Robust Selection of Rationalizability*

Yi-Chun Chen[†] Siyang Xiong[‡]

September 13, 2011

Abstract

We propose and fully characterize a notion of selecting rationalizable actions via perturbing players' higher-order beliefs, which we call *the robust selection*. Like *the WY selection* (Weinstein and Yildiz (2007)), the robust selection generalizes the idea behind the equilibrium selection in the email game (Rubinstein (1989)) and the global game (Carlsson and Van Damme (1993)). Different from the WY selection, the robust selection captures two robustness features of the selection in the global game (and the email game). The robust selection is a strong notion in the sense that, among types with multiple rationalizable actions, *almost all* WY selections are not robust; but it is also a weak notion in the sense that *any* strictly rationalizable action can be robustly selected.

*We thank Eddie Dekel, Songying Fang, Qingmin Liu, George Mailath, Marcin Pęski, Johnathan Weinstein, Muhamet Yildiz, and seminar participants at University of Pennsylvania and University of Western Ontario for very helpful comments. All remaining errors are our own.

[†]Department of Economics, National University of Singapore, Singapore 117570, ecsycc@nus.edu.sg

[‡]Department of Economics, Rice University, Houston, TX 77251, xiong@rice.edu

1 Introduction

One of the challenges that game theory faces is the prevalence of multiple equilibria, which substantially limits the theory's predictive power. Two seminal papers started a literature that pursues the idea of refining predictions by perturbing players' higher-order beliefs: Rubinstein (1989) on the email game and Carlsson and Van Damme (1993) on the global game. Both papers demonstrate that in a complete-information coordination game with two strict equilibria, we can perturb players' higher-order beliefs to select one equilibrium which is uniquely rationalizable. We review the idea in the following example.

Example 1—The Global Game. There are two players whose payoffs depend on an unknown parameter $\theta \in \mathbb{R}$ as summarized in the following matrix.

G_1 :

	Attack	No Attack
Attack	θ, θ	$\theta - 1, 0$
No Attack	$0, \theta - 1$	$0, 0$

Suppose each player i receives a noisy signal $x_i \equiv \theta + \xi_i$, where ξ_i is i.i.d. uniformly distributed on the interval $[-\alpha, \alpha]$ for some $\alpha \geq 0$. Let $\left(\frac{3}{4}\right)_i^\alpha$ denote the type of player i who observes $x_i = \frac{3}{4}$ given α. When $\alpha = 0$, type $\left(\frac{3}{4}\right)_i^0$ has common knowledge of $\theta = \frac{3}{4}$ and it has two rationalizable actions, "Attack" and "No Attack." As $\alpha \downarrow 0$, type $\left(\frac{3}{4}\right)_i^\alpha$ can be viewed as a small perturbation of the common knowledge scenario, and "Attack" is the unique rationalizable action for every $\left(\frac{3}{4}\right)_i^\alpha$ with $\alpha > 0$. In this sense, we say "Attack" is selected for type $\left(\frac{3}{4}\right)_i^0$.

However, Morris and Shin (2003) observe that we may also select "No Attack" for type $\left(\frac{3}{4}\right)_i^0$ if we take a different perturbation. Weinstein and Yildiz (2007) (hereafter, WY) substantially generalize the idea. In particular, WY consider an incomplete-information game in which players' higher-order beliefs are modeled as (Harsanyi) types and each type identifies a belief about the payoffs (the first-order belief), a belief about the opponents' beliefs about the payoffs (the second-order belief), and so on. WY adopt the following notion of selection which generalizes the global-game selection[1] to types with

[1]Throughout the paper, we use "the global-game selection" to represent the idea of selection displayed in Rubinstein (1989) and Carlsson and Van Damme (1993).

incomplete information about the payoffs. An action a is said to be *WY-selected*[2] for a type t if there is a sequence of types $\{t_n\}$ such that (i) $\{t_n\}$ (weak*-)approximates t up to any finite order; (ii) a is the unique rationalizable action for every t_n. WY prove a surprising result: if we relax all common-knowledge assumptions on payoffs, *any* rationalizable action can be WY-selected for *any* (finite) type.[3] This result implies that multiplicity of equilibria is an artifact of modeling assumptions and selecting any outcome does not yield robust prediction beyond rationalizability.

However, we observe an essential difference between the WY selection and the global-game selection: The global-game selection satisfies two important robustness features that are not shared by the WY selection. Specifically, the global-game selection is robust to slight misspecifications of best replies of the players as well as small measurement errors in payoffs.

In this paper, we propose a new notion of selection — *the robust selection*, which not only generalizes the global-game selection, but also captures the two robustness features.

These two robustness features are important because every game in economic models is at best an approximation of reality. The exact specification of best replies and the precise measurement of payoffs are often costly and sometimes impossible. It is therefore unrealistic to assume that players can keep track of all payoffs and figure out their exact best replies, especially when the game is complicated. A modeler faces a similar difficulty and may not be able to measure precisely players' payoffs. These features are also important in prominent applications of global games.[4] For instance, it is difficult to believe that economists can precisely measure the payoff of every investor in every possible state, when they build models of bank runs or currency attacks.

To capture the two robustness features, we use $\varepsilon-$*best replies* to define the robust selection. In light of the issues mentioned above, many authors have argued that we should consider solution concepts based on $\varepsilon-$best replies, such as $\varepsilon-$equilibrium or $\varepsilon-$rationalizability with $\varepsilon > 0$, instead of those based on $0-$best replies, such as ratio-

[2]Throughout the paper, we use "the WY selection" to represent the selection studied in Weinstein and Yildiz (2007).

[3]WY prove the statement for any finite type and Chen (2011) generalizes it to any type. Also Penta (2011) and Chen (2011) gerenalize WY's structure theorem to dynamic settings.

[4]See Morris and Shin (2003) for a survey of applications of the global-game selection.

nalizability or equilibrium.[5] For any $\varepsilon \geq 0$, we say an action is $\varepsilon-$rationalizable iff it survives iterated deletion of actions which are never $\varepsilon-$best replies.[6] Then, an action a is said to be *robustly selected* for a type t if there exist $\varepsilon > 0$ and a sequence of types $\{t_n\}$ such that (i) $\{t_n\}$ (weak*-)approximates t up to any finite order; (ii′) a is the unique $\varepsilon-$rationalizable action for every t_n. So the difference between the WY selection and the robust selection is that $\varepsilon = 0$ is considered in the former but $\varepsilon > 0$ is required by the latter. We demonstrate below that the robust selection shares the two robustness features of the global-game selection, while the WY selection does not.

First, the robustness to misspecifications of best replies is encoded in the definition of the robust selection. Note that an $\varepsilon'-$rationalizable action is also $\varepsilon-$rationalizable if $\varepsilon' < \varepsilon$. As a result, if a is robustly selected for t, conditions (i) and (ii′) must continue to hold for any $\varepsilon' \in (0, \varepsilon)$. Indeed, the global-game selection in Example 1 exhibits this feature, i.e., for any $\varepsilon' \in \left(0, \frac{1}{10}\right)$, "Attack" is the unique $\varepsilon'-$rationalizable action for $\left(\frac{3}{4}\right)_i^\alpha$ with α sufficiently close to 0. We now illustrate by the following example that this is not the case for some WY selections.

Example 2—WY Selection. There is one player and he chooses between two actions $\{a, b\}$.[7] The player's payoff depends on an unknown parameter $\theta \in \{\theta_0, \theta_a\}$ and the action chosen, as illustrated below.

G_2 :

action	payoff
a	0
b	0

$\theta = \theta_0$

action	payoff
a	0
b	-1

$\theta = \theta_a$

In this single-person decision problem, a type is identified by the player's belief about the

[5]The idea of $\varepsilon-$equilibrium is first introduced by Radner (1980). Recently, Levine and Zheng (2010) make the forceful statement that "the only meaningful theory of Nash equilibrium is Radner's notion of epsilon-equilibrium." See also Mailath, Postlewaite, and Samuelson (2005), Dekel, Fudenberg, and Morris (2006), Jackson, Rodriguez-Barraquer, and Tan (2011).

[6]Recall that an action is rationalizable iff it survives iterated deletion of actions which are never $(0-)$best replies. Here we follow Weinstein and Yildiz (2007) and Dekel, Fudenberg, and Morris (2007) to adopt $\varepsilon-$interim correlated rationalizability for incomplete-information games.

[7]For simplicity, we consider a single-person game in Example 2. The idea can be easily extended to multi-person games.

parameter θ. For $n \in \mathbb{N} \cup \{\infty\}$, define type t_n as

$$t_n[\theta_0] = 1 - \frac{1}{n} \text{ and } t_n[\theta_a] = \frac{1}{n},$$

where $t_n[\theta]$ denotes the probability that type t_n assigns to θ. Clearly, a and b are both rationalizable for t_∞ and $\{t_n\}$ converges t_∞, but a is uniquely rationalizable for every t_n, i.e., a is WY-selected for t_∞. However, for any $\varepsilon > 0$, both a and b are $\varepsilon-$rationalizable for t_n with sufficient large n, i.e., the WY selection is not robust to misspecifications of best replies.

Second, the robust selection is also robust to (small) measurement errors in payoffs. To see this, we say a game G is a $\gamma-$approximation of another game G' (with the same sets of actions and players) iff the payoffs of any outcome in G and G' differ at most by $\gamma \geq 0$. Due to measurement errors, a game in an economic model may only be a $\gamma-$approximation of the true strategic situation, where γ represents the measurement error. The modeler may improve the approximation by reducing γ, but a perfect modeling ($\gamma = 0$) is unlikely to be achieved. We say a selection notion is robust to measurement errors iff an action a being selected for a type t in a game G implies that there exists $\gamma > 0$ such that a continues to be selected for t in any $\gamma-$approximation of G. Again, the global-game selection is robust to measurement errors, while this is not always the case for the WY selection.

Example 1 Revisited. Suppose we modify the true payoffs in Example 1 as follows (where $-\frac{1}{10} \leq \gamma_m \leq \frac{1}{10}, m = 1, 2, ..., 8$).

G_1' :

	Attack	No Attack
Attack	$\theta + \gamma_1, \theta + \gamma_2$	$\theta - 1 + \gamma_3, \gamma_4$
No Attack	$\gamma_5, \theta - 1 + \gamma_6$	γ_7, γ_8

Both "Attack" and "No Attack" are rationalizable for $\left(\frac{3}{4}\right)_i^0$, but "Attack" is the unique rationalizable action for $\left(\frac{3}{4}\right)_i^\alpha$ with α sufficiently close to 0.

Example 2 Revisited. Suppose we modify the true payoffs in Example 2 as follows (where

$\gamma_9 > 0$).

G_2' :

action	payoff	action	payoff
a	0	a	0
b	γ_9	b	-1
$\theta = \theta_0$		$\theta = \theta_a$	

Clearly, a cannot be WY-selected for t_∞.

In this paper, we first prove that the robust selection is robust to measurement errors (Theorem 1). Second, following the same setup as in WY, we show that the theory based on the robust selection is very different from that based on the WY selection. Recall that WY's main result is that any rationalizable action can be WY-selected for any finite type. We show that this result changes dramatically when we replace the WY selection with the robust selection: Among types with more than one rationalizable actions (i.e., types for which the prediction needs to be refined), the robust selection is *generically impossible* (Theorem 2). We prove this by showing that, generically, multiplicity occurs due to a payoff tie, as in the single-person decision problem in Example 2.

In contrast to the negative result above, we prove that any strictly rationalizable (or strict equilibrium) action can be robustly selected (Theorem 3). This result sheds light on why the strict equilibria selected in the email game and the global game are robust.

Finally, we propose a notion called $\varepsilon-$*strict curb collection* ($\varepsilon-SCC$), and we show that it fully characterizes the robust selection for any complete-information types (Theorem 4), and more generally, for any finite types (Theorem 5).

The rest of the paper is organized as follows. Section 2 presents preliminaries. Section 3 proves the generic impossibility of the robust selection. Section 4 fully characterizes the robust selection. Section 5 discusses the implications of our results and related issues. Some technical proofs are relegated to the Appendix.

2 Preliminaries

For any metric space Y, let $\Delta(Y)$ denote the space of all probability measures on the Borel σ-algebra of Y endowed with the weak*-topology. Every product space is endowed with the product topology and every subspace is endowed with the relative topology. Every finite or countable set is endowed with the discrete topology. For a finite set E, let $|E|$ denote the cardinality of E. For any $\mu \in \Delta(Y)$, let $\text{supp}\mu$ denote the support of μ, i.e., the intersection of all closed sets with $\mu-$measure 1.

2.1 The model of incomplete information

Fix a finite set of players $N = \{1, 2, ..., n\}$ and a finite set of payoff-relevant parameters Θ. By a *model*, we mean a pair (T, κ), where $T = T_1 \times T_2 \times \cdots \times T_n$ is a compact metric space. Each $t_i \in T_i$ is called a type of player i and it is associated with a belief $\kappa_{t_i} \in \Delta(\Theta \times T_{-i})$. Assume that $t_i \mapsto \kappa_{t_i}$ is a continuous mapping. A model (T, κ) with $|T| < \infty$ is called a *finite model*.

Given any type t_i in a model (T, κ), we can compute the first-order belief of t_i (i.e., his belief on Θ) by setting t_i^1 equal to the marginal distribution of κ_{t_i} on Θ. We then compute the second-order belief of t_i (i.e., his belief about (θ, t_{-i}^1)) by setting

$$t_i^2(F) = \kappa_{t_i}\left(\left\{(\theta, t_{-i}) : \left(\theta, t_{-i}^1\right) \in F\right\}\right)$$

for every measurable $F \subseteq \Theta \times [\Delta(\Theta)]^{n-1}$. We can compute the entire hierarchy of beliefs $(t_i^1, t_i^2, ..., t_i^k, ...)$ by proceeding in this way and write $h_i(t_i) = (t_i^1, t_i^2, ..., t_i^k, ...)$ for the resulting hierarchy of beliefs.

We endow Θ with the discrete metric. Let $Y^0 = \Theta$ and $Y^{k+1} = Y^k \times \left[\Delta\left(Y^k\right)\right]^{n-1}$ for every $k \geq 0$. We will work with the universal type space T_i^* constructed in Mertens and Zamir (1985) which is a subset of $\times_{k=0}^{\infty}\Delta\left(Y^k\right)$. Mertens and Zamir (1985) show that for any type t_i in any model, there is some $t_i' \in T_i^*$ such that t_i' and t_i have the same hierarchy of beliefs (i.e., $h_i(t_i) = h_i(t_i')$), and moreover, T_i^* (endowed with the product topology) is a compact metric space homeomorphic to $\Delta\left(\Theta \times T_{-i}^*\right)$. We use κ_i^* to denote the homeomorphism. Then, (T^*, κ^*) is itself a model where $\kappa_{t_i}^* \equiv \kappa_i^*(t_i)$ for every $t_i \in T_i^*$. Let d_i be the metric on T_i^*.

A type $t_i \in T_i^*$ is said to be a complete-information type if there exist some $\theta \in \Theta$ and some $t_{-i} \in T_{-i}^*$ such that $\kappa_{t_j}^*\left[(\theta, t_{-j})\right] = 1$ for every $j \in N$. We use t_i^θ to denote the complete-information type for which θ is common knowledge. A type $t_i \in T_i^*$ is a finite type if there exists some t'_i in a finite model such that $h_i(t_i) = h_i(t'_i)$. For example, a complete-information type is finite. For a sequence of types $\{t_{i,m}\}$ and a type t_i, we write $t_{i,m} \to t_i$ when $\{t_{i,m}\}$ converges to t_i in the product topology. That is, $t_{i,m} \to t_i$ iff $t^k_{i,m} \to t^k_i$ in the weak*-topology for every $k \geq 1$.

2.2 The game and solution concept

Each player i has a finite set of actions A_i, which we fix throughout the paper. Let $A = \Pi_{i \in N} A_i$ denote the set of action profiles. A game is a tuple $G = (u_i)_{i \in N}$, where $u_i : \Theta \times A \to \mathbb{R}$ is player i's utility function. WY's Richness assumption can be stated as follows.

Definition 1 *A game $G = (u_i)_{i \in N}$ satisfies the Richness assumption if for every $i \in N$ and every $a_i \in A_i$, there exists $\theta^{a_i} \in \Theta$ such that $u_i(\theta^{a_i}, a_i, a_{-i}) > u_i(\theta^{a_i}, a'_i, a_{-i})$, $\forall a'_i \neq a_i$, $\forall a_{-i}$.*

Except for Section 2.3, we consider a fixed game which satisfies the Richness assumption.

Following WY, we adopt the solution concept of interim correlated rationalizability (ICR) proposed in Dekel, Fudenberg, and Morris (2006, 2007) and restrict our attention to the universal type space.[8] For any $\pi \in \Delta(\Theta \times A_{-i})$ and any $\varepsilon \geq 0$, we use $BR_i(\pi, G, \varepsilon)$ to denote the set of $\varepsilon-$best replies to π in game $G = (u_i)_{i \in N}$. That is,

$$BR_i(\pi, G, \varepsilon) = \left\{ a_i \in A_i : \sum_{\theta, a_{-i}} \left[u_i(\theta, a_i, a_{-i}) - u_i(\theta, a'_i, a_{-i}) \right] \pi[(\theta, a_{-i})] \geq -\varepsilon, \forall a'_i \in A_i \right\}.$$

We can similarly define the set of *strict* $\varepsilon-$best replies to π in game $G = (u_i)_{i \in N}$. That is,

$$BR^\circ_i(\pi, G, \varepsilon) = \left\{ a_i \in A_i : \sum_{\theta, a_{-i}} \left[u_i(\theta, a_i, a_{-i}) - u_i(\theta, a'_i, a_{-i}) \right] \pi[(\theta, a_{-i})] > -\varepsilon, \forall a'_i \neq a_i \right\}.$$

In any model (T, κ), a type $t_i \in T_i$ and a measurable function $\sigma_{-i} : \Theta \times T_{-i} \to \Delta(A_{-i})$ define a distribution on $\Theta \times A_{-i}$ as follows, which is denoted by $\pi_{t_i, \sigma_{-i}} \in \Delta(\Theta \times A_{-i})$.

$$\pi_{t_i, \sigma_{-i}}[(\theta, a_{-i})] = \int_{T_{-i}} \sigma_{-i}(\theta, t'_{-i})[a_{-i}] \, \kappa_{t_i}\left[(\theta, dt'_{-i})\right], \forall (\theta, a_{-i}) \in \Theta \times A_{-i}$$

[8]This is without loss of generality because the ($\varepsilon-$)ICR actions of a type are fully determined by its belief hierarhy (see Dekel, Fudenberg, and Morris (2007)) and the universal type space contains all belief hierarchies (see Mertens and Zamir (1985)).

Given any model (T, κ) and any $\varepsilon \geq 0$, the $\varepsilon-$ICR actions of a type t_i in game G, denoted by $S_i^\infty [t_i, G, \varepsilon]$, is defined as

$$S_i^\infty [t_i, G, \varepsilon] = \bigcap_{k=0}^{\infty} S_i^k [t_i, G, \varepsilon]$$

where

$$S_i^0 [t_i, G, \varepsilon] \equiv A_i,$$

and inductively, for each integer $k \geq 1$, $a_i \in S_i^k [t_i, G, \varepsilon]$ if and only if there is a measurable function $\sigma_{-i} : \Theta \times T_{-i} \to \Delta(A_{-i})$ such that[9]

$$\begin{aligned} a_i &\in BR_i \left(\pi_{t_i,\sigma_{-i}}, G, \varepsilon\right) \text{ and} \\ \text{supp}\sigma_{-i} (\theta, t_{-i}) &\subset \Pi_{j \neq i} S_j^{k-1} \left[t_j, G, \varepsilon\right] \text{ for all } (\theta, t_{-i}) \in \Theta \times T_{-i}. \end{aligned}$$

Define $S_{-i}^\infty [t_{-i}, G, \varepsilon] \equiv \Pi_{j \neq i} S_j^\infty \left[t_j, G, \varepsilon\right]$. A measurable function $\sigma_{-i} : \Theta \times T_{-i} \to \Delta(A_{-i})$ is called a *conjecture*. We say σ_{-i} is an $\varepsilon-$*valid* conjecture in G if $\text{supp}\sigma_{-i} (\theta, t_{-i}) \subset S_{-i}^\infty [t_{-i}, G, \varepsilon]$ for all $(\theta, t_{-i}) \in \Theta \times T_{-i}$. Dekel, Fudenberg, and Morris (2006, 2007) show that $a_i \in S_i^\infty [t_i, G, \varepsilon]$ iff $a_i \in BR_i \left(\pi_{t_i,\sigma_{-i}}, G, \varepsilon\right)$ for some $\varepsilon-$valid conjecture σ_{-i}, and moreover, $S_i^\infty [t_i, G, \varepsilon] \neq \varnothing$ for any t_i and any $\varepsilon \geq 0$.

2.3 WY selection and robust selection

We define the WY selection and the robust selection as follows.

Definition 2 *An action a_i is WY-selected for $t_i \in T_i^*$ in G, if there exists a sequence of types $\{t_{i,m}\}$ such that $t_{i,m} \to t_i$ and $S_i^\infty [t_{i,m}, G, 0] = \{a_i\}$ for every m.*

Definition 3 *An action a_i is robustly selected for $t_i \in T_i^*$ in G, if there exist $\varepsilon > 0$ and a sequence of types $\{t_{i,m}\}$ such that $t_{i,m} \to t_i$ and $S_i^\infty [t_{i,m}, G, \varepsilon] = \{a_i\}$ for every m.*

We say a type t admits a robust selection (resp. WY-selection) if there is some action which can be robustly selected (resp. WY-selected) for t. Our first result shows that the robust selection is robust to measurement errors.

[9] See, for example, footnote 16 in Chen, Di Tillio, Faingold, and Xiong (2010) for an explanation why we can make $\text{supp}\sigma_{-i} (\theta, t_{-i}) \subset S_{-i}^\infty [t_{-i}]$ "for *all* (θ, t_{-i})" instead of "for $\kappa_{t_i}^*-$almost all (θ, t_{-i})."

Theorem 1 *If a_i is robustly selected for t_i in $G = (u_i)$, then there is some $\gamma > 0$ such that a_i can still be robustly selected for t_i in any $G' = (u'_i)$ such that*

$$\left|u'_i(\theta, a) - u_i(\theta, a)\right| \leq \gamma, \forall (i, \theta, a) \in N \times \Theta \times A.$$

Proof. Since a_i is robustly selected for t_i in G, there exists some $\varepsilon > 0$ and some sequence of types $\{t_{i,m}\}$ such that $t_{i,m} \to t_i$ and $\{a_i\} = S_i^\infty[t_{i,m}, \varepsilon]$ for every m. Hence, every $a'_i (\neq a_i)$ is not $\varepsilon-$rationalizable for $t_{i,m}$ in G. Let $\gamma = \varepsilon/4$. By (Dekel, Fudenberg, and Morris, 2006, Lemma 10), every $a'_i (\neq a_i)$ is not $\gamma-$rationalizable for $t_{i,m}$ in G'. Since $S_i^\infty[t_{i,m}, G', \varepsilon] \neq \varnothing$, it follows that $S_i^\infty[t_{i,m}, G', \gamma] = \{a_i\}$. Thus, a_i is robustly selected for t_i in G'.■

3 Generic impossibility of robust selection

Hereafter, we fixed a game $G = (u_i)_{i \in N}$ which satisfies the Richness assumption. Thus, we simplify the notation by writing $S_{-i}^\infty[t_{-i}, \varepsilon]$ for $S_{-i}^\infty[t_{-i}, G, \varepsilon]$. Furthermore, if $\varepsilon = 0$, we will write $S_i^\infty[t_i]$ for $S_i^\infty[t_i, 0]$. Similarly, all notations without an explicit reference to ε should be understood as having an implicit reference to $\varepsilon = 0$.

In this section, we demonstrate that the robust selection is generically impossible among types with multiple rationalizable actions. To achieve this, we first follow WY to partition the universal type space into two parts: the set of types with multiple rationalizable actions, denoted by M_i, and the set of types with unique rationalizable actions, denoted by $T_i^* \backslash M_i$.

$$M_i = \{t_i \in T_i^* : |S_i^\infty[t_i]| > 1\} \text{ and } T_i^* \backslash M_i = \{t_i \in T_i^* : |S_i^\infty[t_i]| = 1\}.$$

Our analyses focus on M_i because there is no need to refine the prediction for types outside M_i who have a unique rationalizable action.[10] Let $M_i^{rs} (\subset M_i)$ denote the set of types that admit robust selections. Let $M_i^{wy} (\subset M_i)$ denote the set of types that admit WY selections. WY prove that $M_i^{wy} = M_i$.

[10] By (Dekel, Fudenberg, and Morris, 2006, Lemma 1), if $S_i^\infty[t_i] = \{a_i\}$, there is some $\varepsilon > 0$ such that $S_i^\infty[t_i, \varepsilon] = \{a_i\}$. It follows that every type with a unique rationalizable action admits a (trivial) robust selection (of its only rationalizable action) by taking the sequence $\{t_{i,m}\}$ with $t_{im} = t_i$ for all m.

A *meager* set is a countable union of nowhere dense sets and the complement of a meager set is called a residual set. That is, a residual set is a countable intersection of open and dense sets. A set is said to be generic if it contains a residual set; a set is said to be non-generic if its complement is generic.[11] Theorem 2 says that types in M_i generically do not admit any robust selection, which is in sharp contrast to the fact that $M_i^{wy} = M_i$.

Theorem 2 *M_i^{rs} is a non-generic set in M_i.*

As $M_i^{wy} = M_i$, Theorem 2 implies that "almost all" WY selections for types in M_i are not robust and thus provides a sense in which the phenomenon exhibited in Example 2 prevails. To prove Theorem 2, we show that $M_i \backslash M_i^{rs}$ contains a residual set. Consider the following sets.

$$B_{i,\infty} \equiv \left\{t_i \in M_i : \kappa_{t_i}^* \left[\Theta \times \left(T_{-i}^* \backslash M_{-i}\right)\right] = 1\right\};$$

$$B_{i,n} \equiv \left\{t_i \in M_i : \kappa_{t_i}^* \left[\Theta \times \left(T_{-i}^* \backslash M_{-i}\right)\right] > 1 - \frac{1}{n}\right\}, \forall n \in \mathbb{N}.$$

Theorem 2 is then a direct consequence of the following three lemmas. The proofs can be found in Appendix A.1.

Lemma 1 *$B_{i,n}$ is open in M_i.*

Lemma 2 *$B_{i,\infty}$ is dense in M_i.*

Lemma 3 *$B_{i,\infty} \subset M_i \backslash M_i^{rs}$.*

Proof of Theorem 2. Clearly, $B_{i,\infty} \subset B_{i,n}$ and $\cap_{n=1}^{\infty} B_{i,n} = B_{i,\infty}$. Hence, by Lemma 1 and 2, $B_{i,n}$ is open and dense in M_i. Consequently, $B_{i,\infty}$ is a residual set in M_i. Therefore, by Lemma 3, M_i^{rs} is non-generic in M_i.■

Lemma 1 is a technical result. The intuition of Lemma 3 is similar to the idea of Example 2 in Section 1. Facing opponents with unique rationalizable actions, every type

[11] This is a standard notion of topological genericity and it is adopted by several recent papers, for example, Barelli (2009), Chen and Xiong (2011b), Dekel, Fudenberg, and Morris (2006), Ely and Pęski (2011).

in B_∞ has a unique conjecture about their opponents' rationalizable actions, and hence, its multiplicity of rationalizability is due solely to payoff ties, just like the player in Example 2. As a result, this type does not admit any robust selection.

To illustrate Lemma 2, we revisit G_1 in Example 1 with $\theta \in \{-2/5, 2/5, 6/5\}$.[12]

G_1 :

	Attack	No Attack
Attack	θ, θ	$\theta - 1, 0$
No Attack	$0, \theta - 1$	$0, 0$

Recall that we use t_j^θ to denote the complete-information type for which θ is common knowledge. First, observe that both actions are rationalizable for $t_i^{2/5}$. In their Example 3, WY recap the idea of Rubinstein (1989) to show that we can select either "Attack" or "No Attack" for $t_i^{2/5}$. The selection is in fact a robust one.

We show below that there exists a sequence of types $\{t_{i,k}\}$ such that $t_{i,k} \to t_i^{2/5}$ and $t_{i,k} \in B_{i,\infty}$ for every k. Consider a type $t_{i,1} \in B_{i,\infty}$ defined as follows.

$$\kappa^*_{t_{i,1}} \left[\left\{ \left(\theta = 2/5, t_{-i}^{-2/5} \right) \right\} \right] = 2/5;$$
$$\kappa^*_{t_{i,1}} \left[\left\{ \left(\theta = 2/5, t_{-i}^{6/5} \right) \right\} \right] = 3/5.$$

Note that the unique rationalizable action for $t_{-i}^{-2/5}$ is "No Attack", while the unique rationalizable action for $t_{-i}^{6/5}$ is "Attack." Thus, every rationalizable action of $t_{i,1}$ must be a best reply to the belief which assigns probability $2/5$ to $(\theta = 2/5, \text{No Attack})$ and probability $3/5$ to $(\theta = 2/5, \text{Attack})$. Given such a belief, both "Attack" and "No Attack" deliver the same payoff and they are both rationalizable for $t_{i,1}$. Therefore, $t_{i,1} \in B_{i,\infty}$ and $t_{i,1}$ shares the same first-order belief with $t_i^{2/5}$, i.e., they both believe that $\theta = 2/5$ occurs with probability 1.

Similar, we can construct $t_{i,k} \in B_{i,\infty}$ and $t_{i,k}$ approximates $t_i^{2/5}$ up to the kth-order belief. To see this, recall that $t_i^{2/5}$ assigns probability one to $\left(\theta = 2/5, t_{-i}^{2/5}\right)$. Since both "Attack" and "No Attack" are rationalizable for $t_{-i}^{2/5}$, we can apply WY's result to find types $t^A_{-i,k-1}$ and $t^{NA}_{-i,k-1}$ who have beliefs approximating $t_i^{2/5}$ up to order $k-1$ and "Attack" and "No Attack" are their unique rationalizable actions, respectively. Consider a type $t_{i,k} \in B_{i,\infty}$ defined as follows.

$$\kappa^*_{t_{i,k}} \left[\left\{ \left(\theta = 2/5, t^{NA}_{-i,k-1} \right) \right\} \right] = 2/5;$$

[12] This is WY's example 3 which is modified from Rubinstein (1989) and Carlsson and Van Damme (1993).

$$\kappa^*_{t_{i,k}}\left[\left\{\left(\theta = 2/5, t^A_{-i,k-1}\right)\right\}\right] = 3/5.$$

Then, again, every rationalizable action of $t_{i,k}$ must be a best reply to the belief which assigns probability $2/5$ to $(\theta = 2/5$, No Attack$)$ and probability $3/5$ to $(\theta = 2/5$, Attack$)$. Thus, both actions are rationalizable for $t_{i,k}$. To sum up, we find a sequence of types $\{t_{i,k}\}$ in $B_{i,\infty}$ such that $t_{i,k} \rightarrow t_i^{2/5}$.

Observe that the multiplicity of rationalizable actions of $t_i^{2/5}$ and $t_{i,k}$ occurs for different reasons: for $t_i^{2/5}$, there is a coordination problem because his opponent also has two rationalizable actions; for $t_{i,k}$, multiplicity occurs simply due to a payoff tie.

4 A full characterization of robust selection

In light of the negative result that "almost all" WY selections are not robust, we may wonder when we are able to obtain a robust selection. In this section, we provide a full characterization of the robust selection. We first restrict attention to complete-information scenarios, i.e., types under which payoffs are commonly known among players. Indeed, this is one of the most commonly imposed common-knowledge assumptions in economic models. Also, the idea of our characterization is most transparent in these scenarios.

In Section 4.1.1, we show first that every strictly rationalizable action can be robustly selected. In Section 4.1.2, we propose a notion called *$\varepsilon-$strict curb collection* which fully characterizes the robust selection. In Section 4.2, we extend these results to finite types, for which WY prove that every rationalizable action can be WY-selected.

4.1 Complete-information types

We first study complete-information types. Recall that a complete-information type is a type $\bar{t}_i \in T_i^*$ such that there exists some $\left(\bar{\theta}, \bar{t}_{-i}\right) \in \Theta \times T^*_{-i}$ with $\kappa^*_{\bar{t}_j}\left[\left(\bar{\theta}, \bar{t}_{-j}\right)\right] = 1$ for every $j \in N$. For the complete-information types $\left(\bar{t}_i, \bar{t}_{-i}\right)$, they are facing the complete-information game $\overline{G} = \left(\overline{u}_j\right)_{j \in N}$ with $\overline{u}_j(a) = u_j\left(\bar{\theta}, a\right)$ for every $a \in A$. Note that in this case, since $\kappa^*_{\bar{t}_j}\left[\left(\bar{\theta}, \bar{t}_{-j}\right)\right] = 1$ for every $j \in N$, any $\pi_{\bar{t}_j, \sigma_{-i}} \in \Delta\left(\left\{\bar{\theta}\right\} \times A_{-i}\right)$ can be identified by a belief $\pi \in \Delta\left(A_{-j}\right)$ and the meaning of notations, such as $BR_j(\pi, \varepsilon)$, $BR_j(\pi)$,

$BR_j^\circ(\pi,\varepsilon)$ and $BR_j^\circ(\pi)$, should be clear.

4.1.1 Strict rationalizability and robust selection

We define strict rationalizability and present the main result of this sub-section as follows.

Definition 4 *An action a_i is strictly rationalizable in $\overline{G}$ if there exists some $\left(R_j^\circ\right)_{j\in N} \subset \left(A_j\right)_{j\in N}$ such that*

i) $a_i \in R_i^\circ$ and

ii) for every $j \in N$ and every $a_j \in R_j^\circ$, there exists $\pi \in \Delta\left(R_{-j}^\circ\right)$ such that $a_j \in BR_j^\circ(\pi, 0)$.

Theorem 3 *An action a_i can be robustly selected for $\bar{t}_i$ if a_i is strictly rationalizable in $\overline{G}$.*

Similar to WY's result that every rationalizable action can be WY-selected, Theorem 3 says that every strictly rationalizable action can be robustly selected. In fact, the proof of Theorem 3 is similar to the proof of WY's Lemma 7: By induction, we show that there exists $\gamma > 0$ such that for any positive integer m, any player j and any strictly rationalizable action a_j in $\overline{G}$, we can find some $t_{j,m}$ such that $t_{j,m}$ and $\bar{t}_j$ have the same beliefs up to order m and $S_j^\infty\left[t_{j,m},\gamma\right] = \left\{a_j\right\}$. The initial step (i.e., $m = 1$) is implied by the Richness assumption. In fact, Theorem 3 holds for any finite types, which is a corollary of our full characterization below (see Theorem 6).

However, we may have an action such that it can be robustly selected, but it is not strictly rationalizable. This is illustrated in the following example.

Example 3. Consider the following two-player game.

G_3 :

$\theta = \theta_3$

	C	D	D′
A	1, 1	0, 0	0, 0
B	0, 0	1, 1	1, 1

$\theta = \theta_D$

	C	D	D′
A	0, 0	0, 1	0, 0
B	0, 0	0, 1	0, 0

When $\theta = \theta_3$ is commonly known, neither D nor D′ is strictly rationalizable for player 2. As a result, B is not strictly rationalizable for player 1. Nevertheless, B can be robustly selected for $t_1^{\theta_3}$. Define a sequence of types $\{t_m\}$ as follows, with t_m being player 1's (resp. player 2's) type if m is odd (resp. even).

$$t_0 \equiv t_2^{\theta_D} \text{ and } \kappa^*_{t_m}\left[(\theta_3, t_{m-1})\right] \equiv 1, \forall m \geq 1.$$

Clearly, $t_{2m-1} \to t_1^{\theta_3}$ and $t_{2m} \to t_2^{\theta_3}$. Moreover, $S_1^\infty\left[t_{2m-1}, 1/2\right] = \{\text{B}\}$ and $S_2^\infty\left[t_{2m}, 1/2\right] = \left\{\text{D,D}'\right\}$ for any positive integer m. Therefore, B can be robustly selected for $t_1^{\theta_3}$.

4.1.2 The full characterization

Recall that an action is $\varepsilon-$rationalizable for player i iff it is a $\varepsilon-$best reply to some $\varepsilon-$valid conjecture. In the complete-information scenario, a conjecture is simply a belief about A_{-i}. To fully characterize the robust selection, we consider a particular way to form conjectures. First, player i has a *theory*, denoted by η_{-i}, which is a distribution on *subsets* of A_{-i}, i.e., $\eta_{-i} \in \Delta\left(2^{A_{-i}} \backslash \{\varnothing\}\right)$. Second, a *conditional conjecture* $\varphi_{-i} : 2^{A_{-i}} \backslash \{\varnothing\} \to \Delta\left(A_{-i}\right)$ is a function mapping each subset $R_{-i} \subset A_{-i}$ to $\Delta\left(R_{-i}\right)$, i.e., $\varphi_{-i}\left(R_{-i}\right) \in \Delta\left(R_{-i}\right)$. That is, conditional on each $R_{-i} \subset A_{-i}$, player i further forms a belief on the distribution of actions in R_{-i}. Then, each pair $\left(\eta_{-i}, \varphi_{-i}\right)$ naturally induces a (composite) conjecture, denoted by $\sigma_{-i}^{\left(\eta_{-i}, \varphi_{-i}\right)} \in \Delta\left(A_{-i}\right)$, as follows.

$$\sigma_{-i}^{\left(\eta_{-i}, \varphi_{-i}\right)}\left[a_{-i}\right] \equiv \sum_{R_{-i} \in 2^{A_{-i}} \backslash \{\varnothing\}} \eta_{-i}\left[R_{-i}\right] \times \varphi_{-i}\left(R_{-i}\right)\left[a_{-i}\right], \forall a_{-i} \in A_{-i}.$$

Definition 5 *For any $\varepsilon > 0$, an $\varepsilon-$*strict curb collection *($\varepsilon-$SCC) in $\overline{G}$ is a profile of collections of actions $\left(\mathcal{R}_i\right)_{i \in N}$ (i.e., $\mathcal{R}_i \subset 2^{A_i} \backslash \{\varnothing\}$) such that for every $i \in N$ and every $R_i \in \mathcal{R}_i$, there exists a theory $\eta_{-i} \in \Delta\left(\mathcal{R}_{-i}\right)$ (where $\mathcal{R}_{-i} \equiv \left\{\left(R_j\right)_{j \neq i} : \forall j \neq i, R_j \in \mathcal{R}_j\right\}$) such that*

$$R_i \supset BR_i^\circ\left(\sigma_{-i}^{\left(\eta_{-i}, \varphi_{-i}\right)}, \varepsilon\right), \forall \textit{conditional conjecture } \varphi_{-i}. \tag{1}$$

The notion of $\varepsilon-$SCC is closely related to the notion of *curb set* proposed in Basu and Weibull (1991) for complete-information games.[13] We now state our full characterization of the robust selection.

[13] If $\left(B_i\right)_{i \in N}$ with $B_i \subset A_i$ is a curb set in $\overline{G}$ in the sense of Basu and Weibull (1991), then $\left\{\left\{\{a_i\} : a_i \in B_i\right\}\right\}_{i \in N}$ is a $\varepsilon-$SCC in $\overline{G}$ for some $\varepsilon > 0$.

Theorem 4 *An action a_i can be robustly selected for $\bar{t}_i$ iff $\{a_i\} \in \mathcal{R}_i$ for some $\varepsilon-$SCC $\left(\mathcal{R}_j\right)_{j\in N}$ in $\overline{G}$ with $\varepsilon > 0$.*

We use two propositions below to prove Theorem 4. The "if" and "only if" directions of Theorem 4 are direct consequences of Propositions 1 and 2, respectively.

Proposition 1 *If $\left(\mathcal{R}_j\right)_{j\in N}$ is an $\varepsilon-$SCC in $\overline{G}$ for some $\varepsilon > 0$, then for any $R_i \in \mathcal{R}_i$, there exist $\gamma > 0$ and a sequence of types $\{t_{i,m}\}$ such that $t_{i,m} \to \bar{t}_i$ and $S_i^\infty [t_{i,m}, \gamma] \subset R_i$ for every m.*

Proof. By the Richness assumption and finiteness of the action sets, there exists some $\gamma \in (0, \varepsilon)$ such that for each $j \in N$ and each $a_j \in A_j$, we can find some $\theta^{a_j} \in \Theta$ with $S_j^\infty \left[t_j^{\theta^{a_j}}, \gamma\right] = \{a_j\}$.

For notational ease, we assume that the statement – "types t_i and t_i' have the same belief up to order 0" – is always true for any two types t_i and t_i'. Then, Proposition 1 is an immediate consequence of the following claim.

Claim. *For each integer $m \geq 0$, each $j \in N$ and each $R_j \in \mathcal{R}_j$, we can find some finite type $t_{j,m,R_j} \in T_j^*$ such that (I) $S_j^\infty \left[t_{j,m,R_j}, \gamma\right] \subset R_j$; (II) t_{j,m,R_j} and $\bar{t}_j$ have the same beliefs up to order m.*

Proof the Claim. We prove the claim by induction. The claim for $m = 0$ holds by picking any $a_j \in R_j$ and $t_{j,0,R_j} = t_j^{\theta_{a_j}}$. We now assume the claim is true for m and prove below the case of $m + 1$.

By $R_j \in \mathcal{R}_j$ and the definition of $\varepsilon-$SCC, there exists a theory $\eta_{-i} \in \Delta(\mathcal{R}_{-i})$ such that

$$R_j \supset BR_j^\circ \left(\sigma_{-i}^{(\eta_{-i},\varphi_{-i})}, \varepsilon\right), \forall \text{conditional conjecture } \varphi_{-j}.$$

Since $\gamma \in (0, \varepsilon)$, we have

$$R_j \supset BR_j^\circ \left(\sigma_{-i}^{(\eta_{-i},\varphi_{-i})}, \varepsilon\right) \supset BR_j \left(\sigma_{-i}^{(\eta_{-i},\varphi_{-i})}, \gamma\right), \forall \text{conditional conjecture } \varphi_{-j}. \quad (2)$$

Recall that $\bar{t}_j = t_j^{\bar{\theta}}$. By the induction hypothesis, for each $j' \neq j$ and each $R_{j'} \in \mathcal{R}_{j'}$, there is some $t_{j',m,R_{j'}}$ satisfying (I_m) $S_{j'}^\infty \left[t_{j',m,R_{j'}}, \gamma\right] \subset R_{j'}$; ($\mathrm{II}_m$) $t_{j',m,R_{j'}}$ and $\bar{t}_{j'}$ have the same beliefs up to order m. Define $t_{-j,m,R_{-j}} \equiv \left(t_{j',m,R_{j'}}\right)_{j'\neq j}$. We then define $t_{j,m+1,R_j} \in T_j^*$ as the type which has the following belief (recall T_j^* is homeomorphic to $\Delta\left(\Theta \times T_{-j}^*\right)$).

$$\kappa^*_{t_{j,m+1,R_j}}\left[\left(\theta, t_{-j,m,R_{-j}}\right)\right] = \begin{cases} \eta_{-j}\left(R_{-j}\right), & \text{if } \theta = \overline{\theta}; \\ 0, & \text{if } \theta \neq \overline{\theta}, \end{cases}, \forall j \in N, \forall R_{-j} \in \mathcal{R}_{-j}. \tag{3}$$

By condition (II$_m$), condition (II) is satisfied for $t_{j,m+1,R_j}$.

We prove condition (I) for $t_{j,m+1,R_j}$ as follows. Pick any $a_j \in S_j^\infty\left[t_{j,m+1,R_j}, \gamma\right]$, and we show $a_j \in R_j$. By $a_j \in S_j^\infty\left[t_{j,m+1,R_j}, \gamma\right]$, we can find a $\gamma-$valid conjecture $\sigma_{-j} : \Theta \times T_{-j}^* \to \Delta\left(A_{-j}\right)$ such that

$$a_j \in BR_j\left(\pi_{t_{j,m+1,R_j},\sigma_{-j}}, \gamma\right). \tag{4}$$

Since σ_{-j} is $\gamma-$valid, by condition (I$_m$), we have

$$\sigma_{-j}\left(\overline{\theta}, t_{-j,m,R_{-j}}\right) \in \Delta\left(R_{-j}\right).$$

Define a conditional conjecture φ_{-j} as

$$\varphi_{-j}\left(R_{-j}\right) \equiv \sigma_{-j}\left(\overline{\theta}, t_{-j,m,R_{-j}}\right) \in \Delta\left(R_{-j}\right), \forall j \in N, \forall R_{-j} \in \mathcal{R}_{-j}. \tag{5}$$

We then have

$$\sigma_{-j}^{\left(\eta_{-j},\varphi_{-j}\right)}\left[a_{-j}\right] = \pi_{t_{j,m+1,R_j},\sigma_{-j}}\left[\left(\overline{\theta}, a_{-j}\right)\right] \text{ for any } a_{-j} \in A_{-j}, \tag{6}$$

which further implies

$$BR_j\left(\sigma_{-i}^{\left(\eta_{-i},\varphi_{-i}\right)}, \gamma\right) = BR_j\left(\pi_{t_{j,m+1,R_j},\sigma_{-j}}, \gamma\right). \tag{7}$$

Combining (4), (7) and (2), we have

$$a_j \in BR_j\left(\pi_{t_{j,m+1,R_j},\sigma_{-j}}, \gamma\right) = BR_j\left(\sigma_{-i}^{\left(\eta_{-i},\varphi_{-i}\right)}, \gamma\right) \subset BR_j^\circ\left(\sigma_{-i}^{\left(\eta_{-i},\varphi_{-i}\right)}, \varepsilon\right) \subset R_j,$$

which implies that condition (I) holds for $t_{j,m+1,R_j}$.

Finally, to see (6) is true, fix and $a_{-j} \in A_{-j}$, and we get

$$\begin{aligned} & \sigma_{-j}^{\left(\eta_{-j},\varphi_{-j}\right)}\left[a_{-j}\right] \\ = & \sum_{R_{-j}\in 2^{A_{-j}}\setminus\{\varnothing\}} \eta_{-j}\left[R_{-j}\right] \times \varphi_{-j}\left(R_{-j}\right)\left[a_{-j}\right] \\ = & \sum_{R_{-j}\in 2^{A_{-j}}\setminus\{\varnothing\}} \kappa^*_{t_{j,m+1,R_j}}\left[\left(\overline{\theta}, t_{-j,m,R_{-j}}\right)\right] \times \sigma_{-j}\left(\overline{\theta}, t_{-j,m,R_{-j}}\right)\left[a_{-j}\right] \\ = & \pi_{t_{j,m+1,R_j},\sigma_{-j}}\left[\left(\overline{\theta}, a_{-j}\right)\right], \end{aligned}$$

where the first equality follows from the definition of $\sigma_{-i}^{(\eta_{-i},\varphi_{-i})}$; the second equality follows from (3) and (5); the last equality follows from the definition of $\pi_{t_{j,m+1,R_j},\sigma_{-j}}$.■

Proposition 2 *For any $\varepsilon > 0$ and every $j \in N$, define*

$$\overline{\mathcal{R}}_j \equiv \left\{ R_j \subset A_j : \exists \text{ a sequence of types } \{t_{j,m}\} \text{ s.t. } t_{j,m} \to \bar{t}_j \text{ and } S_j^{\infty}\left[t_{j,m}, \varepsilon\right] = R_j \text{ for every } m \right\}.$$

Then, $\left(\overline{\mathcal{R}}_j\right)_{j\in N}$ is an $\varepsilon-$SCC in $\overline{G}$.

Proof. We show that $\left(\overline{\mathcal{R}}_j\right)_{j\in N}$ is an $\varepsilon-$SCC. Take any $R_j \in \overline{\mathcal{R}}_j$. By definition, there is some sequence of types $\{t_{j,m}\}$ such that $t_{j,m} \to \bar{t}_j$ and $S_j^{\infty}\left[t_{j,m}, \varepsilon\right] = R_j$ for all m. We will show that

$$\exists \text{ a theory } \eta_{-j} \text{ s.t. } R_j \supset BR_j^{\circ}\left(\sigma_{-j}^{(\eta_{-j},\varphi_{-j})}, \varepsilon\right), \forall \text{conditional conjecture } \varphi_{-j}. \qquad (\bigstar)$$

For any $r > 0$ and any $j' \in N$, let $\mathcal{B}\left(t_{j'}, r\right)$ denote the open ball around $t_{j'}$ with radius r. We then show ($\bigstar$) holds in three steps.

Step 1. *There exists $\zeta > 0$ such that $S_{j'}^{\infty}\left[t_{j'}, \varepsilon\right] \in \overline{\mathcal{R}}_{j'}$ for every $j' \in N$ and every $t_{j'} \in \mathcal{B}\left(\bar{t}_{j'}, \zeta\right)$.*

Suppose that step 1 does not hold. Then, for each m, there exist some $j' \in N$ and some $t_{j',m} \in \mathcal{B}\left(\bar{t}_{j'}, \frac{1}{m}\right)$ such that $S_{j'}^{\infty}\left[t_{j',m}, \varepsilon\right] \notin \overline{\mathcal{R}}_{j'}$. By finiteness of players and actions, there exist some $j'' \in N$, some $R_{j''} \subset A_{j''}$ and a subsequence of $\left\{t_{j'',m_k}\right\}$ such that $S_{j''}^{\infty}\left[t_{j'',m_k}, \varepsilon\right] = R_{j''} \notin \overline{\mathcal{R}}_{j''}$ for all k. Since $t_{j'',m_k} \in \mathcal{B}\left(\bar{t}_{j''}, \frac{1}{m_k}\right)$ for all k, we have $t_{j'',m_k} \to \bar{t}_{j''}$, which, together with $S_{j''}^{\infty}\left[t_{j'',m_k}, \varepsilon\right] = R_{j''}$ for all k, implies $R_{j''} \in \overline{\mathcal{R}}_{j''}$ by the definition of $\overline{\mathcal{R}}_{j''}$. This contradicts $R_{j''} \notin \overline{\mathcal{R}}_{j''}$.

Step 2. *Construction of the theory η_{-j}.*

Partition T_{-j}^{*} as follows.

$$T_{-j}^{R_{-j}} = \left\{ t_{-j} \in T_{-j}^{*} : S_{-j}^{\infty}\left[t_{-j}, \varepsilon\right] = R_{-j} \right\}, \forall R_{-j} \in 2^{A_{-j}} \backslash \{\varnothing\}. \qquad (8)$$

For each $t_{j,m}$, the belief $\kappa_{t_{j,m}}^{*}$ induces a distribution on $2^{A_{-j}}$, i.e.,

$$\eta_{-j,m}\left[R_{-j}\right] \equiv \kappa_{t_{j,m}}^{*}\left[\Theta \times T_{-j}^{R_{-j}}\right], \forall R_{-j} \in 2^{A_{-j}} \backslash \{\varnothing\}. \qquad (9)$$

By finiteness of A_{-j}, $\left\{\eta_{-j,m}\right\}$ has a convergent subsequence, say itself. Then, define the theory η_{-j} as follows.

$$\eta_{-j}\left[R_{-j}\right] = \lim_{m\to\infty} \eta_{-j,m}\left[R_{-j}\right], \forall R_{-j} \in 2^{A_{-j}} \backslash \{\varnothing\}. \tag{10}$$

Note that

$$\eta_{-j}\left[R_{-j}\right] = 0, \forall R_{-j} \notin \overline{\mathcal{R}}_{-j}. \tag{11}$$

By finiteness of actions, (11) implies $\eta_{-j} \in \Delta\left(\overline{\mathcal{R}}_{-j}\right)$, i.e., η_{-j} is a valid theory.

To see (11) is true, fix any $R_{-j} \notin \overline{\mathcal{R}}_{-j}$. By step 1, for any $t_{-j} \in \Pi_{j'\neq j}\mathcal{B}\left(\bar{t}_{j'},\zeta\right)$, $S^{\infty}_{-j}\left[t_{-j},\varepsilon\right] \neq R_{-j}$. Hence,

$$\left(\Pi_{j'\neq j}\mathcal{B}\left(\bar{t}_{j'},\zeta\right)\right) \bigcap T^{R_{-j}}_{-j} = \varnothing. \tag{12}$$

Furthermore,

$$\begin{aligned} \lim_{m\to\infty} \eta_{-j,m}\left[\Theta \times \Pi_{j'\neq j}\mathcal{B}\left(\bar{t}_{j'},\zeta\right)\right] &= \lim_{m\to\infty} \kappa^*_{t_{j,m}}\left[\Theta \times \Pi_{j'\neq j}\mathcal{B}\left(\bar{t}_{j'},\zeta\right)\right] \\ &\geq \kappa^*_{\bar{t}_j}\left[\Theta \times \Pi_{j'\neq j}\mathcal{B}\left(\bar{t}_{j'},\zeta\right)\right] \\ &= 1, \end{aligned} \tag{13}$$

where the first equality follows from (9); the inequality follows from the facts that $\kappa^*_{t_{j,m}} \to \kappa^*_{\bar{t}_j}$ and $\left[\Theta \times \Pi_{j'\neq j}\mathcal{B}\left(\bar{t}_{j'},\zeta\right)\right]$ is open (see Lemma 4); the last equality follows the facts that $\left(\bar{\theta},\bar{t}_{-j}\right) \in \Theta \times \Pi_{j'\neq j}\mathcal{B}\left(\bar{t}_{j'},\zeta\right)$ and $\kappa^*_{\bar{t}_j}\left[\left(\bar{\theta},\bar{t}_{-j}\right)\right] = 1$. Finally, (12) and (13) imply (11).

Moreover, the following facts will be useful for our proof. Since $t_{j,m} \to \bar{t}_j$ and $\kappa^*_{\bar{t}_j}\left[\Theta\backslash\left\{\bar{\theta}\right\} \times T^*_{-j}\right] = 0$, we have

$$\lim_{m\to\infty} \kappa^*_{t_{j,m}}\left[\left(\Theta\backslash\left\{\bar{\theta}\right\}\right) \times T^*_{-j}\right] = 0. \tag{14}$$

(10) and (14) imply

$$\lim_{m\to\infty} \eta_{-j,m}\left[R_{-j}\right] = \lim_{m\to\infty} \kappa^*_{t_{j,m}}\left[\Theta \times T^{R_{-j}}_{-j}\right] = \lim_{m\to\infty} \kappa^*_{t_{j,m}}\left[\left\{\bar{\theta}\right\} \times T^{R_{-j}}_{-j}\right]. \tag{15}$$

Step 3. *$a_j \in R_j$ if $a_j \in BR^{\circ}_j\left(\sigma^{\left(\eta_{-j},\varphi_{-j}\right)}_{-j},\varepsilon\right)$ for some conditional conjecture φ_{-j}, i.e., (★) holds.*

Fix any $a_j \in BR_j^\circ \left(\sigma_{-j}^{\left(\eta_{-j}, \varphi_{-j}\right)}, \varepsilon \right)$ for some conditional conjecture φ_{-j}, and we show $a_j \in R_j$. By finiteness of actions, we have

$$a_j \in BR_j^\circ \left(\sigma_{-j}^{\left(\eta_{-j}, \varphi_{-j}\right)}, \varepsilon \right) \subset BR_j \left(\sigma_{-j}^{\left(\eta_{-j}, \varphi_{-j}\right)}, \varepsilon' \right) \text{ for some } \varepsilon' < \varepsilon. \tag{16}$$

Define an $\varepsilon-$valid conjecture $\sigma_{-j} : \Theta \times T_{-j}^* \to \Delta\left(A_{-j}\right)$ as follow.

$$\sigma_{-j}\left(\theta, t_{-j}\right) = \varphi_{-j}\left(R_{-j}\right) \text{ iff } \left(\theta, t_{-j}\right) \in T_{-j}^{R_{-j}}. \tag{17}$$

By the definition of $T_{-j}^{R_{-j}}$, σ_{-j} is a $\varepsilon-$valid conjecture. We thus have

$$\lim_{m \to \infty} \pi_{t_{j,m}, \sigma_{-j}} \left[\left(\overline{\theta}, a_{-j}\right)\right] = \sigma_{-j}^{\left(\eta_{-j}, \varphi_{-j}\right)} \left[a_{-j}\right], \forall a_{-j} \in A_{-j}; \tag{18}$$

$$\lim_{m \to \infty} \pi_{t_{j,m}, \sigma_{-j}} \left[\left(\theta, a_{-j}\right)\right] = 0, \forall \theta \neq \overline{\theta}, \forall a_{-j} \in A_{-j}. \tag{19}$$

(19) is implied by (14), and (18) will be proved later. Then, for any $a_j' \in A_j'$, we have

$$\begin{aligned}
& \lim_{m \to \infty} \sum_{\left(\theta, a_{-j}\right) \in \Theta \times A_{-j}} \pi_{t_{j,m}, \sigma_{-j}} \left[\left(\theta, a_{-j}\right)\right] \times \left(u_j\left(\theta, a_j, a_{-j}\right) - u_j\left(\theta, a_j', a_{-j}\right) \right) & (20) \\
= & \lim_{m \to \infty} \sum_{a_{-j} \in A_{-j}} \pi_{t_{j,m}, \sigma_{-j}} \left[\left(\overline{\theta}, a_{-j}\right)\right] \times \left(u_j\left(\overline{\theta}, a_j, a_{-j}\right) - u_j\left(\overline{\theta}, a_j', a_{-j}\right) \right) \\
= & \sum_{a_{-j} \in A_{-j}} \sigma_{-j}^{\left(\eta_{-j}, \varphi_{-j}\right)} \left[a_{-j}\right] \times \left(u_j\left(\overline{\theta}, a_j, a_{-j}\right) - u_j\left(\overline{\theta}, a_j', a_{-j}\right) \right) \\
\geq & -\varepsilon' \\
> & -\varepsilon, & (21)
\end{aligned}$$

where the first equality follows from (19); the second equality follows from (18); the first inequality follows from (16); the last inequality follows from $\varepsilon' < \varepsilon$.

By (20), we have the following inequality for sufficiently large m.

$$\sum_{\left(\theta, a_{-j}\right) \in \Theta \times A_{-j}} \pi_{t_{j,m}, \sigma_{-j}} \left[\left(\theta, a_{-j}\right)\right] \times \left(u_j\left(\theta, a_j, a_{-j}\right) - \overline{u}_j\left(\theta, a_j', a_{-j}\right) \right) \geq -\varepsilon, \forall a_j' \in A_j',$$

i.e., $a_j \in BR_j\left(\pi_{t_{j,m}, \sigma_{-j}}, \varepsilon\right)$. Finally, recall that $S_j^\infty\left[t_{j,m}, \varepsilon\right] = R_j$ for all m, and we conclude that $a_j \in R_j$.

The last step is to show (18). For each $a_{-j} \in A_j$, we have

$$\begin{aligned}
\lim_{m\to\infty} \pi_{t_{j,m},\sigma_{-j}}\left[\left(\bar{\theta}, a_{-j}\right)\right] &= \lim_{m\to\infty} \int_{T_{-j}} \sigma_{-j}\left(\bar{\theta}, t'_{-j}\right)\left[a_{-j}\right] \kappa_{t_{j,m}}\left[\left(\bar{\theta}, dt'_{-j}\right)\right] \\
&= \lim_{m\to\infty} \sum_{R_{-j}\in 2^{A_{-j}}\backslash\{\varnothing\}} \int_{T_{-j}^{R_{-j}}} \sigma_{-j}\left(\bar{\theta}, t'_{-j}\right)\left[a_{-j}\right] \kappa_{t_{j,m}}\left[\left(\bar{\theta}, dt'_{-j}\right)\right] \\
&= \lim_{m\to\infty} \sum_{R_{-j}\in 2^{A_{-j}}\backslash\{\varnothing\}} \kappa^*_{t_{j,m}}\left[T_{-j}^{R_{-j}}\right] \times \varphi_{-j}\left(R_{-j}\right)\left[a_{-j}\right] \\
&= \sum_{R_{-j}\in 2^{A_{-j}}\backslash\{\varnothing\}} \eta_{-j}\left[R_{-j}\right] \times \varphi_{-j}\left(R_{-j}\right)\left[a_{-j}\right] \\
&= \sigma_{-j}^{\left(\eta_{-j},\varphi_{-j}\right)}\left[a_{-j}\right]
\end{aligned}$$

where the first equality follows from the definition of $\pi_{t_{j,m},\sigma_{-j}}$; the second equality follows from the fact that $\left\{T_{-j}^{R_{-j}} : R_{-j} \in 2^{A_{-j}}\backslash\{\varnothing\}\right\}$ partitions T_{-j}; the third equality follows from (17); the fourth equality follows from (9) and (10); the last equality follows from the definition of $\sigma_{-j}^{\left(\eta_{-j},\varphi_{-j}\right)}$.■

4.2 Finite types

We now extend the full characterization of the robust selection to finite types. Here, given a finite model (T, κ), a theory for player i is a (measurable) function $\eta_{-i} : \Theta \times T_{-i} \to \Delta\left(2^{A_{-i}}\backslash\{\varnothing\}\right)$; a conditional conjecture $\varphi_{-i} : \Theta \times T_{-i} \times \left[2^{A_{-i}}\backslash\{\varnothing\}\right] \to \Delta\left(A_{-i}\right)$ is a (measurable) function mapping each $\left(\theta, t_{-i}, R_{-i}\right)$ to $\Delta\left(R_{-i}\right)$, i.e., $\varphi_{-i}\left(\theta, t_{-i}, R_{-i}\right) \in \Delta\left(R_{-i}\right)$. Then, each pair $\left(\eta_{-i}, \varphi_{-i}\right)$ defines a (composite) conjecture, denoted by $\sigma_{-i}^{\left(\eta_{-i},\varphi_{-i}\right)} : \Theta \times T_{-i} \to \Delta\left(A_{-i}\right)$ such that

$$\sigma_{-i}^{\left(\eta_{-i},\varphi_{-i}\right)}\left(\theta, t_{-i}\right)\left[a_{-i}\right] \equiv \sum_{R_{-i}\in 2^{A_{-i}}\backslash\{\varnothing\}} \eta_{-i}\left(\theta, t_{-i}\right)\left[R_{-i}\right] \times \varphi_{-i}\left(\theta, t_{-i}, R_{-i}\right)\left[a_{-i}\right], \forall a_{-i} \in A_{-i}.$$

Definition 6 *Given a finite model (T, κ) and $\varepsilon > 0$, an $\varepsilon-$SCC in (T, κ) is a profile of correspondences*[14] $\left(\mathcal{R}_i : T_i \rightrightarrows 2^{A_i}\backslash\{\varnothing\}\right)_{i\in N}$ *such that for every $i \in I$, every $t_i \in T_i$, and every $R_i \in$*

[14]We use "$\rightrightarrows$" to denote correspondence, i.e., $f : X \rightrightarrows Y$ means $f(x) \subset 2^Y\backslash\{\varnothing\}$ for every $x \in X$.

$\mathcal{R}_i(t_i)$, *there exists a theory* $\eta_{-i} : \Theta \times T_{-i} \to \Delta\left(2^{A_{-i}} \backslash \{\varnothing\}\right)$ *with* $\eta_{-i}(\theta, t_{-i}) \in \Delta(\mathcal{R}_{-i}(t_{-i}))$ *for all* $(\theta, t_{-i}) \in \Theta \times T_{-i}$ *(where* $\mathcal{R}_{-i}(t_{-i}) \equiv \left\{(R_j)_{j \neq i} : \forall j \neq i, R_j \in \mathcal{R}_j(t_j)\right\}$*), such that*

$$R_i \supset BR_i^{\circ}\left(\pi_{t_i, \sigma_{-i}^{(\eta_{-i}, \varphi_{-i})}}, \varepsilon\right), \forall \text{conditional conjecture } \varphi_{-i}.$$

Definition 6 corresponds to Definition 5 for complete-information types. Similarly, Theorem 5 and Propositions 3 and 4 below correspond to Theorem 4 and Propositions 1 and 2 for complete-information types, respectively.

Theorem 5 *Given a finite model* (T, κ) *and* $t_i \in T_i$*, an action* a_i *can be robustly selected for* t_i *iff* $\{a_i\} \in \mathcal{R}_i(t_i)$ *for some* $\varepsilon-$*SCC* $(\mathcal{R}_j)_{j \in N}$ *in* (T, κ) *with* $\varepsilon > 0$.

The "if" and "only if" directions of Theorem 5 are immediate consequences of Propositions 3 and 4, respectively.

Proposition 3 *Given a finite model* (T, κ) *and* $\varepsilon > 0$*, if* $(\mathcal{R}_j)_{j \in N}$ *is an* $\varepsilon-$*SCC in* (T, κ) *and* $R_i \in \mathcal{R}_i(t_i)$ *for some* $t_i \in T_i$*, then there exist* $\gamma > 0$ *and a sequence of types* $\{t_{i,m}\}$ *such that* $t_{i,m} \to t_i$ *and* $S_i^{\infty}[t_{i,m}, \gamma] \subset R_i$ *for every* m.

Proposition 4 *Given a finite model* (T, κ) *and* $\varepsilon > 0$*, for every* $j \in N$ *and every* $t_j \in T_j$*, define*

$$\mathcal{R}_j^T(t_j) \equiv \left\{R_j \subset A_j : \exists \text{ a sequence of types } \{t_{j,m}\} \text{ s.t. } t_{j,m} \to t_j \text{ and } S_j^{\infty}[t_{j,m}, \varepsilon] = R_j, \forall m\right\}.$$

Then, $\left(\mathcal{R}_j^T\right)_{j \in N}$ *is an* $\varepsilon-$*SCC.*

We relegate the proofs of Propositions 3 and 4 to Appendices A.2 and A.3 as they are very similar to the proofs of Propositions 1 and 2. For any finite type t_i, the basic idea is to apply the arguments for the complete-information type to each of the finitely-many types on the support of t_i. In fact, the arguments in Propositions 3 and 4 go through as long as each type in the model (T, κ) assigns positive probability to finitely-many types. There are infinite types that satisfy this property, e.g., the email game sequence types constructed in (Dekel, Fudenberg, and Morris, 2006, Figure 1 of Example 1).[15]

We now generalize Theorem 3 to finite types as a corollary of Theorem 5.

[15] The characerization for general infinite types remains an open question.

Definition 7 *An action a_i is strictly rationalizable for a type t_i in a model (T, κ) if there exists $\left(R_j^{\circ}\right)_{j \in N}$ with $R_j^{\circ} : T_j \to 2^{A_j} \backslash \{\varnothing\}$ such that*

i) $a_i \in R_i^{\circ}(t_i)$ and

ii) for each $j \in N$, each $t_j \in T_j$, and each $a_j \in R_j^{\circ}(t_j)$, there exists some conjecture σ_{-j} such that $a_j \in BR_j^{\circ}\left(\pi_{t_j, \sigma_{-j}}\right)$ and $\text{supp}\sigma_{-j}(\theta, t_{-j}) \subset R_{-j}^{\circ}(t_{-j})$ for every (θ, t_{-j}).

Theorem 6 *An action a_i can be robustly selected for t_i in a finite model (T, κ) if a_i is strictly rationalizable for t_i in (T, κ).*

Proof. Since a_i is strictly rationalizable for t_i in (T, κ), by ii) in Definition 7, for each $j \in N$, each $t_j \in T_j$, and each $a_j \in R_j^{\circ}(t_j)$, there exists some conjecture σ_{-j, a_j, t_j} such that $a_j \in BR_j^{\circ}\left(\pi_{t_j, \sigma_{-j, a_j, t_j}}\right)$ and $\text{supp}\sigma_{-j, a_j, t_j}(\theta, t_{-j}) \subset R_{-j}^{\circ}(t_{-j})$ for every (θ, t_{-j}). Since both T and A are finite, there exists $\varepsilon > 0$ such that for any $j \in I$, any $t_j \in T_j$ and any $a_j \in R_j^{\circ}(t_j)$, we have

$$\sum_{(\theta, a_{-j}) \in A_{-j}} \pi_{t_j, \sigma_{-j, a_j, t_j}}\left[(\theta, a_{-j})\right]\left(u_j(\theta, a_j, a_{-j}) - u_j\left(\theta, a_j', a_{-j}\right)\right) > \varepsilon, \forall a_j' \in A_j \backslash \{a_j\}.$$

This implies

$$BR_j^{\circ}\left(\pi_{t_j, \sigma_{-j, a_j, t_j}}, \varepsilon\right) = \{a_j\}. \tag{22}$$

Define $\mathcal{R}_j(t_j) = \left\{\{a_j\} : a_i \in R_j^{\circ}(t_j)\right\}$. We verify that $(\mathcal{R}_j)_{j \in N}$ is an $\varepsilon-$SCC. For any $\{a_j\}$, define a theory $\eta_{-j} : \Theta \times T_{-j} \to \Delta\left(2^{A_{-j}} \backslash \{\varnothing\}\right)$ as follows.

$$\eta_{-j}(\theta, t_{-j})\left[\{a_{-j}\}\right] = \sigma_{-j, a_j, t_j}(\theta, t_{-j})\left[a_{-j}\right], \forall a_{-j} \in R_{-j}^{\circ}(t_{-j}).$$

Pick any conditional conjecture φ_{-j}. Since each $R_j \in \mathcal{R}_j(t_j)$ is a singleton, i.e. $R_j = \{a_j\}$, then $\varphi_{-j}(R_j)$ must be the Dirac measure on a_j. As a result, we have

$$\sigma_{-j}^{(\eta_{-j}, \varphi_{-j})} \equiv \sigma_{-j, a_j, t_j}. \tag{23}$$

Thus,

$$BR_j^{\circ}\left(\pi_{t_j, \sigma_{-j}^{(\eta_{-j}, \varphi_{-j})}}, \varepsilon\right) = BR_j^{\circ}\left(\pi_{t_j, \sigma_{-j, a_j, t_j}}, \varepsilon\right) = \{a_j\} \in \mathcal{R}_j(t_j),$$

where the first equality follows from (23) and the second equality follows from (22). Therefore, $(\mathcal{R}_j)_{j\in N}$ is an $\varepsilon-$SCC and a_i can be robustly selected for t_i by Theorem 5.■

5 Discussion

5.1 Robust selection and robust predictions

WY show that every rationalizable action can be WY-selected. Conversely, by the upper hemicontinuity of the rationalizable correspondence (see (Dekel, Fudenberg, and Morris, 2006, Theorem 2)), every action that is WY-selected must also be rationalizable. That is, an action is WY-selected if and only if it is rationalizable. Our results show that the robust selection is strictly stronger than rationalizability and strictly weaker than strict rationalizability.

Though the robust selection refines rationalizability, some types may still have multiple actions which can be robustly selected. For example, the complete-information type in the global game has two strict equilibrium actions, both of which can be robustly selected. Thus, with one perturbation of higher-order beliefs, we can robustly select one action, and with another perturbation of higher-order beliefs, we can robustly select the other action.

With a lack of the exact specification of best replies or the precise measurement of payoffs, the idea of the robust selection is that any actions which cannot be robustly selected are fragile predictions. They are predictions inferior to those actions that can be robustly selected regarding the two robustness features.

Among those actions that can be robustly selected, which one should we pick? Or equivalently, what perturbation of higher-order beliefs is the most sensible approximation of the strategic situation that we are analyzing? The answer to this question is subject to the judgement of the modeler. For example, if we think that the global game (i.e., Example 1 in Section 1) is a good approximation of the strategic situation that we are facing, we predict that player i who observes $x_i = \frac{3}{4}$ will choose "Attack," whereas if we regard the alternative perturbation studied in Morris and Shin (2003) is more sensible, we predict

"No Attack." In the case that we do not have a good way to tell one perturbation is more appealing than the other, we may follow WY to say that a prediction is robust only when it holds under both of the actions.

5.2 Robust selection, strategic discontinuity, and critical types

The notion of robust selection is closely related to the notions of strategic discontinuity and critical types studied in Ely and Pęski (2011) and Chen and Xiong (2011a). In these papers, a type t_i is said to display *strategic discontinuity* in a game G if there exist $\varepsilon > 0$, an action a_i, and a sequence of types $\{t_{i,m}\}$ with $t_{i,m} \rightarrow t_i$ such that $a_i \in S_i^{\infty}[t_i, G, 0]$ and $a_i \notin S_i^{\infty}[t_{i,m}, G, \varepsilon]$ for every m. For a set of games $\mathcal{G}$, a type is $\mathcal{G}$-critical iff it displays strategic discontinuity in some game in $\mathcal{G}$. A type is $\mathcal{G}$-regular iff it is not $\mathcal{G}$-critical.

Consider a type t_i who has multiple rationalizable actions. If t_i admits a robust selection in game G, t_i must display strategic discontinuity in G.[16] Indeed, the complete-information types in Rubinstein (1989) and Carlsson and Van Damme (1993) are prominent examples of strategic discontinuity. In contrast, a type that admits a WY selection in a game G may not display strategic discontinuity in G, for example, the type t_{∞} in Example 2 and every type in the set $B_{i,\infty}$ defined in Section 3. In this sense, the robust selection generalizes the global-game selection regarding strategic discontinuity, but WY selection does not.

Let $\mathcal{G}^f$ denote the set of all finite action game. Ely and Pęski (2011) fully characterize $\mathcal{G}^f$-critical types. For a fixed game G that satisfies the Richness assumption, we fully characterize finite G-critical types using $\varepsilon-$SCC as follows.

Proposition 5 *Given a finite model (T, κ), a type $t_i \in T_i$ is G-critical iff there exist some $\varepsilon-$SCC $(\mathcal{R}_j)_{j \in N}$ with $\varepsilon > 0$, and some $R_i \in \mathcal{R}_i(t_i)$ such that $S_i^{\infty}[t_i] \setminus R_i \neq \varnothing$.*

[16] Suppose that b_i is robustly selected for t_i. That is, there exist $\varepsilon > 0$ and a sequence of types $\{t_{i,m}\}$ such that $t_{i,m} \rightarrow t_i$ and $S_i^{\infty}[t_{i,m}, G, \varepsilon] = \{b_i\}$ for every m. Since $S_i^{\infty}[t_i, G, 0]$ contains at least two actions, there is some $a_i \in S_i^{\infty}[t_i, G, 0]$ and $a_i \notin S_i^{\infty}[t_{i,m}, G, \varepsilon]$ for every m. That is, t_i displays strategic discontinuity in G.

5.3 Robust selection and common$-p$ belief

Let $C^p(\cdot)$ be the common$-p$ belief operator defined in Ely and Pęski (2011). That is, for any measurable set $E \subset \times_{i\in N} T_i^*$, $C^p(E)$ occurs iff E occurs; every player assigns at least probability p that E occurs; every player assigns at least probability p that every player assigns at least probability p that E occurs, and so on *ad infinitum*.

Recall that $\mathcal{G}^f$ is the set of all finite action games. Ely and Pęski (2011) prove that a type is $\mathcal{G}^f$-critical types iff it has common$-p$ belief on a closed proper subset of $\times_{i\in N} T_i^*$. The following proposition shows that the set of finite G-critical types is a fixed point of the common$-p$ belief operator.

Proposition 6 *For any $i \in N$, let $T_i^{f,c}$ $\left(\subset T_i^*\right)$ denote the set of all finite G-critical types. Then, $T^{f,c} = \cup_{p>0} C^p\left(T^{f,c}\right)$, where $T^{f,c} = \times_{i\in N} T_i^{f,c}$.*

Therefore, Proposition 5 and 6 build a connection between the common$-p$ belief and $\varepsilon-$SCC, hence also the robust selection.

5.4 Ex ante selection versus interim selection

Both the WY selection and the robust selection are interim notions. Namely, an action is selected for a given type according to the interim (higher-order) beliefs of the type. An alternative approach is to study selections from an ex ante viewpoint. In this case, a type is replaced by a type space, which is usually associated with a common prior. Jackson, Rodriguez-Barraquer, and Tan (2011) independently show that for any ex ante Bayesian Nash equilibrium and any sequence of (ex ante) perturbations of the game, there is a corresponding sequence of ex ante $\varepsilon-$equilibria converging to that equilibrium.[17] The authors conclude from the result that no refinement of equilibrium is robust to slight perturbations to best replies or underlying preferences.

Clearly, both Jackson, Rodriguez-Barraquer, and Tan (2011) and this paper employ solution concepts based on $\varepsilon-$best replies to approximate solution concepts that are based

[17] Aside from this result, Jackson, Rodriguez-Barraquer, and Tan (2011) also show that under a weaker notion of perturbation, the approximating equilibria can be made $\varepsilon-$interim equiliria.

on $0-$best replies. However, Jackson, Rodriguez-Barraquer, and Tan (2011) study ex ante equilibrium, while we study interim rationalizability. Because of the difference, Jackson, Rodriguez-Barraquer, and Tan (2011) can approximate *any* equilibrium with $\varepsilon-$equilibria along any sequence, while we only obtain a generic set of types for which approximation with $\varepsilon-$rationalizable actions along arbitrary sequence is available. Technically, the difference between the two papers is that we require *all* types play interim $\varepsilon-$best replies, while in Jackson, Rodriguez-Barraquer, and Tan (2011), a set of types may play actions different from interim $\varepsilon-$best replies. As long as the set has a small probability, players are playing ex ante $\varepsilon-$best replies.

The ex ante approach is also adopted by Kajii and Morris (1997). They define *the robust equilibrium* as a Nash equilibrium in a complete-information game such that it is played with a sufficiently high probability in some Bayesian Nash equilibrium on any common priors which are sufficiently close to the complete-information scenario.[18] The solution concept in Kajii and Morris (1997) is based on $0-$best replies.

6 Conclusion

We propose the notion of the robust selection that not only strictly refines rationalizability, but also shares the two robustness features with the global-game selection. We show that the robust selection is generically impossible among types with multiple rationalizable actions, because multiplicity is generically due to a payoff tie, as in a single-person decision problem. Furthermore, we provide a full characterization of the robust selection. The characterization also sheds light on the strategic impact of higher-order beliefs, as studied in the recent papers by Dekel, Fudenberg, and Morris (2006), Ely and Pęski (2011), and Chen, Di Tillio, Faingold, and Xiong (2010, 2011).

[18]This approach has been extended by Oyama and Tercieux (2010) to allow for non-common prior perturbations.

A Appendix

A.1 Proofs of Lemma 1, 2 and 3

The proofs of Lemmas 1 and 3 need the following result.

Lemma 4 *For any i, if $t_{i,m} \to t_i$, then*

$$\liminf_{m\to\infty} \kappa^*_{t_{i,m}}(G) \geq \kappa^*_{t_i}(G) \text{ for any open set } G \subset \Theta \times T^*_{-i}; \tag{24}$$

$$\limsup_{m\to\infty} \kappa^*_{t_{i,m}}(F) \leq \kappa^*_{t_i}(F) \text{ for any closed set } F \subset \Theta \times T^*_{-i}. \tag{25}$$

Proof. Recall that $\kappa^*_{t_i} = \kappa^*_i(t_i)$ and κ^*_i is the homeomorphism between T^*_i and $\Delta\left(\Theta \times T^*_{-i}\right)$, where T^*_i is endowed with the product topology and $\Delta\left(\Theta \times T^*_{-i}\right)$ is endowed with the weak*-topology. Since $t_{i,m} \to t_i$, we have $\kappa^*_{t_{i,m}} \to \kappa^*_{t_i}$ in weak* topology. Then, (24) and (25) follow from the definition of weak*-topology (see (Dudley, 2002, 11.1.1. Theorem)).■

A.1.1 Proof of Lemma 1

Lemma 1. *$B_{i,n}$ is open in M_i.*

Proof. Recall

$$B_{i,n} \equiv \left\{t_i \in M_i : \kappa^*_{t_i}\left[\Theta \times \left(T^*_{-i} \backslash M_{-i}\right)\right] > 1 - \frac{1}{n}\right\}.$$

By WY's Proposition 2, $T^*_{-i} \backslash M_{-i}$ is open. Suppose $B_{i,n}$ is not open in M_i. That is, there is some $t_i \in B_{i,n}$ and some sequence $\{t_{i,m}\}$ with $t_{i,m} \to t_i$ such that $t_{i,m} \in M_i$ and $\kappa^*_{t_{i,m}}\left[\Theta \times \left(T^*_{-i} \backslash M_{-i}\right)\right] \leq 1 - \frac{1}{n}$ for all m. Hence,

$$\liminf_{m\to\infty} \kappa^*_{t_{i,m}}\left[\Theta \times \left(T^*_{-i} \backslash M_{-i}\right)\right] \leq 1 - \frac{1}{n} < \kappa^*_{t_i}\left[\Theta \times \left(T^*_{-i} \backslash M_{-i}\right)\right]. \tag{26}$$

where the last inequality follows because $t_i \in B_{i,n}$. Then, (26) contradicts to (24) in Lemma 4.■

A.1.2 Proof of Lemma 2

Lemma 2. *$B_{i,\infty}$ is dense in M_i.*

Proof. Recall that T_i^* is a compact metric space and d_i is the metric on T_i^*. We now divide the proof into three steps.

Step 1 *For any finite type $t_i \in M_i$ and*

$$T_{-i} = \left\{ t_{-i} \in T^*_{-i} : \kappa^*_{t_i} \left(\{ (\theta, t_{-i}) \} \right) > 0 \text{ for some } \theta \in \Theta \right\},$$

there is some conjecture $\sigma_{-i} : \Theta \times T_{-i} \to \Delta(A_{-i})$ which is valid for t_i and $BR_i\left(\pi_{t_i,\sigma_{-i}}\right)$ has more than one action.

T_{-i} is a finite set because t_i is a finite type. Define

$$\begin{aligned} \Sigma_{-i} &\equiv \Pi_{(\theta,t_{-i})\in\Theta\times T_{-i}} \Delta\left(S^\infty_{-i}[t_{-i}]\right); \\ P_{a_i} &\equiv \left\{ \sigma_{-i} \in \mathbf{\Sigma}_{-i} : \{a_i\} = BR_i\left(\pi_{t_i,\sigma_{-i}}\right) \right\}, \forall a_i \in A_i. \end{aligned}$$

Σ_{-i} is the set of all valid conjectures of t_i. Observe that Σ is convex, and hence also a connected set in the Euclidean space $\mathbb{R}^{|\Theta\times T_{-i}|}$ (recall T_{-i} is a finite set). P_{a_i} is the set of valid conjectures of t_i to which a_i is the unique best reply.

Since $t_i \in M_i$, there are at least two distinct actions a_i' and a_i'' in $S_i^\infty[t_i]$. Thus, there are $\sigma'_{-i}, \sigma''_{-i} \in \Sigma_{-i}$ such that $a_i' \in BR_i\left(\pi_{t_i,\sigma'_{-i}}\right)$ and $a_i'' \in BR_i\left(\pi_{t_i,\sigma''_{-i}}\right)$. Step 1 holds if either $BR_i\left(\pi_{t_i,\sigma'_{-i}}\right)$ or $BR_i\left(\pi_{t_i,\sigma''_{-i}}\right)$ has more than one action. Now suppose that $\left|BR_i\left(\pi_{t_i,\sigma'_{-i}}\right)\right| = \left|BR_i\left(\pi_{t_i,\sigma''_{-i}}\right)\right| = 1$.

Note that for every $a_i \in A_i$, P_{a_i} is an (Euclidean-)open set in Σ and $P_{a_i} \cap P_{b_i} = \varnothing$ if $a_i \neq b_i$. Moreover, $P_{a_i'} \neq \varnothing$ and $P_{a_i''} \neq \varnothing$ because $\sigma'_{-i} \in P_{a_i'}$ and $\sigma''_{-i} \in P_{a_i''}$. Since $\mathbf{\Sigma}$ is connected, we have $\cup_{a_i\in A_i} P_{a_i} \subsetneq \mathbf{\Sigma}$.[19] Thus, there is some $\sigma_{-i} \in \mathbf{\Sigma}_{-i}$ such that $\sigma_{-i} \notin P_{a_i}$ for all $a_i \in A_i$. Since $BR_i\left(\pi_{t_i,\sigma_{-i}}\right) \neq \varnothing$, $BR_i\left(\pi_{t_i,\sigma_{-i}}\right)$ has more than one action. Furthermore, σ_{-i} is valid because $\sigma_{-i}(\theta, t_{-i}) \in \Delta\left(S^\infty_{-i}[t_{-i}]\right)$ for every $(\theta, t_{-i}) \in \Theta \times T_{-i}$.

Step 2 *For any finite type $t_i \in M_i$, there is a sequence of finite types $\{t_{i,m}\}$ such that $t_{i,m} \to t_i$ and $t_{i,m} \in B_{i,\infty}$ for all m.*

[19] $\mathbf{\Sigma}$ is connected, but $\cup_{a_i\in A_i} P_{a_i}$ is not connected. Thus, $\cup_{a_i\in A_i} P_{a_i} \neq \mathbf{\Sigma}$.

Since t_i is a finite type, the set T_{-i} defined below is a finite set.

$$T_{-i} = \left\{t_{-i} \in T^*_{-i} : \kappa^*_{t_i}\left(\{(\theta, t_{-i})\}\right) > 0 \text{ for some } \theta \in \Theta\right\}.$$

By step 1, there is some valid conjecture $\sigma_{-i} : \Theta \times T_{-i} \to \Delta(A_{-i})$ for t_i such that $BR_i\left(\pi_{t_i,\sigma_{-i}}\right)$ has at least two actions, i.e., $\left|BR_i\left(\pi_{t_i,\sigma_{-i}}\right)\right| > 1$. Define

$$\eta = \min\left\{d_{-i}\left(t_{-i}, s_{-i}\right) : t_{-i}, s_{-i} \in T_{-i} \text{ and } t_{-i} \neq s_{-i}\right\}.$$

Thus, $\eta > 0$ because T_{-i} is a finite set.

Fix any $m \geq 1$. By Proposition 1 in Weinstein and Yildiz (2007), for each $t_{-i} \in T_{-i}$ and each $a_{-i} \in S^\infty_{-i}[t_{-i}]$, there is some type $\widetilde{t}_{-i}(t_{-i}, a_{-i}) \in T^*_{-i}$ such that

$$S^\infty_{-i}\left[\widetilde{t}_{-i}(a_{-i}, t_{-i})\right] = \{a_{-i}\} \text{ and} \tag{27}$$

$$d_{-i}\left(t_{-i}, \widetilde{t}_{-i}(t_{-i}, a_{-i})\right) < \min\{\eta, 1/m\}. \tag{28}$$

Consider $\kappa^*_{t_{i,m}} \in \Delta\left(\Theta \times T^*_{-i}\right)$ defined as follows.

$$\kappa^*_{t_{i,m}}\left[\left(\theta, \widetilde{t}_{-i}(t_{-i}, a_{-i})\right)\right] = \kappa^*_{t_i}\left[(\theta, t_{-i})\right] \times \sigma_{-i}(\theta, t_{-i})[a_{-i}], \forall \theta \in \Theta, \forall t_{-i} \in T_{-i}, \forall a_{-i} \in S^\infty_{-i}[t_{-i}]. \tag{29}$$

Since σ_{-i} is valid, we have $\sigma_{-i}(\theta, t_{-i}) \in \Delta\left(S^\infty_{-i}[t_{-i}]\right)$ for every $(\theta, t_{-i}) \in \Theta \times T_{-i}$. Consequently, $\kappa^*_{t_{i,m}}$ is well-defined in (29). Then, $\kappa^*_{t_{i,m}} \in \Delta\left(\Theta \times T^*_i\right)$ uniquely determines a finite type $t_{i,m}$ in T^*_i because T^*_i is homeomorphic to $\Delta\left(\Theta \times T^*_{-i}\right)$. Clearly, $\kappa^*_{t_{i,m}}\left(T^*_{-i} \backslash M_{-i}\right) = 1$ by (27). Then, because (28) holds for all $t_{-i} \in T_{-i}$, $\kappa^*_{t_{i,m}} \to \kappa^*_{t_i}$ and hence $t_{i,m} \to t_i$ as $m \to \infty$.

We now show that $t_{i,m}$ has multiple rationalizable actions. By our construction and (27), any valid conjecture $\sigma_{-i,m} : \Theta \times T^*_{-i} \to \Delta(A_{-i})$ for $t_{i,m}$ must satisfy

$$\sigma_{-i,m}\left(\theta, \widetilde{t}_{-i}(t_{-i}, a_{-i})\right)[a_{-i}] = 1. \tag{30}$$

As a result, $\pi_{t_{i,m},\sigma_{-i,m}} \equiv \pi_{t_i,\sigma_{-i}}$. Therefore, $\left|BR_i\left(\pi_{t_{i,m},\sigma_{-i,m}}\right)\right| = \left|BR_i\left(\pi_{t_i,\sigma_{-i}}\right)\right| > 1$. To see $\pi_{t_{i,m},\sigma_{-i,m}} \equiv \pi_{t_i,\sigma_{-i}}$, observe first that

$$\pi_{t_{i,m},\sigma_{-i,m}}\left[(\theta, a_{-i})\right] = \pi_{t_i,\sigma_{-i}}\left[(\theta, a_{-i})\right] = 0, \forall a_{-i} \notin \cup_{t_{-i} \in T_{-i}} S^\infty_{-i}[t_{-i}].$$

Second, for any $\theta \in \Theta$, any $a_{-i} \in \cup_{t_{-i} \in T_{-i}} S^\infty_{-i}[t_{-i}]$, we have

$$
\begin{aligned}
&\pi_{t_{i,m},\sigma_{-i,m}}\left[(\theta,a_{-i})\right] && (31)\\
=\ & \sum_{\left\{t_{-i}\in T_{-i}:a_{-i}\in S^{\infty}_{-i}[t_{-i}]\right\}} \kappa^{*}_{t_{i,m}}\left[\left(\theta,\widetilde{t}_{-i}\left(t_{-i},a_{-i}\right)\right)\right]\times\sigma_{-i,m}\left(\theta,\widetilde{t}_{-i}\left(t_{-i},a_{-i}\right)\right)\left[a_{-i}\right]\\
=\ & \sum_{\left\{t_{-i}\in T_{-i}:a_{-i}\in S^{\infty}_{-i}[t_{-i}]\right\}} \kappa^{*}_{t_{i,m}}\left[\left(\theta,\widetilde{t}_{-i}\left(t_{-i},a_{-i}\right)\right)\right]\\
=\ & \sum_{\left\{t_{-i}\in T_{-i}:a_{-i}\in S^{\infty}_{-i}[t_{-i}]\right\}} \kappa^{*}_{t_{i}}\left[\left(\theta,t_{-i}\right)\right]\times\sigma_{-i}\left(\theta,t_{-i}\right)\left[a_{-i}\right]\\
=\ & \pi_{t_i,\sigma_{-i}}\left[(\theta,a_{-i})\right],
\end{aligned}
$$

where the first equality follows from the definition of $\pi_{t_{i,m},\sigma_{-i,m}}$; the second equality follows from (30); the third equality follows from (29); the last equality follows from the definition of $\pi_{t_i,\sigma_{-i}}$.

To sum up, we find a sequence of finite types $\{t_{i,m}\}$ such that $t_{i,m}\to t_i$ and $t_{i,m}\in M_i$, $\kappa^{*}_{t_{i,m}}\left(T^{*}_{-i}\backslash M_{-i}\right)=1$ for all m, i.e., $t_{i,m}\in B_{i,\infty}$ for all m.

Step 3 *$B_{i,\infty}$ is dense in M_i.*

Take any $t_i\in M_i$ and any $\varepsilon>0$. First, by Lemma 3 in Chen (2011), there is some finite type $t'_i\in T^{*}_i$ such that $S^{\infty}_i[t_i]=S^{\infty}_i\left[t'_i\right]$ and $d_i\left(t_i,t'_i\right)<\varepsilon/2$. Then, for any $\varepsilon>0$, by step 2, there is some $t''_i\in B_{i,\infty}$ such that $d_i\left(t'_i,t''_i\right)<\varepsilon/2$. Hence, $d_i\left(t_i,t''_i\right)<\varepsilon$. Therefore, $B_{i,\infty}$ is dense in M_i.■

A.1.3 Proof of Lemma 3

Lemma 3. $B_{i,\infty}\subset M_i\backslash M^{rs}_i$.

Proof. Pick any $t_i\in B_{i,\infty}$ and we will show $t_i\in M_i\backslash M^{rs}_i$. Since $B_{i,\infty}\subset M_i$ by the definition of $B_{i,\infty}$, it remains to prove $t_i\notin M^{rs}_i$. To see this, fix any $\varepsilon>0$, any sequence of types $\{t_{i,m}\}$ with $t_{i,m}\to t_i$ and any $a_i\in S^{\infty}_i[t_i]$. We will prove that $a_i\in S^{\infty}_i[t_{i,m},\varepsilon]$ for all sufficiently large m. This implies $t_i\notin M^{rs}_i$.

For $a_{-i}\in A_{-i}$, define

$$
U^{a_{-i}}_{-i}:=\left\{t_{-i}\in T^{*}_{-i}:S^{\infty}_{-i}\left[t_{-i}\right]=\left\{a_{-i}\right\}\right\}.
$$

Moreover, for every $\theta \in \Theta$, $\{\theta\} \times U_{-i}^{a_{-i}}$ is open by Proposition 2 of WY. Observe that $\{\Theta \times M_{-i}\} \cup \left\{\{\theta\} \times U_{-i}^{a_{-i}}\right\}_{\theta \in \Theta, a_{-i} \in A_{-i}}$ is a partition of $\Theta \times T_{-i}^*$. The proof of Lemma 3 will use the following claim. The proof of Claim 1 will be presented later.

Claim 1

$$\lim_{m \to \infty} \kappa_{t_{i,m}}^* \left[\Theta \times M_i\right] = \kappa_{t_i}^* \left[\Theta \times M_i\right] = 0. \tag{32}$$

$$\lim_{m \to \infty} \kappa_{t_{i,m}}^* \left[\{\theta\} \times U_{-i}^{a_{-i}}\right] = \kappa_{t_i}^* \left[\{\theta\} \times U_{-i}^{a_{-i}}\right]. \tag{33}$$

Recall that $a_i \in S_i^\infty [t_i]$ implies that there is some $\sigma_{-i} : \Theta \times T_{-i}^* \to \Delta(A_{-i})$ such that

$$\text{supp}\sigma_{-i}(\theta, t_{-i}) \subset S_{-i}^\infty [t_{-i}], \forall (\theta, t_{-i}); \tag{34}$$

$$\int_{\Theta \times T_{-i}^*} \sum_{a_{-i} \in A_{-i}} \left[u_i(\theta, a_i, a_{-i}) - u_i\left(\theta, a_i', a_{-i}\right)\right] \sigma_{-i}(\theta, t_{-i})[a_{-i}] \kappa_{t_i}^* [(\theta, t_{-i})] \geq 0. \tag{35}$$

Since $t_i \in B_{i,\infty}$, we have $\kappa_{t_i}^* [\Theta \times M_i] = 0$. Furthermore, (34) implies that $\sigma_{-i}(\theta, t_{-i})[a_{-i}] = 1$ for every $t_{-i} \in U_{-i}^{a_{-i}}$. Recall $\{\Theta \times M_{-i}\} \cup \left\{\{\theta\} \times U_{-i}^{a_{-i}}\right\}_{\theta \in \Theta, a_{-i} \in A_{-i}}$ is a partition of $\Theta \times T_{-i}^*$. We thus have

$$\begin{aligned} &\int_{\Theta \times T_{-i}^*} \sum_{a_{-i} \in A_{-i}} \left[u_i(\theta, a_i, a_{-i}) - u_i\left(\theta, a_i', a_{-i}\right)\right] \sigma_{-i}(\theta, t_{-i})[a_{-i}] \kappa_{t_i}^* [(\theta, t_{-i})] \\ = &\sum_{(\theta, a_{-i}) \in \Theta \times A_{-i}} \left[u_i(\theta, a_i, a_{-i}) - u_i\left(\theta, a_i', a_{-i}\right)\right] \kappa_{t_i}^* \left[\{\theta\} \times U_{-i}^{a_{-i}}\right]. \end{aligned} \tag{36}$$

Let $\Psi = \max_{i,a,\theta} |u_i(\theta, a)|$. The incentive constraints for $t_{i,m}$ under σ_{-i} is

$$\begin{aligned} &\int_{\Theta \times T_{-i}^*} \sum_{a_{-i} \in A_{-i}} \left[u_i(\theta, a_i, a_{-i}) - u_i\left(\theta, a_i', a_{-i}\right)\right] \sigma_{-i}(\theta, t_{-i})[a_{-i}] \kappa_{t_{i,m}}^* [(\theta, t_{-i})] \\ \geq &\sum_{(\theta, a_{-i}) \in \Theta \times A_{-i}} \left[u_i(\theta, a_i, a_{-i}) - u_i\left(\theta, a_i', a_{-i}\right)\right] \kappa_{t_{i,m}}^* \left[\{\theta\} \times U_{-i}^{a_{-i}}\right] - 2\Psi \kappa_{t_{i,m}}^* [\Theta \times M_i]. \end{aligned}$$

Since $\kappa_{t_{i,m}}^* [\Theta \times M_i] \to 0$ as $m \to \infty$ by (32) in Claim 1, it follows that

$$\begin{aligned} &\lim_{m \to \infty} \int_{\Theta \times T_{-i}^*} \sum_{a_{-i} \in A_{-i}} \left[u_i(\theta, a_i, a_{-i}) - u_i\left(\theta, a_i', a_{-i}\right)\right] \sigma_{-i}(\theta, t_{-i})[a_{-i}] \kappa_{t_{i,m}}^* [(\theta, t_{-i})] \\ \geq &\lim_{m \to \infty} \sum_{(\theta, a_{-i}) \in \Theta \times A_{-i}} \left[u_i(\theta, a_i, a_{-i}) - u_i\left(\theta, a_i', a_{-i}\right)\right] \kappa_{t_{i,m}}^* \left[\{\theta\} \times U_{-i}^{a_{-i}}\right] \\ = &\sum_{(\theta, a_{-i}) \in \Theta \times A_{-i}} \left[u_i(\theta, a_i, a_{-i}) - u_i\left(\theta, a_i', a_{-i}\right)\right] \kappa_{t_i}^* \left[\{\theta\} \times U_{-i}^{a_{-i}}\right] \geq 0 \end{aligned} \tag{37}$$

where the equality follows from (33) in Claim 1 and the last inequality follows from (35) and (36). By (37), for sufficiently large m, $a_i \in BR_i\left(\pi_{t_{i,m}, \sigma_{-i}}, \varepsilon\right)$. By (34), σ_{-i} is valid and hence $\varepsilon-$valid for $t_{i,m}$. Therefore, $a_i \in S_i^\infty [t_{i,m}, \varepsilon]$ for sufficiently large m.■

Proof of Claim 1 First, we prove (32). By the definition of $B_{i,\infty}$, $t_i \in B_{i,\infty}$ implies

$$\kappa^*_{t_i}[\Theta \times M_i] = 0. \tag{38}$$

By Proposition 2 in WY, $\Theta \times M_i$ is closed in $\Theta \times T^*_{-i}$. Hence,

$$0 \leq \liminf_{m\to\infty} \kappa^*_{t_{i,m}}[\Theta \times M_i] \leq \limsup_{m\to\infty} \kappa^*_{t_{i,m}}[\Theta \times M_i] \leq \kappa^*_{t_i}[\Theta \times M_i] = 0$$

where the last inequality follows from (25) in Lemma 4. Hence, (32) holds.

Second, we prove (33). Since $\{\theta\} \times U^{a_{-i}}_{-i}$ is open in $\Theta \times T^*_{-i}$, by (24) in Lemma 4,

$$\liminf_{m\to\infty} \kappa^*_{t_{i,m}}\left[\{\theta\} \times U^{a_{-i}}_{-i}\right] \geq \kappa^*_{t_i}\left[\{\theta\} \times U^{a_{-i}}_{-i}\right]. \tag{39}$$

Moreover, $(\Theta \times M_i) \cup \left(\{\theta\} \times U^{a_{-i}}_{-i}\right)$ is closed, because $\{\theta'\} \times U^{a'_{-i}}_{-i}$ is open for any $\left(\theta', a'_{-i}\right) \in \Theta \times A_{-i}$ and

$$(\Theta \times M_i) \cup \left(\{\theta\} \times U^{a_{-i}}_{-i}\right) = \left(\Theta \times T^*_{-i}\right) \setminus \left[\bigcup_{\left(\theta', a'_{-i}\right) \neq (\theta, a_{-i})} \left(\{\theta'\} \times U^{a'_{-i}}_{-i}\right)\right].$$

Then,

$$\begin{aligned} \limsup_{m\to\infty} \kappa^*_{t_{i,m}}\left[\{\theta\} \times U^{a_{-i}}_{-i}\right] &\leq \limsup_{m\to\infty} \kappa^*_{t_{i,m}}\left[(\Theta \times M_i) \cup \left(\{\theta\} \times U^{a_{-i}}_{-i}\right)\right] \\ &\leq \kappa^*_{t_i}\left[(\Theta \times M_i) \cup \left(\{\theta\} \times U^{a_{-i}}_{-i}\right)\right] \\ &= \kappa^*_{t_i}[\Theta \times M_i] + \kappa^*_{t_i}\left[\{\theta\} \times U^{a_{-i}}_{-i}\right] \\ &= \kappa^*_{t_i}\left[\{\theta\} \times U^{a_{-i}}_{-i}\right], \end{aligned} \tag{40}$$

where the second inequality follows from (25) in Lemma 4 and the fact that $(\Theta \times M_i) \cup \left(\{\theta\} \times U^{a_{-i}}_{-i}\right)$ is closed; the last equality follows from (38). Finally, (39) and (40) imply (33).■

A.2 Proof of Proposition 3

Proposition 3. *Given a finite model (T, κ) and $\varepsilon > 0$, if $\left(\mathcal{R}_j\right)_{j\in N}$ is an $\varepsilon-$SCC in (T, κ) and $R_i \in R_i(t_i)$ for some $t_i \in T_i$, then there exist $\gamma > 0$ and a sequence of types $\{t_{i,m}\}$ such that $t_{i,m} \to t_i$ and $S^\infty_i[t_{i,m}, \gamma] \subset R_i$ for every m.*

Proof. Suppose that $(\mathcal{R}_j)_{j\in N}$ is an $\varepsilon-$SCC in (T,κ). First, by the Richness assumption and the finiteness of the action sets, there is some $\gamma \in (0,\varepsilon)$ such that $S_j^\infty\left[t_j^{\theta_{a_j}},\gamma\right] = \{a_j\}$. Proposition 3 is implied by Claim 2 below.

Claim 2 *For any player j, any $m \geq 0$, any $t_j \in T_j$, and $R_j \in \mathcal{R}_j(t_j)$, there is a finite type $\widetilde{t}_j \in T_j^*$ satisfying (A) $S_j^\infty\left[\widetilde{t}_j,\gamma\right] \subset R_j$; (B) $\widetilde{t}_j$ and t_j have the same belief up to order m.*

Proof of Claim 2. We prove the claim by induction. The claim for $m=0$ holds by picking some $a_j \in R_j$ and $\widetilde{t}_j = t_j^{\theta_{a_j}}$. We now assume the claim is true for m and prove the case for $m+1$. Since $R_j \in \mathcal{R}_j(t_j)$, there exists a theory η_{-j} which maps each $(\theta,t_{-j}) \in \Theta\times T_{-i}$ to some probability distribution over $\mathcal{R}_{-j}(t_{-j})$ such that

$$R_j \supset BR_j^\circ\left(\pi_{t_j,\sigma_{-j}^{(\varphi_{-j},\eta_{-j})}}\right), \forall\text{conditional conjecture } \varphi_{-j}.$$

Since t_j is a finite type, suppt_j is a finite set. By the induction hypothesis, for each $t_{-j} = \left(t_{j'}\right)_{j'\neq j} \in$supp$t_j$ and each $R_{j'} \in \mathcal{R}_{j'}\left(t_{j'}\right)$, there is some finite type $\widetilde{t}_{j'}\left(t_{j'},R_{j'}\right) \in T_{j'}^*$ such that $\widetilde{t}_{j'}^l\left(t_{j'},R_{j'}\right) = t_{j'}^l$ for all $l \leq m$ and $S_{j'}^\infty\left[\widetilde{t}_{j'}\left(t_{j'},R_{j'}\right),\gamma\right] \subset R_{j'}$. Let $\widetilde{t}_{-j}(t_{-j},R_{-j}) = \left(\widetilde{t}_{j'}\left(t_{j'},R_{j'}\right)\right)_{j'\neq j}$ and

$$T'_{-j} = \left\{\widetilde{t}_{-j}(t_{-j},R_{-j}) \in T_{-j}^* : R_{-j} \in \mathcal{R}_{-j}(t_{-j}), t_{-j} \in \text{supp}t_j\right\}.$$

Consider $\widetilde{\mu} \in \Delta\left(\Theta\times T_{-j}^*\right)$ such that for any $(\theta,t_{-j}) \in \Theta\times T_{-j}^*$,

$$\widetilde{\mu}\left[(\theta,t_{-j})\right] = \begin{cases} \eta_{-j}(\theta,t_{-j})\left[R_{-j}\right]\kappa_{t_j}\left[(\theta,t_{-j})\right], & \text{if } t_{-j} = \widetilde{t}_{-j}(t_{-j},R_{-j}); \\ 0, & \text{otherwise.}\end{cases} \tag{41}$$

Since T_j^* is homeomorphic to $\Delta\left(\Theta\times T_{-j}^*\right)$, there is some $\widetilde{t}_j$ such that $\kappa_{\widetilde{t}_j}^* = \widetilde{\mu}$. We show that $\widetilde{t}_j$ is the desired type.

Since every $\widetilde{t}_{-j}(t_{-j},R_{-j})$ is a finite type, $\widetilde{t}_j$ is a finite type. Moreover, since $\widetilde{t}_{j'}^l\left(t_{j'},R_{j'}\right) = t_{j'}^l$ for all $l \leq m$, $\widetilde{t}_j^l = t_j^l$ for all $l \leq m+1$. Finally, we verify that $S_j^\infty\left[\widetilde{t}_j,\gamma\right] \subset R_j$. Let $a_j \in S_j^\infty\left[\widetilde{t}_j,\gamma\right]$. Then, $a_j \in BR_j\left(\pi_{\widetilde{t}_j,\sigma'_{-j}},\gamma\right)$ for some conjecture $\sigma'_{-j} : \Theta\times T_{-j}^* \to$

$\Delta\left(A_{-j}\right)$ which is $\gamma-$valid for $\widetilde{t}_j$. For each $\left(\theta, t_{-j}\right) \in \Theta \times T_{-j}$ and $R_{-j} \in \mathcal{R}_{-j}\left(t_{-j}\right)$, define $\varphi_{-j}\left(\theta, t_{-j}, R_{-j}\right) \in \Delta\left(A_{-j}\right)$ by

$$\varphi_{-j}\left(\theta, t_{-j}, R_{-j}\right) = \sigma'_{-j}\left(\theta, \widetilde{t}_{-j}\left(t_{-j}, R_{-j}\right)\right), \forall\left(\theta, \widetilde{t}_{-j}\left(t_{-j}, R_{-j}\right)\right) \in \Theta \times T'_{-j}. \tag{42}$$

Observe that φ_{-j} is indeed a conditional conjecture since

$$\operatorname{supp}\varphi_{-j}\left(\theta, t_{-j}, R_{-j}\right) = \operatorname{supp}\sigma'_{-j}\left(\theta, \widetilde{t}_{-j}\left(t_{-j}, R_{-j}\right)\right) \subset S^{\infty}_{-j}\left[\widetilde{t}_{-j}\left(t_{-j}, R_{-j}\right), \gamma\right] \subset R_{-j}$$

where the first inclusion is because σ'_{-j} is a $\gamma-$valid conjecture and the second inclusion follows from the induction hypothesis.

We then obtain

$$\begin{aligned}
&\pi_{t_j, \sigma_{-j}^{\left(\varphi_{-j}, \eta_{-j}\right)}}\left[\left(\theta, a_{-j}\right)\right] \\
= &\sum_{t_{-j} \in T_{-j}}\left[\sum_{R_{-j} \in \mathcal{R}_{-j}\left(t_{-j}\right)} \varphi_{-j}\left(\theta, t_{-j}, R_{-j}\right)\left[a_{-j}\right] \eta_{-j}\left(\theta, t_{-j}\right)\left[R_{-j}\right]\right] \kappa_{t_j}\left[\left(\theta, t_{-j}\right)\right] \\
= &\sum_{\widetilde{t}_{-j}\left(t_{-j}, R_{-j}\right) \in T'_{-j}} \varphi_{-j}\left(\theta, t_{-j}, R_{-j}\right)\left[a_{-j}\right] \kappa^*_{\widetilde{t}_j}\left[\left(\theta, \widetilde{t}_{-j}\left(t_{-j}, R_{-j}\right)\right)\right] \\
= &\sum_{\widetilde{t}_{-j}\left(t_{-j}, R_{-j}\right) \in T'_{-j}} \sigma'_{-j}\left(\theta, \widetilde{t}_{-j}\left(t_{-j}, R_{-j}\right)\right)\left[a_{-j}\right] \kappa^*_{\widetilde{t}_j}\left[\left(\theta, \widetilde{t}_{-j}\left(t_{-j}, R_{-j}\right)\right)\right] \\
= &\pi_{\widetilde{t}_j, \sigma'_{-j}}\left[\left(\theta, a_{-j}\right)\right]
\end{aligned} \tag{43}$$

where the second equality follows from (41) and the third equality follows from (42). Since $a_j \in BR_j\left(\pi_{\widetilde{t}_j, \sigma'_{-j}}, \gamma\right)$, it follows from (43) that $a_j \in BR_j\left(\pi_{t_j, \sigma_{-j}^{\left(\varphi_{-j}, \eta_{-j}\right)}}, \varepsilon\right)$. Since $\gamma < \varepsilon$, $a_j \in BR^{\circ}_j\left(\pi_{t_j, \sigma_{-j}^{\left(\varphi_{-j}, \eta_{-j}\right)}}, \varepsilon\right)$ and thus $a_j \in R_j$.■

A.3 Proof of Proposition 4

Proposition 4. *Given a finite model (T, κ) and $\varepsilon > 0$, for every $j \in N$ and every $t_j \in T_j$, define*

$$\mathcal{R}^T_j\left(t_j\right) \equiv \left\{R_j \subset A_j : \exists \text{ a sequence of types } \left\{t_{j,m}\right\} \text{ s.t. } t_{j,m} \to t_j \text{ and } S^{\infty}_j\left[t_{j,m}, \varepsilon\right] = R_j, \forall m\right\}.$$

Then, $\left(\mathcal{R}^T_j\right)_{j \in N}$ is an $\varepsilon-$SCC.

Proof. Let $t_j \in T_j$. Since t_j is in a finite type space, suppt_j is a finite set. Consider $R_j \in \mathcal{R}_j^T(t_j)$. We will show that

$$\exists \text{ a theory } \eta_{-j} \text{ s.t. } R_j \supset BR_j^{\circ}\left(\pi_{t_j, \sigma_{-j}^{(\varphi_{-j}, \eta_{-j})}}, \varepsilon\right), \forall \text{conditional conjecture } \varphi_{-j}. \qquad (\bigstar)$$

Since $R_j \in \mathcal{R}_j^T(t_j)$, there is a sequence of types $\{t_{j,m}\}$ with $t_{j,m} \to t_j$ such that $S_j^{\infty}[t_{j,m}, \varepsilon] = R_j$ for all m. Let $\lambda_m = d_j(t_{j,m}, t_j)$. Note that $\lambda_m \to 0$ since $t_{j,m} \to t_j$. We prove ($\bigstar$) in three steps.

Step 1. *There is some $\beta > 0$ such that*

(1) $\{(\theta, t_{-j})\}^{\beta} \cap \left\{\left(\theta', t'_{-j}\right)\right\}^{\beta} =$ *for any* (θ, t_{-j}) *and* $\left(\theta', t'_{-j}\right)$ *in* supp$\kappa_{t_j}^*$;

(2) *for each* $\left(\theta, \left(t_{j'}\right)_{j' \neq j}\right) \in$supp$\kappa_{t_j}^*$, $S_{j'}^{\infty}\left[t'_{j'}, \varepsilon\right] \in \mathcal{R}_{j'}\left(t_{j'}\right)$ *for all* $j' \neq j$ *and for all* $\left(\theta', t'_{-j}\right) \in \{(\theta, t_{-j})\}^{\beta}$.

We can find $\beta > 0$ such that (1) holds because supp$\kappa_{t_j}^*$ is a finite set. To see (2), suppose to the contrary that for every m, there is some $\left(\theta', t'_{-j,m}\right) = \left(\theta, \left(t_{j',m}\right)_{j' \neq j}\right) \in \{(\theta, t_{-j})\}^{1/m}$ such that $S_{j'}^{\infty}\left[t'_{j',m}, \varepsilon\right] \notin \mathcal{R}_{j'}\left(t_{j'}\right)$. Since the number of players and the number of actions are both finite, there is some j' and some $R_{j'} \subset A_{j'}$ and a subsequence of $\left\{t'_{-j,m}\right\}$, say itself, such that $S_{j'}^{\infty}\left[t'_{j',m}, \varepsilon\right] = R_{j'}$ for all m. However, since $t'_{j',m} \to t_{j'}$, this implies $S_{j'}^{\infty}\left[t'_{j',m}, \varepsilon\right] \in \mathcal{R}_{j'}^T\left(t_{j'}\right)$, which is a contradiction to $S_{j'}^{\infty}\left[t'_{j',m}, \varepsilon\right] \notin \mathcal{R}_{j'}\left(t_{j'}\right)$.

Step 2. *Defining the theory η_{-j}.*

Consider $(\theta, t_{-j}) \in$supp$\kappa_{t_j}^*$. Then, since $t_{j,m} \to t_j$, $\kappa_{t_{j,m}}^*\left[\{(\theta, t_{-j})\}^{\lambda_m}\right] > 0$ for sufficiently large m. Hereafter, we consider m large enough such that

$$\begin{aligned} \min\{\beta, 1\} &> \lambda_m; \\ \kappa_{t_{j,m}}^*\left[\{(\theta, t_{-j})\}^{\lambda_m}\right] &> 0, \end{aligned} \qquad (44)$$

where β is as specified in step 1 and hence (1) and (2) in step 1 hold.

By (2) in step 1,

$$S_{-j}^{\infty}\left[\widetilde{t}_{-j}, \varepsilon\right] \in \mathcal{R}_{-j}\left(\widetilde{t}_{-j}\right), \forall \left(\theta', t'_{-j}\right) \in \{(\theta, t_{-j})\}^{\lambda_m}. \qquad (45)$$

Define a distribution $\eta_{-j,m}(\theta, t_{-j}) \in \Delta(\mathcal{R}_{-j}(t_{-j}))$ as follows. For each $R_{-j} \in \mathcal{R}_{-j}(t_{-j})$, denote

$$T_{-j}^{R_{-j}} = \left\{ \left(\theta', t'_{-j}\right) \in \Theta \times T_{-j}^{*} : S_{-j}^{\infty}\left[t'_{-j}, \varepsilon\right] = R_{-j} \right\}.$$

For each $(\theta, t_{-j}) \in \Theta \times T_{-j}$ and $R_{-j} \in \mathcal{R}_{-j}(t_{-j})$,

$$\eta_{-j,m}(\theta, t_{-j})[R_{-j}] \equiv \frac{\kappa_{t_{j,m}}^{*}\left[\left\{(\theta, t_{-j})\right\}^{\lambda_m} \cap T_{-j}^{R_{-j}}\right]}{\kappa_{t_{j,m}}^{*}\left[\left\{(\theta, t_{-j})\right\}^{\lambda_m}\right]}. \tag{46}$$

Observe that $\eta_{-j,m} \in \Delta(\mathcal{R}_{-j}(t_{-j}))$ is well defined by (44). For any $(\theta, t_{-j}) \in \Theta \times T_{-j}$, since $\mathcal{R}_{-j}(t_{-j})$ is a finite family, $\eta_{-j,m}(\theta, t_{-j})$ has a convergent subsequence in $\Delta(\mathcal{R}_{-j}(t_{-j}))$, say itself, converging to some $\eta_{-j}(\theta, t_{-j}) \in \Delta(\mathcal{R}_{-j}(t_{-j}))$. That is, for each $(\theta, t_{-j}) \in \Theta \times T_{-j}$ and $R_{-j} \in \mathcal{R}_{-j}(t_{-j})$,

$$\eta_{-j}(\theta, t_{-j})[R_{-j}] = \lim_{m \to \infty} \eta_{-j,m}(\theta, t_{-j})[R_{-j}]. \tag{47}$$

Step 3. *$a_j \in R_j$ if $a_j \in BR_j^{\circ}\left(\pi_{t_j, \sigma_{-j}^{(\varphi_{-j}, \eta_{-j})}}, \varepsilon\right)$ for some conditional conjecture φ_{-j}.*

Since $a_j \in BR_j^{\circ}\left(\pi_{t_j, \sigma_{-j}^{(\varphi_{-j}, \eta_{-j})}}, \varepsilon\right)$ for some conditional conjecture φ_{-j} and the game is finite, $a_j \in BR_j\left(\pi_{t_j, \sigma_{-j}^{(\varphi_{-j}, \eta_{-j})}}, \varepsilon'\right)$ for some $\varepsilon' < \varepsilon$. We prove that $a_j \in S_j^{\infty}[t_{j,m}, \varepsilon] = R_j$ for sufficiently large m. Let $\sigma'_{-j} : \Theta \times T_{-j} \to \Delta(A_{-j})$ be a fixed measurable selection from $S_{-j}^{\infty}[\cdot, \varepsilon]$ (see footnote 9). We define $\sigma_{-j,m} : \Theta \times T_{-j}^{*} \to \Delta(A_{-j})$ as follows.

$$\sigma_{-j,m}\left(\theta', t'_{-j}\right) = \begin{cases} \varphi_{-j}(\theta, t_{-j}, R_{-j}), & \text{if } \left(\theta', t'_{-j}\right) \in \left\{(\theta, t_{-j})\right\}^{\lambda_m} \cap T_{-j}^{R_{-j}}; \\ \sigma'_{-j}\left(\theta', t'_{-j}\right), & \text{otherwise.} \end{cases}$$

By the definitions of $T_{-j}^{R_{-j}}$ and σ'_{-j}, $\sigma_{-j,m}$ is $\varepsilon-$valid for $t_{j,m}$. We will conclude step 3 by showing that

$$\lim_{m \to \infty} \pi_{t_{j,m}, \sigma_{-j,m}}\left[(\theta, a_{-j})\right] = \pi_{t_j, \sigma_{-j}^{(\varphi_{-j}, \eta_{-j})}}\left[(\theta, a_{-j})\right], \forall (\theta, a_{-j}) \in \Theta \times A_{-j} \tag{48}$$

where the limit is taken in the Euclidean space because $\Theta \times A_{-j}$ is finite. Since $a_j \in BR_j\left(\pi_{t_j, \sigma_{-j}^{(\varphi_{-j}, \eta_{-j})}}, \varepsilon'\right)$ and $\varepsilon' < \varepsilon$, (48) implies that $a_j \in BR_j\left(\pi_{t_{j,m}, \sigma_{-j,m}}, \varepsilon\right)$ for sufficiently large m. Since $\sigma_{-j,m}$ is $\varepsilon-$valid for $t_{j,m}$, $a_j \in S_j^{\infty}[t_{j,m}, \varepsilon] = R_j$.

It remains to prove (48). To save the notation, for each θ, let

$$T_{-j}^{\theta} \equiv \left\{ t_{-j} : \left(\theta, t_{-j}\right) \in \text{supp}\kappa_{t_j}^{*} \right\}.$$

First, observe that for each $\left(\theta, a_{-j}\right) \in \Theta \times A_{-j}$,

$$\begin{aligned} & \pi_{t_{j,m},\sigma_{-j,m}} \left[\left(\theta, a_{-j}\right) \right] \\ = & \sum_{t_{-j} \in T_{-j}^{\theta}} \left[\sum_{\left\{ t'_{-j} : \left(\theta, t'_{-j}\right) \in \left\{ \left(\theta, t_{-j}\right) \right\}^{\lambda_m} \right\}} \sigma_{-j,m} \left(\theta, t'_{-j}\right) \left[a_{-j}\right] \kappa_{t_{j,m}}^{*} \left[\left(\theta, t'_{-j}\right) \right] \right] \\ & + \int_{\left(\{\theta\} \times T_{-j}^{*}\right) \setminus \left(\text{supp}\kappa_{t_j}^{*}\right)^{\lambda_m}} \sigma_{-j,m} \left(\theta, t'_{-j}\right) \left[a_{-j}\right] \kappa_{t_{j,m}}^{*} \left[\left(\theta, dt'_{-j}\right) \right]. \end{aligned} \tag{49}$$

Second, since $t_{j,m} \to t_j$, $\kappa_{t_{j,m}}^{*} \left[\left(\{\theta\} \times T_{-j}^{*}\right) \setminus \left(\text{supp}\kappa_{t_j}^{*}\right)^{\lambda_m} \right] \to 0$ and hence

$$\lim_{m \to \infty} \int_{\left(\{\theta\} \times T_{-j}^{*}\right) \setminus \left(\text{supp}\kappa_{t_j}^{*}\right)^{\lambda_m}} \sigma_{-j,m} \left(\theta, t'_{-j}\right) \left[a_{-j}\right] \kappa_{t_{j,m}}^{*} \left[\left(\theta, dt'_{-j}\right) \right] = 0. \tag{50}$$

Third, for each $t_{-j} \in T_{-j}^{\theta}$,

$$\begin{aligned} & \sum_{\left\{ t'_{-j} : \left(\theta, t'_{-j}\right) \in \left\{ \left(\theta, t_{-j}\right) \right\}^{\lambda_m} \right\}} \sigma_{-j,m} \left(\theta, t'_{-j}\right) \left[a_{-j}\right] \kappa_{t_{j,m}}^{*} \left[\left(\theta, t'_{-j}\right) \right] \\ = & \sum_{R_{-j} \in \mathcal{R}_{-j}\left(t_{-j}\right)} \left[\varphi_{-j} \left(\theta, t_{-j}, R_{-j}\right) \left[a_{-j}\right] \kappa_{t_{j,m}}^{*} \left[\left\{ \left(\theta, t_{-j}\right) \right\}^{\lambda_m} \cap T_{-j}^{R_{-j}} \right] \right] \\ = & \sum_{R_{-j} \in \mathcal{R}_{-j}\left(t_{-j}\right)} \left[\varphi_{-j} \left(\theta, t_{-j}, R_{-j}\right) \left[a_{-j}\right] \eta_{-j,m} \left(\theta', t_{-j}\right) \left[R_{-j}\right] \right] \kappa_{t_{j,m}}^{*} \left[\left\{ \left(\theta, t_{-j}\right) \right\}^{\lambda_m} \right] \end{aligned} \tag{51}$$

where the second equality follows from (46). Finally, since $\text{supp}\kappa_{t_j}^{*}$ is finite and $t_{j,m} \to t_j$, we also know that $\lim_{m \to \infty} \kappa_{t_{j,m}}^{*} \left[\left\{ \left(\theta, t_{-j}\right) \right\}^{\lambda_m} \right] = \kappa_{t_j}^{*} \left[\left(\theta, t_{-j}\right) \right]$. Hence, by (47) and (51), for each t_{-j} such that $\left(\theta, t_{-j}\right) \in \text{supp}\kappa_{t_j}^{*}$,

$$\begin{aligned} & \lim_{m \to \infty} \sum_{\left\{ t'_{-j} : \left(\theta, t'_{-j}\right) \in \left\{ \left(\theta, t_{-j}\right) \right\}^{\lambda_m} \right\}} \sigma_{-j,m} \left(\theta, t'_{-j}\right) \left[a_{-j}\right] \kappa_{t_{j,m}}^{*} \left[\left(\theta, t'_{-j}\right) \right] \\ = & \sum_{R_{-j} \in \mathcal{R}_{-j}\left(t_{-j}\right)} \left[\varphi_{-j} \left(\theta, t_{-j}, R_{-j}\right) \left[a_{-j}\right] \eta_{-j} \left(\theta, t_{-j}\right) \left[R_{-j}\right] \right] \kappa_{t_j}^{*} \left[\left(\theta, t_{-j}\right) \right]. \end{aligned} \tag{52}$$

Hence, by (51) and (52),

$$\lim_{m\to\infty}\sum_{t_{-j}\in T^{\theta}_{-j}}\left[\sum_{\left\{t'_{-j}:\left(\theta,t'_{-j}\right)\in\left\{\left(\theta,t_{-j}\right)\right\}^{\lambda_m}\right\}}\sigma_{-j,m}\left(\theta,t'_{-j}\right)\left[a_{-j}\right]\kappa^*_{t_{j,m}}\left[\left(\theta,t'_{-j}\right)\right]\right] \tag{53}$$
$$=\sum_{t_{-j}\in T^{\theta}_{-j}}\left[\sum_{R_{-j}\in\mathcal{R}_{-j}\left(t_{-j}\right)}\left[\varphi_{-j}\left(\theta,t_{-j},R_{-j}\right)\left[a_{-j}\right]\eta_{-j}\left(\theta,t_{-j}\right)\left[R_{-j}\right]\right]\kappa^*_{t_j}\left[\left(\theta,t_{-j}\right)\right]\right]$$
$$=\pi_{t_j,\sigma^{\left(\varphi_{-j},\eta_{-j}\right)}_{-j}}\left[\left(\theta,a_{-j}\right)\right].$$

Therefore, (48) follows from (49), (50), and (53).■

A.4 Proof of Proposition 5

Proposition 5. *Given a finite model (T,κ), a type $t_i\in T_i$ is G-critical iff there exist some $\varepsilon-$SCC $\left(\mathcal{R}_j\right)_{j\in N}$ with $\varepsilon>0$, and some $R_i\in R_i\left(t_i\right)$ such that $S^{\infty}_i\left[t_i\right]\setminus R_i\neq\varnothing$.*

Proof. If there is some $\varepsilon-$SCC $\left(\mathcal{R}_j\right)_{j\in N}$ and some $R_i\in\mathcal{R}_i\left(t_i\right)$ such that some action $a_i\in S^{\infty}_i\left[t_i\right]$ and $a_i\notin R_i$, then by Proposition 3, there exist $\gamma>0$ and a sequence of types $\{t_{i,m}\}$ such that $t_{i,m}\to t_i$ and $S^{\infty}_i\left[t_{i,m},\gamma\right]\subset R_i$ for every m. Since $S^{\infty}_i\left[t_{i,m},\gamma\right]\subset R_i$ for every m, it follows that $a_i\in S^{\infty}_i\left[t_i\right]$ and $a_i\notin S^{\infty}_i\left[t_{i,m},\gamma\right]$ for every m. That is, t_i is G-critical.

Conversely, if t_i is G-critical, there exist $\varepsilon>0$, an action a_i, and a sequence of types $\{t_{i,m}\}$ with $t_{i,m}\to t_i$ such that $a_i\in S^{\infty}_i\left[t_i\right]$ and $a_i\notin S^{\infty}_i\left[t_{i,m},\varepsilon\right]$ for every m. By finiteness of actions, there exist some $R_i\subset A_i$ and some subsequence of $\{t_{i,m_k}\}$ such that $t_{i,m_k}\to t_i$ and $S^{\infty}_i\left[t_{i,m_k},\varepsilon\right]=R_i$ for all k. Hence, by Proposition 4, $R_i\in\mathcal{R}^T_i\left(t_i\right)$ for the $\varepsilon-$SCC $\left(\mathcal{R}^T_j\right)_{j\in N}$, and moreover, $a_i\in S^{\infty}_i\left[t_i\right]$ and $a_i\notin R_i$.■

A.5 Proof of Proposition 6

Proposition 6. *For any $i\in N$, let $T^{f,c}_i\left(\subset T^*_i\right)$ denote the set of all finite G-critical types. Then, $T^{f,c}=\cup_{p>0}C^p\left(T^{f,c}\right)$, where $T^{f,c}=\times_{i\in N}T^{f,c}_i$.*

Proof. By the definition of $C^p_i\left(\cdot\right)$ in (Ely and Pęski, 2011, Section 4.2), $C^p_i\left(T^{f,c}\right)\subset T^{f,c}$ for all $p>0$. Conversely, suppose that $t_i\notin C^p_i\left(T^{f,c}\right)$ for all $p>0$. Then, by the definition of $C^p(\cdot)$, for any $p > 0$, $t_i \notin B_i^p\left([B]^k\left(T^{f,c}\right)\right)$ for some k. Let (T,κ) be the finite model which contains t_i. Since T is a finite set, there is some $\gamma > 0$ such that for any $t_j \in T_j$, $\kappa_{t_j}\left(\{t_{-j}\}\right) > 0$ implies that $\kappa_{t_j}\left(\{t_{-j}\}\right) > \gamma$. Fix $p < \gamma$. It suffices to show that

$$\text{for any non-negative integer } k,\ t_i \notin B_i^p\left([B]^k\left(T^{f,c}\right)\right) \text{ implies } t_i \notin T_i^{f,c}. \qquad (\blacklozenge)$$

For any $i \in N$, let $T_i^{f,r}\ (\subset T_i^*)$ denote the set of all finite G-regular types. Note that $T_i^{f,c}$ and $T_i^{f,r}$ are disjoint and they partition the set of all finite type, i.e., for any finite types t_i,

$$\kappa_{t_i}^*\left[\Theta \times T_{-i}^{f,c}\right] + \kappa_{t_i}^*\left[\Theta \times T_{-i}^{f,r}\right] = 1.$$

By the results in Ely and Pęski (2011) and Chen and Xiong (2011a), we know that

$$\text{for any } j \in N, \text{ if type } \kappa_{t_j}^*\left[\Theta \times T_{-j}^{f,r}\right] = 1, \text{ then } t_j \in T_j^{f,r}, \text{ i.e., } t_j \notin T_j^{f,c}. \qquad (\blacktriangle)$$

We now prove ($\blacklozenge$) by induction. For $k = 0$, suppose $t_i \notin B_i^p\left(T^{f,c}\right)$, $t_i \in T_i^{f,c}$ and we derive a contradiction. By the definition of γ and $p < \gamma$, we have $\kappa_{t_i}^*\left[\Theta \times T_{-i}^{f,c}\right] = 0$, or equivalently, $\kappa_{t_i}^*\left[\Theta \times T_{-i}^{f,r}\right] = 1$. Hence, $t_i \notin T_i^{f,c}$ by ($\blacktriangle$), which contradicts to our assumption $t_i \in T_i^{f,c}$.

Assume ($\blacklozenge$) is true for k and we prove the case of $k+1$. Suppose $t_i \notin B_i^p\left([B]^{k+1}\left(T^{f,c}\right)\right)$. There are two cases to consider: i) $t_i \notin B_i^p\left([B]^k\left(T^{f,c}\right)\right)$; ii) $\kappa_{t_i}^*\left[\Theta \times B_{-i}^p\left([B]^k\left(T^{f,c}\right)\right)\right] < p$. In case i), by induction hypothesis, we have $t_i \notin T_i^{f,c}$. In case ii), by the definition of γ and $p < \gamma$, we have $\kappa_{t_i}^*\left[\Theta \times B_{-i}^p\left([B]^k\left(T^{f,c}\right)\right)\right] = 0$. Note that the induction hypothesis implies $T_{-i}^{f,c} \subset B_{-i}^p\left([B]^k\left(T^{f,c}\right)\right)$. We thus have $\kappa_{t_i}^*\left[\Theta \times T_{-i}^{f,c}\right] = 0$, or equivalently, $\kappa_{t_i}^*\left[\Theta \times T_{-i}^{f,r}\right] = 1$. Hence, $t_i \notin T_i^{f,c}$ by ($\blacktriangle$).■

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