Supplemental Appendix for "The Strategic Impact of Higher-Order Beliefs"

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Throughout the note, we endow the universal type space (denoted by T_i^*) with product topology. Say $\eta \in \Delta(\Theta \times T_i^* \times T_{-i}^*)$ is a common prior if for every bounded measurable function $\varphi : \Theta \times T_i^* \times T_{-i}^* \to \mathbb{R}$,

$$\int_{T_i^*} \left(\int_{\Theta \times T_{-i}^*} \varphi(\theta, t_i, t_{-i}) \, \mu_i ((\theta, dt_{-i}) | t_i) \right) \eta_i [dt_i] = \int_{\Theta \times T^*} \varphi(\theta, t) \, \eta [d(\theta, t)], \forall i \in I,$$

where η_i denotes the marginal distribution on T_i^* and $\mu_i(\cdot|t_i)$ denotes the interim belief of type t_i. We say that $(T_i)_{i \in I}$ is a common prior space if there is a common prior $\eta \in \Delta(\Theta \times T_1^* \times T_2^*)$ such that T_i is the support of η_i in T_i^* for every player i. Furthermore, we say a type t_i is a common prior type if t_i is contained in a common prior type space $(T_j)_{j \in I}$ with the associated common prior $\eta \in \Delta(\Theta \times T_i^* \times T_{-i}^*)$ such that

$$\eta \left[\Theta \times \{t_i\} \times T_{-i}^* \right] > 0. \tag{1}$$

We call (1) the positivity condition. In this note, we demonstrate two pathological problems if we do not require the positivity condition in the definition of common-prior types.

1 A common prior type who has conflicting belief with his opponents

Our usual understanding is that we should be able to perform Bayesian updating on common prior types. For instance, the law of iterated expectation should hold. That is, under a common prior, if I believe with probability 1 that all of my opponents believe $\theta = 0$ with probability 1, then I

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should believe $\theta = 0$ with probability 1. However, we show this does not hold for a common prior type, if the positivity condition is not required.

Consider the following sets.

$$\begin{aligned} A &= \left\{a^k : k \in Z_{++}\right\}; \\ B &= \left\{b^k : k \in Z_{++}\right\}; \\ C &= \left\{c^k : k \in Z_{++}\right\}; \end{aligned}$$

Elements in $A \cup B$ are player 1's types, and elements in C are player 2's types. Consider a prior η on $(A \cup B) \times C \times \{\theta = 0, \theta = 1\}$ defined as follows. For each $k \in Z_{++}$,

$$\begin{aligned} \eta\left[a^k, c^k, \theta = 0\right] &= \left(1 - \frac{1}{k}\right) \times \left(\frac{1}{2}\right)^k; \\ \eta\left[b^k, c^k, \theta = 1\right] &= \frac{1}{k} \times \left(\frac{1}{2}\right)^k. \end{aligned}$$

a^k, b^k, c^k are types, and their interim beliefs can be derived by applying the Bayes' Rule.

Consider another type space, $\left\langle \left\{t_1^*, t_1^{**}\right\}, \left\{t_2^*\right\}, (\mu_i)\right\rangle$ such that

$$\begin{aligned} \mu_1\left(t_1^*\right)\left[t_2^*, \theta = 0\right] &= 1; \\ \mu_2\left(t_2^*\right)\left[t_1^*, \theta = 0\right] &= 1; \\ \mu_1\left(t_1^{**}\right)\left[t_2^*, \theta = 1\right] &= 1. \end{aligned}$$

Note that $(A \cup B) \times C$ is not closed, and three cluster points are missing: $\left\{a^k\right\} \to t_1^*$, $\left\{b^k\right\} \to t_1^{**}$, $\left\{c^k\right\} \to t_2^*$. Therefore, $\left(A \cup B \cup \left\{t_1^*, t_1^{**}\right\}\right) \times \left(C \cup \left\{t_2^*\right\}\right)$ is a common prior type space with the associated prior η. According to η, only three types has 0 probability, i.e., t_1^*, t_1^{**}, t_2^*. Without the positivity condition, t_1^{**} would be considered as a common prior type, but this simply does not make sense, because this type knows $\theta = 1$, and he knows his opponent knows $\theta = 0$ — the extreme case of violation of Bayesian updating.

2 A full support common prior on the universal type space

Let $\mathcal{F} \subset T^*$ denote the set of finite common prior type spaces. For any profile of positive integers $(n_i)_{i \in I}$, let $\mathcal{F}^{(n_i)_{i \in I}}$ denote the set of common prior type space in which agent i has n_i types. Thus, $\mathcal{F}$ can partitioned into a countable set of $\mathcal{F}^{(n_i)_{i \in I}}$.

For each $\mathcal{F}^{(n_i)_{i \in I}}$, there exists a countable dense subset (in Hausdorff metric), e.g., the following set:

$$\left\{ T \in \mathcal{F}^{(n_i)_{i \in I}} : \begin{array}{l} \eta \text{ is a prior on } T \\ \text{and } \eta(t, \theta) \in Q \backslash \{0\},\ \forall (t, \theta) \in T \times \Theta \end{array} \right\}$$

We say a type space is minimal, iff it does not contain another type space as a strict subset. Clearly, all elements in the set above is a minimal type space.

As a result, we can find a countable set of minimal common prior type spaces, denumerated as $\left\{\left(T_i^1\right),\left(T_i^2\right),...\right\}$, which is dense in $\mathcal{F}$ (in Hausdorff metric). By Lipman's result, types in the type spaces of $\mathcal{F}$ are dense in the universal type spaces. As a result, $\bigcup\limits_k T_i^k$ is dense in the universal type space for every i. Also, note that the minimality condition implies that elements in $\left\{\left(T_i^1\right),\left(T_i^2\right),...\right\}$ are disjoint.

Let η^k denote the prior for $\left(T_i^k\right)$, and hence,

$$\eta^k\left(\Theta\times T_i^k\times T_{-i}^k\right)=1 \tag{2}$$

and for any bounded measurable function $\varphi:\Theta\times T_i^*\times T_{-i}^*\rightarrow R$,

$$\int_{T_i^*}\left(\int_{\Theta\times T_{-i}^*}\varphi\left(\theta,t_i,t_{-i}\right)\mu_i\left(\left(\theta,dt_{-i}\right)|t_i\right)\right)\eta_i^k\left[dt_i\right]=\int_{\Theta\times T^*}\varphi\left(t\right)\eta\left[dt\right],\forall i\in I. \tag{3}$$

We are now ready to define a new prior η^* on the universal type space such that the support is the universal type space itself.

$$\eta^*\left(\Theta\times T_i^k\times T_{-i}^k\right)=\left(\frac{1}{2}\right)^k,\forall k\in Z_{++};$$

$$\eta^*\left(E\right)=\sum_k\eta^*\left(\Theta\times T_i^k\times T_{-i}^k\right)\times\eta^k\left[E\cap\left(\Theta\times T_i^k\times T_{-i}^k\right)\right], \tag{4}$$

$$\forall E\subset\Theta\times T_i^*\times T_{-i}^*.$$

That is, for each k, $\eta^*\left(\Theta\times T_i^k\times T_{-i}^k\right)=\left(\frac{1}{2}\right)^k$, and condition on $\Theta\times T_i^k\times T_{-i}^k$, the probability is distributed according to η^k.

Clearly, the support of η^* is the universal type space. Finally, we check η^* is a prior. For any bounded measurable function $\varphi : \Theta \times T_i^* \times T_{-i}^* \to R$,

$$\begin{aligned}
\int_{\Theta \times T^*} \varphi(t)\, \eta^*[dt] &= \sum_k \int_{\Theta \times T_i^k \times T_{-i}^k} \varphi(\theta, t)\, \eta^*(\theta, dt) \\
&= \sum_k^\infty \eta^*\left(\Theta \times T_i^k \times T_{-i}^k\right) \int_{\Theta \times T_i^k \times T_{-i}^k} \varphi(\theta, t)\, \eta^k(\theta, dt) \\
&= \sum_k^\infty \eta^*\left(\Theta \times T_i^k \times T_{-i}^k\right) \int_{\Theta \times T_i^* \times T_{-i}^*} \varphi(\theta, t)\, \eta^k(\theta, dt) \\
&= \sum_k^\infty \eta^*\left(\{\theta\} \times T_i^k \times T_{-i}^k\right) \times \int_{T_i^*} \left(\int_{\Theta \times T_{-i}^*} \varphi(\theta, t_i, t_{-i})\, \mu_i((\theta, dt_{-i})\,|t_i)\right) \eta_i^k[dt_i] \\
&= \int_{T_i^*} \left(\int_{\Theta \times T_{-i}^*} \varphi(\theta, t_i, t_{-i})\, \mu_i((\theta, dt_{-i})\,|t_i)\right) \eta_i^*[dt_i],
\end{aligned}$$

where the first equality follows from the fact that $\Theta \times T^*$ is partitioned by $\left\{\Theta \times T_i^k \times T_{-i}^k\right\}_{k=1}^\infty$; the second equation follows from (4); the third equation follows (2); the fourth equation follows from (3); the last equation follows from (2) and (4). Therefore, μ^* is a prior on the universal type space.