot]$ is a nonempty and upper hemicontinuous correspondence (see Dekel et al. (2006)). In words, WY start the "infection argument" from the

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dominance regions, while we start it from the "types with unique ICR actions" defined by RURA. The proof of the induction step is similar to WY.

The modification of Lemma 6 is as follows. Let t_i be a finite type contained in the model $(\Theta \times T, \kappa)$ and $s_i \in S_i^{\infty}[t_i]$. Suppose that s_i is a best reply to $\text{marg}_{\Theta \times S_{-i}} \pi^{t_i, s_i}$ for some π^{t_i, s_i} which is valid for t_i. WY make s_i a strict best reply $t_i(m)$ by setting the belief of $t_i(m)$ to be the $\left(1 - \frac{1}{m}, \frac{1}{m}\right)$-mixture between π^{t_i, s_i} and some belief π which assigns probability 1 to θ^{s_i}. In our case, since s_i is the unique rationalizable action for some type by RURA, s_i is also a strict best reply to some belief π^{s_i} whose support contains only uniquely rationalizable actions which by RURA are also strict best replies to some other beliefs, and so on.

A.1 Proof of Lemma 1

To prove Lemma 1, we need the following lemma which is a straightforward consequence of RURA.

Lemma 1 *Under RURA, for each i and each s_i such that $s_i \in S_i^{\infty}[t_i']$ for some $t_i' \in T_i^*$, there is some $\pi^{s_i} \in \Delta(\Theta \times S_{-i})$ such that $\{s_i\} = BR_i(\pi^{s_i})$, and moreover, $\pi^{s_i}(s_{-i}) > 0$ only if $s_{-i} \in S_{-i}^{\infty}[t_{-i}]$ for some $t_{-i} \in T_{-i}^*$.*

Proof. Since $s_i \in S_i^{\infty}[t_i']$ for some t_i', by RURA, $S_i^{\infty}[t_i] = \{s_i\}$ for some type t_i. Hence, $\{s_i\} = BR_i\left(\text{marg}_{\Theta \times S_{-i}} \pi^{t_i, s_i}\right)$ for some valid π^{t_i, s_i} for t_i. Let $\pi^{s_i} = \text{marg}_{\Theta \times S_{-i}} \pi^{t_i, s_i}$ and hence $\{s_i\} = BR_i(\pi^{s_i})$. Since $\pi^{t_i, s_i}\left(\left\{(\theta, t_{-i}, s_{-i}) : s_{-i} \in S_{-i}^{\infty}[t_{-i}]\right\}\right) = 1$, $\pi^{s_i}(s_{-i}) > 0$ only if $s_{-i} \in S_{-i}^{\infty}[t_{-i}]$ for some $t_{-i} \in T_{-i}^*$.■

We now prove Lemma 1.

Lemma 1 *Under RURA, for any finite type $t_i \in T_i^*$ and any action $s_i \in S_i^{\infty}[t_i]$, there exists a sequence of finite models $((\Theta \times T^m, \kappa^m))_{m=1}^{\infty}$ and a sequence of finite types $(t_i(m))_{m=1}^{\infty}$ such that $t_i(m) \in T_i^m$ and $s_i \in V_i^m[t_i(m)]$ for some profile of correspondences $\left(V_j^m\right)_{j \in N}$ with $V_j^m : T_j^m \rightrightarrows S_j$ which satisfies the strict best reply property, for all m, and $\lim_{m \to \infty} t_i(m) = t_i$.*

Proof. Consider any $s_j \in S_j^{\infty}[t_j]$, $s_j \in BR_j\left(\text{marg}_{\Theta \times S_{-j}} \pi^{t_j, s_j}\right)$ for some valid π^{t_j, s_j} for t_j. Moreover, by Lemma 1, there is some $\pi^{s_j} \in \Delta(\Theta \times S_{-j})$ such that $\{s_j\} = BR_j(\pi^{s_j})$, and moreover, $\pi^{s_j}(s_{-j}) > 0$ only if $s_{-j} \in S_{-j}^{\infty}[t_{-j}]$ for some $t_{-j} \in T_{-j}^{*}$.

We now define $(\Theta \times T^m, \kappa^m)$ as follows.[1]

$$\begin{aligned} T_j^m &= \left\{\overline{\tau}_j(t_j, s_j, m) : t_j \in T_j, s_j \in S_j^{\infty}[t_j]\right\} \bigcup \\ &\quad \left\{\overline{\tau}_j(\theta, s_j) : \theta \in \Theta, s_j \in S_j^{\infty}[t_j] \text{ for some } t_j \in T_j^{*}\right\}. \end{aligned}$$

$\kappa^m_{\overline{\tau}_j(t_j,s_j,m)}$ and $\kappa^m_{\overline{\tau}_j(\theta,s_j)}$ are defined respectively by

$$\begin{aligned} \kappa^m_{\overline{\tau}_j(t_j,s_j,m)} &= \left(\frac{1}{m}\right)\pi^{s_j} \circ \widehat{\eta}_{-j}^{-1} + \left(1 - \frac{1}{m}\right)\pi^{t_j,s_j} \circ \widehat{\tau}_{-j,m}^{-1}; \\ \kappa^m_{\overline{\tau}_j(\theta,s_j)} &= \pi^{s_j} \circ \widehat{\eta}_{-j}^{-1}, \end{aligned}$$

where

$$\begin{aligned} \widehat{\tau}_{-j,m} &: (\theta, t_{-j}, s_{-j}) \mapsto (\theta, \overline{\tau}_{-j}(t_{-j}, s_{-j}, m)), \forall (\theta, t_{-j}, s_{-j}) \text{ s.t. } t_{-j} \in T_{-j}, s_{-j} \in S_{-j}^{\infty}[t_{-j}], \\ \widehat{\eta}_{-j} &: (\theta, s_{-j}) \mapsto (\theta, \overline{\tau}_{-j}(\theta, s_{-j})), \forall (\theta, s_{-j}) \text{ s.t. } s_{-j} \in S_{-j}^{\infty}[t_{-j}] \text{ for some } t_{-j} \in T_{-j}^{*}. \end{aligned}$$

For each $\overline{\tau}_j(t_j, s_j, m)$, define the belief

$$\begin{aligned} \widehat{\pi} &= \left(\frac{1}{m}\right)\pi^{s_j} \circ \widehat{\eta}_{-j}^{-1} \circ \xi^{-1} + \left(1 - \frac{1}{m}\right)\pi^{t_j,s_j} \circ \widehat{\tau}_{-j,m}^{-1} \circ \gamma^{-1} \in \Delta\left(\Theta \times T_{-j}^m \times S_{-j}\right) \text{ where} \\ \gamma &: (\theta, \overline{\tau}_{-j}(t_{-j}, s_{-j}, m)) \mapsto (\theta, \overline{\tau}_{-j}(t_{-j}, s_{-j}, m), s_{-j}); \\ \xi &: (\theta, \overline{\tau}_{-j}(\theta, s_{-j})) \mapsto (\theta, \overline{\tau}_{-j}(\theta, s_{-j}), s_{-j}). \end{aligned}$$

That is, $\overline{\tau}_j(t_j, s_j, m)$ believes that s_{-j} is played at each $(\theta, \overline{\tau}_{-j}(t_{-j}, s_{-j}, m))$ and s'_{-j} is played at each $\left(\theta, \overline{\tau}_{-j}\left(\theta, s'_{-j}\right)\right)$. Then, by construction,

$$\text{marg}_{\Theta \times S_{-j}}\widehat{\pi} = \left(\frac{1}{m}\right)\pi^{s_j} + \left(1 - \frac{1}{m}\right)\text{marg}_{\Theta \times S_{-j}}\pi^{t_j,s_j}.$$

Since $s_j \in BR_j\left(\text{marg}_{\Theta \times S_{-j}}\pi^{t_j,s_j}\right)$, $\{s_j\} = BR_j(\pi^{s_j})$, and $\frac{1}{m} \in (0,1]$, $\{s_j\} = BR_j\left(\text{marg}_{\Theta \times S_{-j}}\widehat{\pi}\right)$. Similarly, for each $\overline{\tau}_j(\theta, s_j)$ define the belief

$$\widetilde{\pi} = \pi^{s_j} \circ \widehat{\eta}_{-j}^{-1} \circ \xi^{-1} \in \Delta\left(\Theta \times T_{-j}^m \times S_{-j}\right).$$

Then, by construction, $\text{marg}_{\Theta \times S_{-j}}\widetilde{\pi} = \pi^{s_j}$. Hence, $\{s_j\} = BR_j\left(\text{marg}_{\Theta \times S_{-j}}\widetilde{\pi}\right)$. Thus, if we define

$$\begin{aligned} V_j^m[\overline{\tau}_j(t_j, s_j, m)] &= \{s_j\}, \forall \overline{\tau}_j(t_j, s_j, m), \forall j; \\ V_j^m[\overline{\tau}_j(\theta, s_j)] &= \{s_j\}, \forall \overline{\tau}_j(\theta, s_j), \forall j, \end{aligned}$$

[1] If Θ is an infinite compact metric space and t_i is in a finite model $(\Theta' \times T, \kappa)$, we replace $(\Theta \times T^m, \kappa^m)$ in the proof by $(\Theta'' \times T^m, \kappa^m)$ where $\Theta'' = \overline{\Theta} \cup \left\{\theta \in \Theta' : \kappa_{t_j}[\theta] > 0 \text{ for some } t_j \in T_j \text{ and } j \in N\right\}$ and $\overline{\Theta}$ is defined in RURA′ in (Chen, 2011, Section A.1).

then $V_j^m[\cdot]$ has the strict best reply property stated in the model $(\Theta \times T^m, \kappa^m)$.

It remains to show that $\lim_{m\to\infty} \overline{\tau}_j(t_j, s_j, m) = t_j$. By construction, each probability distribution is continuous in (t_j, s_j, m). Hence, by Lemma 4 of WY, $h_j(\overline{\tau}_j(t_j, s_j, m)) \to h_j(\overline{\tau}_j(t_j, s_j, 0))$ (in product topology) as $m \to \infty$. The proof that $h_j(\overline{\tau}_j(t_j, s_j, 0)) = h_j(t_j)$ for each t_j and j is exactly the same as that in (Weinstein and Yildiz, 2007, Lemma 6).■

A.2 Proof of Lemma 2

Lemma 2 *Let $(\Theta \times T, \kappa)$ be a finite model. Under RURA, for any type $t_i \in T_i$, any action $s_i \in V_i[t_i]$ for some profile of correspondences $(V_j)_{j\in N}$* with $V_j : T_j \rightrightarrows S_j$ *which satisfies the strict best reply property, and any integer $k \geq 1$, there exists a finite type $\widetilde{t}_i$ such that $\widetilde{t}_i^{k'} = t_i^{k'}$ for all $k' \leq k$ and $S_i^\infty\left[\widetilde{t}_i\right] = \{s_i\}$.*

Proof. We prove this claim by induction on k. First, suppose that $s_i \in V_i[t_i]$ for some profile of correspondences $(V_j)_{j\in N}$ with $V_j : T_j \rightrightarrows S_j$ which satisfies the strict best reply property. A correspondence which has the strict best reply property clearly has the best reply property (as defined in Dekel et al. (2007)). Hence, $s_i \in S_i^\infty[t_i]$. By RURA, there is a finite type $\widetilde{t}_i$ such that $S_i^\infty\left[\widetilde{t}_i\right] = \{s_i\}$.

Now fix any $k > 0$ and any $i \in N$. Write each t_{-i} as $t_{-i} = (l, h)$, where

$$l = \left(t_{-i}^1, t_{-i}^2, ..., t_{-i}^{k-1}\right) \text{ and } h = \left(t_{-i}^k, t_{-i}^{k+1}, ...\right)$$

are the lower- and higher-order beliefs, respectively, Let $L = \left\{l | \exists h : (l, h) \in T_{-i}^*\right\}$. The induction hypothesis is that for each finite $t_{-i} = (l, h)$ and each $s_{-i} \in V_{-i}[t_{-i}]$, there exists finite type $\widetilde{t}_{-i}[s_{-i}] = \left(l, \widetilde{h}[l, s_{-i}]\right)$ such that

$$S_{-i}^\infty\left[\widetilde{t}_{-i}[s_{-i}]\right] = \{s_{-i}\}. \tag{IH}$$

Take any $s_i \in V_i[t_i]$ for some finite type $t_i \in T_i^*$. We will construct a finite type $\widetilde{t}_i$ as in the lemma. Since $(V_j)_{j\in N}$ satisfies the strict best reply property, $BR_i\left(\text{marg}_{\Theta\times S_{-i}}\pi\right) = \{s_i\}$ for some $\pi \in \Delta(\Theta \times T_{-i} \times S_{-i})$ such that $\text{marg}_{\Theta\times T_{-i}^*}\pi = \kappa_{t_i}$ and $\pi(s_{-i} \in V_{-i}[t_{-i}]) = 1$.

Using the induction hypothesis, define the mapping μ :support$\left(\text{marg}_{\Theta\times L\times S_{-i}}\pi\right) \to \Theta \times T_{-i}^*$, by

$$\mu : (\theta, l, s_{-i}) \to \left(\theta, \widetilde{t}_{-i}[s_{-i}]\right),$$

where type $\widetilde{t}_{-i}[s_{-i}] = \left(l, \widetilde{h}[l, s_{-i}]\right)$ is as in (IH). Define $\widetilde{t}_i$ by

$$\kappa_{\widetilde{t}_i} \equiv \left(\text{marg}_{\Theta \times L \times S_{-i}} \pi\right) \circ \mu^{-1}.$$

As (Weinstein and Yildiz, 2007, pp.395-396), we can verify that

$$\begin{aligned} \text{marg}_{\Theta \times L} \kappa_{\widetilde{t}_i} &= \pi \circ \text{proj}^{-1}_{\Theta \times L \times S_{-i}} \circ \mu^{-1} \circ \text{proj}^{-1}_{\Theta \times L} \\ &= \pi \circ \text{proj}^{-1}_{\Theta \times L} = \pi \circ \text{proj}^{-1}_{\Theta \times T^*_{-i}} \circ \text{proj}^{-1}_{\Theta \times L} \\ &= \text{marg}_{\Theta \times L} \kappa_{t_i}. \end{aligned}$$

Moreover, by (IH), each (θ, t_{-i}) on the support of $\kappa_{\widetilde{t}_i}$ which is of the form $\left(\theta, \widetilde{t}_{-i}[s_{-i}]\right)$ and $\widetilde{t}_{-i}[s_{-i}]$ has the unique rationalizable action s_{-i}. Thus, there exists a unique $\widetilde{\pi}$ which is valid for $\widetilde{t}_i$. This belief is $\widetilde{\pi} = \kappa_{\widetilde{t}_i} \circ \gamma^{-1}$ where $\gamma : \left(\theta, \widetilde{t}_{-i}[s_{-i}]\right) \mapsto \left(\theta, \widetilde{t}_{-i}[s_{-i}], s_{-i}\right)$. By construction,

$$\begin{aligned} \text{marg}_{\Theta \times L \times S_{-i}} \widetilde{\pi} &= \pi \circ \text{proj}^{-1}_{\Theta \times L \times S_{-i}} \circ \mu^{-1} \circ \gamma^{-1} \circ \text{proj}^{-1}_{\Theta \times L \times S_{-i}} \\ &= \pi \circ \text{proj}^{-1}_{\Theta \times L \times S_{-i}} \\ &= \text{marg}_{\Theta \times L \times S_{-i}} \pi \end{aligned}$$

where the second equality follows because $\text{proj}_{\Theta \times L \times S_{-i}} \circ \gamma \circ \mu$ is the identity mapping on $\text{support}\left(\text{marg}_{\Theta \times L \times S_{-i}} \pi\right)$. However, s_i is the only best reply to the belief $\text{marg}_{\Theta \times S_{-i}} \pi$ which is the same as $\text{marg}_{\Theta \times S_{-i}} \widetilde{\pi}$. Hence, $S_i^\infty\left[\widetilde{t}_i\right] = \{s_i\}$. Finally, $\widetilde{t}_i$ is indeed a finite type since $\text{support}\kappa_{\widetilde{t}_i}$ is a finite set and consists entirely of finite types.■

A.3 Proof of Lemma 3

In this proof, we only require that Θ is a compact metric space equipped with metric d^0. Let $j \in \{1, 2, ..., n\}$ denote a generic player. Recall that the universal type space T_j^* endowed with the product topology is a compact metrizable space. The compatible metric d_j on T_j^* used in the proof is the one obtained from the Prohorov distance between beliefs of the same order.[2] Specifically, for any $t_j, t_j' \in T_j^*$, let $d_j^1\left(t_j^1, t_j'^1\right)$ be the Prohorov distance between t_j^1 and

[2] Let Y be an arbitrary compact metric space endowed with metric ρ and the Borel σ-algebra. For any two $\mu, \mu' \in \Delta(Y)$, the Prohorov distance between μ and μ' is defined as

$$d(\mu, \mu') = \inf\{\varepsilon > 0 : \mu(E) \leq \mu'(E^\varepsilon) + \varepsilon \text{ for all Borel set } E \subseteq Y\}$$

$t_j'^1$ (recall $t_j^1, t_j'^1 \in \Delta(\Theta)$). Recursively, for any integer $k \geq 2$, and $t_j, t_j' \in T_j^*$, let $d_j^k\left(t_j^k, t_j'^k\right)$ be the Prohorov distance between t_j^k and $t_j'^k$ where $t_j^k, t_j'^k \in \Delta\left(\Theta \times T_{-j}^{k-1}\right)$ in which T_{-j}^{k-1} is the space of all $(k-1)^{th}$-order beliefs of player j's opponents and $\Theta \times T_{-j}^{k-1}$ is equipped with the metric ρ_{-j}^{k-1} defined as $\rho_{-j}^{k-1}\left(\left(\theta, t_{-j}^{k-1}\right), \left(\theta', t_{-j}'^{k-1}\right)\right) \equiv \max\left(d^0(\theta, \theta'), \max_{j' \neq j} d_{j'}^{k-1}\left(t_{j'}', t_{j'}\right)\right)$. Let $d_j\left(t_j, t_j'\right) \equiv \sum_{k=1}^{\infty} 2^{-k} d_j^k\left(t_j^k, t_j'^k\right)$, i.e., d_j is the product metric which metrizes the product topology on T_j^*.

Lemma 3 *For any type $\bar{t}_i \in T_i^*$, there is a sequence of finite types $(t_i(m))_{m=1}^{\infty}$ such that $S_i^{\infty}[t_i(m)] = S_i^{\infty}\left[\bar{t}_i\right]$ for all m and $\lim_{m\to\infty} t_i(m) = \bar{t}_i$.*

Proof. We divide the proof into three steps.

Step 1. Construct the sequence of finite types.

Since T_j^* is a compact metric space, for each natural number m, T_j^* can be covered by finitely many open balls with radius $1/2m$. Let $\mathcal{T}_{j,m}$ be the finite measurable partition of T_j^* induced from these open balls and thus for any $T_j \in \mathcal{T}_{j,m}$, and t_j and t_j' in T_j, $d_j\left(t_j, t_j'\right) < 1/m$. Second, let $\mathcal{T}_{j,0}$ be the finite measurable partition induced by rationalizable sets, i.e., for any $T_j \in \mathcal{T}_{j,0}$, $t_j, t_j' \in T_j$ iff $S_j^{\infty}[t_j] = S_j^{\infty}\left[t_j'\right]$.[3] Let $\widetilde{\mathcal{T}}_{j,m}$ be the join (coarsest common refinement) of $\mathcal{T}_{j,0}$ and $\mathcal{T}_{j,m}$. Let $f_{j,m} : T_j^* \to \widetilde{\mathcal{T}}_{j,m}$ be the mapping such that $f_{j,m}(t_j) = \widetilde{t}_{j,m}$ iff $t_j \in \widetilde{t}_{j,m}$. Moreover, for each $\widetilde{t}_{j,m} \in \widetilde{\mathcal{T}}_{j,m}$, select arbitrarily a type $t_{j,m} \in \widetilde{t}_{j,m}$. It follows that

$$d_j(t_j, t_{j,m}) < 1/m, \forall t_j \in \widetilde{t}_{j,m}. \tag{1}$$

Define a sequence of finite models $\left(\left(\Theta \times \widetilde{T}^m, \widetilde{\kappa}^m\right)\right)_{m=1}^{\infty}$ by letting $\widetilde{T}_j^m \equiv \widetilde{\mathcal{T}}_{j,m}$, and for each $\widetilde{t}_{j,m} \in \widetilde{T}_j^m$,

$$\widetilde{\kappa}_{\widetilde{t}_{j,m}}^m\left[\left(\theta, \widetilde{t}_{-j,m}\right)\right] \equiv \kappa_{t_{j,m}}^*\left[\left\{(\theta, t_{-j}) : f_{-j,m}(t_{-j}) = \widetilde{t}_{-j,m}\right\}\right], \forall\left(\theta, \widetilde{t}_{-j,m}\right) \in \Theta \times \widetilde{T}_{-j}^m. \tag{2}$$

Note $\widetilde{t}_{j,m}$ denotes both a subset of T_j^* and a type in the model $\left(\Theta \times \widetilde{T}^m, \widetilde{\kappa}^m\right)$. We will write $\widetilde{t}_{j,m} \in \widetilde{\mathcal{T}}_{j,m}$ for the former and $\widetilde{t}_{j,m} \in \widetilde{T}_j^m$ for the latter when necessary. Let $\bar{t}_i(m) \equiv f_{i,m}\left(\bar{t}_i\right)$

where $E^{\varepsilon} \equiv \{y \in Y : \inf_{y' \in E} \rho(y, y') < \varepsilon\}$. It is known that the Prohorov metric metrizes the weak*-topology on $\Delta(Y)$ (see (Dudley, 2002, 11.3.3. Theorem)).

[3] Measurability follows from upper hemicontinuity (u.h.c.) of $S_j^{\infty}[\cdot]$: If $A_j' \subseteq A_j$ is $1-$minimal in the sense that there is no type t_j with $S_j^{\infty}[t_j] \subsetneq A_j'$, then u.h.c. implies $\left\{t_j : S_j^{\infty}[t_j] = A_j'\right\} = \left\{t_j : S_j^{\infty}[t_j] \subseteq A_j'\right\}$ is open and hence measurable; if $A_j' \subseteq A_j$ is $2-$minimal in the sense that $S_j^{\infty}[t_j] \subsetneq A_i'$ iff $S_j^{\infty}[t_j]$ is $1-$minimal then $\left\{t_j : S_j^{\infty}[t_j] = A_j'\right\} = \left\{t_j : S_j^{\infty}[t_j] \subseteq A_j'\right\} \setminus \left\{t_j : S_j^{\infty}[t_j] \text{ is } 1-\text{minimal}\right\}$ is measurable, and so on. Since A_j is a finite set, every $S_j^{\infty}[t_j]$ is $k-$minimal for some k and thus $\left\{t_j : S_j^{\infty}[t_j] = A_j'\right\}$ is measurable,$\forall A_j' \subseteq A_j$.

for every m. Step 2 and Step 3 below show that $\lim_{m\to\infty} \bar{t}_i(m) = \bar{t}_i$ and $S_i^\infty\left[\bar{t}_i(m)\right] \supseteq S_i^\infty\left[\bar{t}_i\right]$ for all m. Since $S_i^\infty[\cdot]$ is upper hemicontinuous and $\lim_{m\to\infty} \bar{t}_i(m) = \bar{t}_i$, it follows that $S_i^\infty\left[\bar{t}_i(m)\right] = S_i^\infty\left[\bar{t}_i\right]$ for sufficiently large m, say $m \geq \overline{m}$. We then define $t_i(m) = \bar{t}_i(\overline{m}+m)$,$\forall m$ and $(t_i(m))_{m=1}^\infty$ is the desired sequence.

Step 2. For each m and each $t_j \in T_j^*$, $S_j^\infty\left[f_{j,m}(t_j)\right] \supseteq S_j^\infty[t_j]$.

First, for each $\widetilde{t}_{j,m} \in \widetilde{T}_j^m$, we define $\overline{S}_j\left[\widetilde{t}_{j,m}\right] = S_j^\infty[t_{j,m}]$. We show that $\overline{S}_j[\cdot]$ satisfies the best-reply property on the model $\left(\Theta \times \widetilde{T}^m, \widetilde{\kappa}^m\right)$ (see (Dekel et al., 2007, Definition 1)). To see this, suppose that $s_j \in \overline{S}_j\left[\widetilde{t}_{j,m}\right]$. Since $\overline{S}_j\left[\widetilde{t}_{j,m}\right] = S_j^\infty[t_{j,m}]$, $s_j \in S_j^\infty[t_{j,m}]$. Thus, $s_j \in BR_j\left(\text{marg}_{\Theta\times S_{-j}}\pi\right)$ for some $\pi \in \Delta\left(\Theta \times T_{-j}^* \times S_{-j}\right)$ which is valid for $t_{j,m}$.

Define $\widetilde{\pi} \in \Delta\left(\Theta \times \widetilde{T}_{-j}^m \times S_{-j}\right)$ such that

$$\widetilde{\pi}\left[\left(\theta, \widetilde{t}_{-j,m}, s_{-j}\right)\right] \equiv \pi\left(\left\{(\theta, t_{-j}, s_{-j}) : f_{-j,m}(t_{-j}) = \widetilde{t}_{-j,m}\right\}\right), \forall\left(\theta, \widetilde{t}_{-j,m}, s_{-j}\right) \tag{3}$$

Since π is valid for $t_{j,m}$, $\text{marg}_{\Theta\times T_{-j}^*}\pi = \kappa_{t_{j,m}}^*$. Hence, by (2), $\text{marg}_{\Theta\times\widetilde{T}_{-j}^m}\widetilde{\pi} = \widetilde{\kappa}_{\widetilde{t}_{j,m}}^m$. Moreover,

$$\begin{aligned}
& \widetilde{\pi}\left(\left\{\left(\theta, \widetilde{t}_{-j,m}, s_{-j}\right) : s_{-j} \in \overline{S}_{-j}\left[\widetilde{t}_{-j,m}\right]\right\}\right) \\
= \; & \pi\left\{(\theta, t_{-j,}, s_{-j}) : f_{-j,m}(t_{-j}) = \widetilde{t}_{-j,m} \text{ and } s_{-j} \in \overline{S}_{-j}\left[\widetilde{t}_{-j,m}\right]\right\} \\
= \; & \pi\left\{(\theta, t_{-j,}, s_{-j}) : f_{-j,m}(t_{-j}) = \widetilde{t}_{-j,m} \text{ and } s_{-j} \in S_{-j}^\infty\left[t_{-j,m}\right]\right\} \\
= \; & \pi\left(\left\{(\theta, t_{-j}, s_{-j}) : s_{-j} \in S_{-j}^\infty[t_{-j}]\right\}\right) \\
= \; & 1
\end{aligned}$$

where the first equality follows from (3), the second follows because $\overline{S}_{-j}\left[\widetilde{t}_{-j,m}\right] = S_{-j}^\infty[t_{-j,m}]$, the third follows because every $t_{-j} \in \widetilde{t}_{-j,m}$ has the same rationalizable set as $t_{-j,m}$, and the last is because π is valid for $t_{j,m}$. Finally, since $s_j \in BR_j\left(\text{marg}_{\Theta\times S_{-j}}\pi\right)$ and $\widetilde{\pi}$ and π have the same marginal distribution on $\Theta \times S_{-j}$, it follows that $s_j \in BR_j\left(\text{marg}_{\Theta\times S_{-j}}\widetilde{\pi}\right)$. Hence, $\overline{S}_j[\cdot]$ satisfies the best-reply property on $\left(\Theta \times \widetilde{T}^m, \widetilde{\kappa}^m\right)$. Thus, by (Dekel et al., 2007, Proposition 4), $\overline{S}_j\left[\widetilde{t}_{j,m}\right] \subseteq S_j^\infty\left[\widetilde{t}_{j,m}\right]$, and moreover, since $\overline{S}_j\left[\widetilde{t}_{j,m}\right] = S_j^\infty[t_{j,m}]$, we obtain $S_j^\infty[t_{j,m}] \subseteq S_j^\infty\left[\widetilde{t}_{j,m}\right]$ and because every $t_j \in \widetilde{t}_{j,m}$ has the same rationalizable set as $t_{j,m}$, we get $S_j^\infty[t_j] \subseteq S_j^\infty[f_{j,m}(t_j)]$.

Step 3. $\lim_{m\to\infty} \sup_{t_j \in T_j^*} d_j\left(f_{j,m}(t_j), t_j\right) = 0$.

For each $t_j \in T_j^*$, let $\widetilde{t}_{j,m} = f_{j,m}(t_j)$. We show that the k^{th}-order belief of $\widetilde{t}_{j,m}$ (viewed as a type in the model $\left(\Theta \times \widetilde{T}^m, \widetilde{\kappa}^m\right)$) converges to t_j^k and the convergence is uniform in t_j. We prove this by induction on k. For $k=1$, observe that by (2) $\widetilde{t}^1_{j,m} = t^1_{j,m}$. By (1), $d_j\left(t_j, \widetilde{t}_{j,m}\right) = d_j\left(t_j, t_{j,m}\right) < 1/m$, and since $d^1_j\left(\widetilde{t}^1_{j,m}, t^1_j\right) \leq d_j\left(\widetilde{t}_{j,m}, t_j\right)$, $\lim_{m\to\infty} \sup_{t_j\in T^*_j} d^1_j\left(\widetilde{t}^1_{j,m}, t^1_j\right) = 0$.

Now consider $k > 1$. Let $\varepsilon \in (0,1)$ and we show that for sufficiently large m, $d^k_j\left(\widetilde{t}^k_{j,m}, t^k_j\right) < \varepsilon$ for all $t_j \in T^*_j$. By the induction hypothesis, there is some $\overline{m}(\varepsilon)$ such that for any $m > \overline{m}(\varepsilon)$, $\max_{j'\neq j} d^{k-1}_{j'}\left(f_{j',m}(t_{j'})^{k-1}, t^{k-1}_{j'}\right) < \varepsilon/2$ for all $t_{-j} = (t_{j'})_{j'\neq j} \in T^*_{-j}$. Consider $m > \{2/\varepsilon, \overline{m}(\varepsilon)\}$. Recall that d^k_j is the Prohorov metric on the space of all k^{th}-order beliefs. Since $\widetilde{t}_{j,m}$ is a finite type, it suffices to verify that for each $\left(\theta, \widetilde{t}^{k-1}_{-j,m}\right)$ in the support of $\widetilde{t}_{j,m}$, we have

$$\begin{aligned}
\widetilde{t}^k_{j,m}\left[\left(\theta, \widetilde{t}^{k-1}_{-j,m}\right)\right] &= \kappa^*_{t_{j,m}}\left(\left\{(\theta, t_{-j}) : f_{-j,m}(t_{-j})^{k-1} = \widetilde{t}^{k-1}_{-j,m}\right\}\right) \\
&\leq \kappa^*_{t_{j,m}}\left(\left(\theta, \widetilde{t}^{k-1}_{-j,m}\right)^{\varepsilon/2}\right) \\
&< t^k_j\left(\left(\theta, \widetilde{t}^{k-1}_{-j,m}\right)^{\varepsilon}\right) + \varepsilon
\end{aligned}$$

where the first equality follows from (2); the first inequality follows because $f_{-j,m}(t_{-j})^{k-1} = \widetilde{t}^{k-1}_{-j,m}$ implies $\max_{j'\neq j} d^{k-1}_{j'}\left(t^{k-1}_{j'}, \widetilde{t}^{k-1}_{j',m}\right) < \varepsilon/2$ (since $m > \overline{m}(\varepsilon)$); the second follows because by (1), $d^k_j\left(t^k_{j,m}, t^k_j\right) < 1/m < \varepsilon/2$ (since $m > 2/\varepsilon$). Thus, for $m > \{2/\varepsilon, \overline{m}(\varepsilon)\}$, $d^k_j\left(f_{j,m}(t_j)^k, t^k_j\right) < \varepsilon$ for all $t_j \in T^*_j$. Since $\varepsilon > 0$ is arbitrary, the induction step follows.■

References

Chen, Y.-C., 2011. A structure theorem for rationalizability in the normal form of dynamic games, mimeo.

Dekel, E., Fudenberg, D., Morris, S., 2006. Topologies on types. Theoretical Econ. 1, 275–309.

Dekel, E., Fudenberg, D., Morris, S., 2007. Interim correlated rationalizability. Theoretical Econ. 2, 15–40.

Dudley, R., 2002. Real Analysis and Probability. Cambridge University Press, Cambridge.

Mertens, J.-F., Zamir, S., 1985. Formulation of Bayesian analysis for games with incomplete information. Int. J. Game Theory 14, 1–29.

Weinstein, J., Yildiz, M., 2007. A structure theorem for rationalizability with application to robust predictions of refinements. Econometrica 75, 365–400.