Equivalence of Stochastic and Deterministic Mechanisms*

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Abstract

We consider a general social choice environment that has multiple agents, a finite set of alternatives, independent types, and atomless type distribution. We show that for any Bayesian incentive compatible mechanism, there exists an equivalent deterministic mechanism that (1) is Bayesian incentive compatible; (2) delivers the same interim expected allocation probabilities and the same interim expected utilities for all agents; and (3) delivers the same ex ante expected social surplus. This result holds in settings with a rich class of utility functions, multi-dimensional types, interdependent valuations, and in settings without monetary transfers. To prove our result, we develop a novel methodology of mutual purification, and establish its link with the mechanism design literature.

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1 Introduction

Myerson (1981) provides the framework that has become the paradigm for the study of optimal auction design. Under a regularity condition, the optimal auction allocates the object to the bidder with the highest virtual value, provided that this virtual value is above the seller's opportunity cost. In other words, the optimal auction in Myerson's setting is deterministic.[1]

A natural conjecture is that the optimality of deterministic mechanisms generalizes beyond Myerson's setting. McAfee and McMillan (1988) claim that under a general regularity condition on consumers' demand, stochastic delivery is not optimal for a multi-product monopolist. However, this result has been proven to be incorrect in settings with a single agent. Several papers have shown that a multi-product monopolist may find it beneficial to include lotteries as part of the selling mechanism; see, for example, Thanassoulis (2004), Manelli and Vincent (2006, 2007), Pycia (2006), Pavlov (2011), and more recently, Hart and Reny (2015) and Rochet and Thanassoulis (2017).[2] In this paper, we prove a mechanism equivalence result that implies the optimality of deterministic mechanisms in remarkably general environments with multiple agents.

We consider a general social choice environment that has multiple agents, a finite set of alternatives, independent types, and atomless type distribution. We show that for any Bayesian incentive compatible mechanism, there exists an equivalent deterministic mechanism that (1) is Bayesian incentive compatible; (2) delivers the same interim expected allocation probabilities and the same interim expected utilities for all agents; and (3) delivers the same ex ante expected social surplus. In addition to the standard social choice environments with linear utilities and one-dimensional, private types, our result holds in settings with a rich class of utility functions, multi-dimensional types, interdependent valuations, and in settings without monetary transfers.

Our result implies that any mechanism, including the optimal mechanism (whether in

[1]Also see Riley and Zeckhauser (1983) who consider a one-good monopolist selling to consumers with unit demand and show that lotteries do not help the one-good monopolist.

[2]In environments in which different types are associated with different risk attitudes, it is known that stochastic mechanisms may perform better; see, for example, Laffont and Martimort (2002, p. 67) and Strausz (2006). Gauthier and Laroque (2014) propose a new technique in solving optimization problems, and apply this technique to study when a deterministic local optimum can be locally improved upon by a stochastic deviation in adverse selection and moral hazard problems. Aggarwal, Fiat, Goldberg, Hartline, Immorlica, and Sudan (2011) also study derandomization of auctions. They focus on prior-free auctions, rather than the Bayesian setting.

terms of revenue or efficiency), can be implemented using a deterministic mechanism and nothing can be gained from designing more intricate mechanisms with possibly more complex randomization in the allocation rule. As pointed out in Hart and Reny (2015, p. 912), Aumann commented that it is surprising that randomization cannot increase revenue when there is only one good. Indeed, aforementioned papers in the screening literature establish that randomization helps when there are multiple goods. Nevertheless, we show that in general social choice environment with multiple agents, the revenue maximizing mechanism can always be deterministically implemented. This is in sharp contrast with the results in the screening literature.

Our result has important implications beyond the revenue contrast. The mechanism design literature essentially builds on the assumption that a mechanism designer can credibly commit to any outcome of a mechanism. This requirement implies that any outcome of the mechanism must be verifiable before it can be employed. In this vein, a stochastic mechanism demands not only that a randomization device be available to the mechanism designer, but also that the outcome of the randomization device be objectively verified. As noted in Laffont and Martimort (2002, p. 67),

> Ensuring this verifiability is a more difficult problem than ensuring that a deterministic mechanism is enforced, because any deviation away from a given randomization can only be statistically detected once a sufficient number of realizations of the contracts have been observed. ... The enforcement of such stochastic mechanisms in a bilateral one-shot relationship is thus particularly problematic. This has led scholars to give up those random mechanisms or, at least, to focus on economic settings where they are not optimal.[3]

Our result implies that every mechanism can in fact be deterministically implemented, and thereby irons out the conceptual difficulties associated with stochastic mechanisms.[4]

Along the lines of implementation, our paper is related to the literature of reduced form

[3]Also see Bester and Strausz (2001) and Strausz (2003).

[4]There are other ways to circumvent the problem that the designer is not able to commit to outcomes induced by randomization devices. For example, probabilities in the selling mechanism can be considered as the discount factor from a temporal interpretation (see, for example, Salant (1989)), and the designer is committing to a delay rather than committing to randomizing. For another example, the designer could use jointly controlled lotteries to achieve verifiable randomization. Our contribution in this paper is to show that we do not have to think about any changes in the model, and that the randomization in the allocation rule can be fully absorbed using the agents' private information. This feature also enables us to establish the revelation principle for deterministic mechanisms; see Remark 3.

implementation (see, for example, Border (1991) for the case of the single-unit auction and Cai, Daskalakis, and Weinberg (2018) for the case of multi-item auction). In many applications of mechanism design, it is convenient to work with interim expected allocation probabilities. The reduced form implementation literature asks what interim expected allocation probabilities can be implemented. Our result implies that whatever interim expected allocation probabilities that can be implemented can actually be implemented in a deterministic manner. As such, even if the mechanism designer does not have access to randomization devices or cannot commit to the outcomes induced by randomization devices, we can rest assured working with interim expected allocation probabilities.

This paper joins the strand of literature that studies mechanism equivalence. Though motivations vary, these results show that it is without loss of generality to consider the various subclasses of mechanisms. As in the case of dominant-strategy mechanisms (see Manelli and Vincent (2010) and Gershkov, Goeree, Kushnir, Moldovanu, and Shi (2013)) and symmetric auctions (see Deb and Pai (2017)), our findings imply that the requirement of deterministic mechanisms is not restrictive in itself.[5]

To prove the existence of an equivalent deterministic mechanism, we develop a new methodology of mutual purification and establish its link with the literature of mechanism design.[6] The notion of mutual purification is both conceptually and technically different from the usual purification principle in the literature related to Bayesian games. We clarify these two different notions of purification in the next three paragraphs.

It follows from the general purification principle in Dvoretzky, Wald, and Wolfowitz (1950) that any behavioral-strategy Nash equilibrium in a finite-action Bayesian game with independent types and atomless type distribution corresponds to some pure-strategy Bayesian Nash equilibrium with the same payoff.[7] In particular, independent types and atomless

[5]Manelli and Vincent (2010) show that for any Bayesian incentive compatible auction, there exists an equivalent dominant-strategy incentive compatible auction that yields the same interim expected utilities for all agents. Gershkov, Goeree, Kushnir, Moldovanu, and Shi (2013) extend this equivalence result to social choice environments with linear utilities and independent, one-dimensional, private types; also see Footnote 11 for related discussion. Deb and Pai (2017) show that restricting the seller to using a symmetric auction imposes virtually no restriction on her ability to achieve discriminatory outcomes. Others papers that study mechanism equivalence include Border (1991), Eso and Futo (1999), Börgers and Norman (2009).

[6]Some of our technical results extend the corresponding mathematical results in Arkin and Levin (1972); see the Supplemental Material (Chen, He, Li, and Sun (2019)) for a detailed discussion.

[7]See Radner and Rosenthal (1982), Milgrom and Weber (1985) and Khan, Rath, and Sun (2006). Furthermore, by applying the purification idea to a sequence of Bayesian games, Harsanyi (1973) provided an interpretation of mixed-strategy equilibrium in complete information games; see Govindan, Reny, and Robson (2003) and Morris (2008) for more discussion.

distributions allow the agents to replace their behavioral strategies by some equivalent pure strategies one-by-one.[8] The point is that under the independent information assumption, any agent whose type has an atomless distribution could purify her own behavioral strategy regardless of whether the other agents' types have atomless distributions. Example 6 in the Supplemental Material (Chen, He, Li, and Sun (2019)) illustrates this idea of self purification. Given a behavioral-strategy Nash equilibrium in a 2-agent Bayesian game with independent information, there is an equivalent pure strategy for the agent whose type has an atomless distribution, while the other agent with an atom in her type space could not purify her behavioral strategy.

The general purification principle in Dvoretzky, Wald, and Wolfowitz (1950) is only applicable in the unconditional (ex ante) setting, and thus does not apply to this paper, as the notion of mechanism equivalence naturally requires us to study purification in the conditional (interim) setting. The purification result of this paper is based on the atomless distributions of types of the other agents. Example 7 in the Supplemental Material (Chen, He, Li, and Sun (2019)) partially illustrates this idea of mutual purification. For a given randomized mechanism in a 2-agent setting with independent information, the agent with an atom in her type space can achieve the same interim payoff by some deterministic mechanism, while there does not exist such a deterministic mechanism for the other agent whose type has an atomless distribution. In other words, our result becomes possible because each agent relies on the atomless distributions of types of the other agents rather than her own. This also explains why a similar result does not hold in the one-agent setting since there is no atomless distributions of types of the other agents for such a single agent to purify the relevant randomized mechanism. In addition, we emphasize that in the settings with multiple agents, the notion of mutual purification requires not only that each agent obtain the same interim payoff under some deterministic mechanism, but also that a single deterministic mechanism deliver the same interim payoffs for all the agents simultaneously.

From a methodological point of view, the general purification principle in Dvoretzky, Wald, and Wolfowitz (1950) is simply a version of the classical Lyapunov Theorem about the convex range of an atomless finite-dimensional vector measure. Our purification result is technically different. First, the problem we consider is infinite-dimensional because we require the same interim expected allocation probabilities/ utilities for the equivalent mechanism at the interim level with a continuum of types. Note that Lyapunov's Theorem fails in an

[8]See the proof of Theorem 1 in Khan, Rath, and Sun (2006).

infinite-dimensional setting.[9] Second, it is clearly impossible to obtain a purified deterministic mechanism that delivers the same interim expected allocation probabilities as the original stochastic mechanism, conditioned on the joint types of all the agents.[10] However, our result on mutual purification shows that such an equivalence becomes possible when the conditioning operation is imposed on the individual types of every agent simultaneously, although the combination of the individual types of every agent is the joint types of all the agents. To the best of our knowledge, this paper is the first to consider the purification of a randomized decision rule that retains the same expected payoffs conditioned on the individual types of every agent in an economic model.

The assumption of atomless type distribution facilitates the development of the novel methodology of mutual purification, which lies at the core of our arguments. Thus, our paper also contributes to the Bayesian mechanism design literature in relying on specific aspects of agents' private information. These information aspects are often crucial in pinning down different properties of the optimal mechanism; see, for example, Myerson (1981) and Crémer and McLean (1988).

The rest of the paper is organized as follows. Section 2 introduces the model. Section 3 presents the mechanism equivalence result. Section 4 discusses the assumptions behind our equivalence result, the structures of the equivalence deterministic mechanisms, and the recent literature on the benefit of randomness in settings with multiple agents. Section 5 concludes. The appendices contain proofs and other technical results omitted from the main body of the paper, and examples delineating the limits of our mechanism equivalence result.

2 Model

2.1 Notation

We consider an environment with a finite set $\mathcal{I} = \{1, 2, \dots, I\}$ of risk-neutral agents ($I \geq 2$) and a finite set $\mathcal{K} = \{1, 2, \dots, K\}$ of social alternatives. The set of possible types V_i of agent i is a closed subset of finite dimensional Euclidean space $\mathbb{R}^l$ with generic element v_i. The set of possible type profiles is $V \equiv V_1 \times V_2 \times \cdots \times V_I$ with generic element $v = (v_1, v_2, \dots, v_I)$.

[9]See, for example, Diestel and Uhl (1977, p. 261).

[10]Since the joint types of all the agents carry the full information, the expected allocation probability of a stochastic mechanism conditioned on the joint types is simply the stochastic mechanism itself.

We write v_{-i} for a type profile of agent i's opponents; that is, $v_{-i} \in V_{-i} = \Pi_{j \neq i} V_j$. The type profile v is distributed according to a probability distribution λ. For each agent $i \in \mathcal{I}$, λ_i is the marginal distribution of λ on V_i and is assumed to be atomless. Types are assumed to be independent. If $(Y, \mathcal{Y})$ is a measurable space, then ΔY is the set of all probability measures on $(Y, \mathcal{Y})$. If Y is a metric space, then we treat it as a measurable space with its Borel σ-algebra.

2.2 Mechanism

We consider direct mechanisms characterized by $K + I$ functions, $\{q^k(v)\}_{k \in \mathcal{K}}$ and $\{t_i(v)\}_{i \in \mathcal{I}}$, where v is the profile of reports, $q^k(v) \geq 0$ is the probability that alternative k is implemented satisfying $\sum_{k \in \mathcal{K}} q^k(v) = 1$, and $t_i(v)$ is the monetary transfer that agent i makes to the designer. Thus, we shall denote a mechanism by (q, t), where $q = \{q^k\}_{k \in \mathcal{K}}$ and $t = \{t_i\}_{i \in \mathcal{I}}$. By the revelation principle, it is without loss of generality to restrict attention to direct mechanisms.

We write $u_i^k(v)$ for agent i's gross utility in alternative k under the type profile v. Given a mechanism (q, t), we write

$$u_i(v) = \sum_{k \in \mathcal{K}} u_i^k(v)\, q^k(v) - t_i(v)$$

for agent i's utility under the type profile v. For notational ease, we only define the following objects under the assumption of truthful reporting. We write

$$Q_i^k(v_i) = \int_{V_{-i}} q^k(v_i, v_{-i}) \lambda_{-i}(\mathrm{d}v_{-i})$$

for the interim expected allocation probability (from agent i's perspective) that alternative k is implemented. Agent i's interim expected utility is

$$U_i(v_i) = \int_{V_{-i}} \left[\sum_{k \in \mathcal{K}} u_i^k(v_i, v_{-i}) q^k(v_i, v_{-i}) - t_i(v_i, v_{-i}) \right] \lambda_{-i}(\mathrm{d}v_{-i}).$$

The ex ante expected social surplus is

$$\int_V \sum_{i \in \mathcal{I}} \sum_{k \in \mathcal{K}} \left[u_i^k(v)\, q^k(v) \right] \lambda(\mathrm{d}v).$$

Definition 1. *A mechanism is Bayesian incentive compatible (BIC) if for all $i \in \mathcal{I}$ and $v_i \in V_i$,*

$$U_i(v_i) \geq \int_{V_{-i}} \left[\sum_{k \in \mathcal{K}} u_i^k(v_i, v_{-i})\, q^k(v_i', v_{-i}) - t_i(v_i', v_{-i}) \right] \lambda_{-i}(\mathrm{d}v_{-i})$$

for any alternative type $v_i' \in V_i$.

A mechanism satisfies Bayesian individual rationality (BIR) if for all $i \in \mathcal{I}$ and $v_i \in V_i$,

$$U_i(v_i) \geq 0.$$

Definition 2. *A mechanism (q, t) is deterministic at $v \in V$ if $q^k(v) = 1$ for some $k \in \mathcal{K}$. A mechanism (q, t) is deterministic if the mechanism is deterministic at all $v \in V$.*

2.3 Mechanism equivalence

We employ the following notion of mechanism equivalence in this paper.

Definition 3. *Two mechanisms (q, t) and $(\tilde{q}, \tilde{t})$ are equivalent if and only if they deliver the same interim expected allocation probabilities and the same interim expected utilities for all agents, and the same ex ante expected social surplus.*

Remark 1. Our equivalence notion is stronger than the prevailing mechanism equivalence notions used in the literature. For example, Manelli and Vincent (2010) and Gershkov, Goeree, Kushnir, Moldovanu, and Shi (2013) define two mechanisms to be equivalent if they deliver the same interim expected utilities for all agents and the same ex ante expected social surplus.[11]

To illustrate our notion of mechanism equivalence, it is best to consider an example. The example is deliberately kept simple. Our result is far more general and the proof is much more complex.

Example 1. Consider a single-unit auction with two bidders. Suppose that bidders' valuations for the object $v = (v_1, v_2)$ are uniformly distributed on the square $[0, 1]^2$. Consider the following

[11] Gershkov, Goeree, Kushnir, Moldovanu, and Shi (2013, Section 4.1) show that their BIC-DIC equivalence result breaks down when requiring the same interim expected allocation probabilities. They also note that "this notion (of interim expected allocation probabilities) becomes relevant when, for instance, the designer is not utilitarian or when preferences of agents outside the mechanism play a role."

stochastic allocation rule $q = (q^1, q^2)$ with

$$q^1(v_1, v_2) = v_1, \quad q^2(v_1, v_2) = 1 - q^1(v_1, v_2)$$

where q^i is the probability of bidder i getting the object for $i \in \{1, 2\}$. In the construction of the equivalent deterministic mechanism below, the transfers are kept unchanged. Thus, for simplicity, we do not specify the transfer scheme t in the mechanism (q, t). The readers may think of t as an arbitrary transfer scheme such that (q, t) is BIC.

The interim expected probability of bidder 1 getting the object is

$$\int_0^1 q^1(v_1, v_2)\,\mathrm{d}v_2 = \int_0^1 v_1\,\mathrm{d}v_2 = v_1$$

for all v_1, and the interim expected probability of bidder 2 getting the object is

$$\int_0^1 q^2(v_1, v_2)\,\mathrm{d}v_1 = \int_0^1 (1 - v_1)\,\mathrm{d}v_1 = \frac{1}{2}$$

for all v_2.

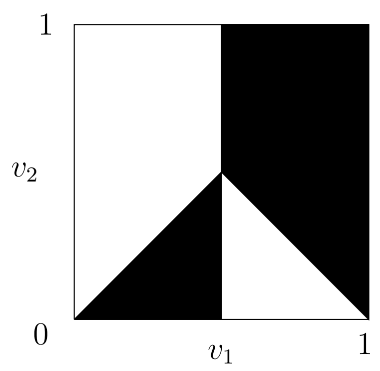

Figure 1: Bidder 1 is allocated the object in the shaded region, and bidder 2 is allocated the object in the unshaded region.

It is easy to verify that the following deterministic mechanism $(\hat{q}, t)$ is equivalent in terms of interim expected allocation probabilities for all agents (Figure 1 provides a graphical illustration of the mechanism $(\hat{q}, t)$). Since the transfers are kept unchanged, the deterministic mechanism $(\hat{q}, t)$ is also equivalent in terms of interim expected utilities for all agents and ex ante expected social surplus.

$$\hat{q}^1(v_1, v_2) = \begin{cases} 1 & \text{if } v_2 \le v_1 \le 1/2, \\ 1 & \text{if } v_1 \ge \max\{1 - v_2, \frac{1}{2}\}, \\ 0 & \text{otherwise,} \end{cases} \quad \hat{q}^2(v_1, v_2) = 1 - \hat{q}^1(v_1, v_2).$$

In Section 3, we show that for whatever stochastic mechanism that the designer may choose to use, however complicated, there always exists an equivalent mechanism that is deterministic.

3 Equivalence result

This section presents our mechanism equivalence result. To make the logic of our arguments and the roles played by the various assumptions clear, we break down our analysis into two steps. In the first step, we show that for any allocation rule, there exists a deterministic allocation rule that delivers the same interim expected allocation probabilities for all agents. This step only requires the assumption of atomless distribution. While the assumption of independent types is not needed, for simplicity of exposition, we present this result in settings with independent types (see Remark 2 below for a detailed discussion). In the second step, under additional assumptions of independent types and separable payoffs, we show that for any BIC and BIR mechanism, there exists a deterministic mechanism that is BIC and BIR, delivers the same interim expected allocation probabilities and the same interim expected utilities for all agents, and delivers the same ex ante expected social surplus.

Interim expected allocation probabilities

We first show that for any allocation rule, there exists a deterministic allocation rule that delivers the same interim expected allocation probabilities for all agents. For this result, we only require that λ_i be atomless for all $i \in \mathcal{I}$. Let

$$\Upsilon = \{g : V \to [0,1]^K \mid \sum_{k \in \mathcal{K}} g^k(v) = 1 \text{ for } \lambda\text{-almost all } v \in V\}.[12]$$

[12]Here, we use a different letter g, becuase g only requires $\sum_{k\in\mathcal{K}} g^k(v) = 1$ for λ-almost all $v \in V$ and thus is not necessarily an allocation rule.

Theorem 1. *For any allocation rule q, there exists a deterministic allocation rule $\hat{q}$ such that q and $\hat{q}$ induce the same interim expected allocation probabilities for all agents. That is, for all $i \in \mathcal{I}$ and $v_i \in V_i$,*

$$\mathbb{E}(\hat{q}|v_i) = \mathbb{E}(q|v_i).^{13} \tag{1}$$

The proof of Theorem 1 is relegated to the appendix. Here, we provide a sketch of the proof. For any allocation rule q, let

$$\Upsilon_q = \{g \in \Upsilon : \mathbb{E}(g|v_i) = \mathbb{E}(q|v_i) \text{ for all } i \in \mathcal{I} \text{ and } \lambda_i\text{-almost all } v_i \in V_i\}.$$

Step (1) shows that the set Υ_q is nonempty, convex, and weakly compact. Therefore, the set Υ_q admits extreme points. Step (2) proceeds to show that all extreme points of Υ_q are deterministic at λ-almost all $v \in V$.[14] Indeed, if g' is not deterministic at λ-almost all $v \in V$, then there exist distinct $\bar{g}$, $\bar{\bar{g}} \in \Upsilon_q$ such that $g' = \frac{1}{2}(\bar{g} + \bar{\bar{g}})$. Thus, g' is not an extreme point of Υ_q. The existence of $\bar{g}$ and $\bar{\bar{g}}$ relies on the assumption that λ_i is atomless for all $i \in \mathcal{I}$. Step (1) and Step (2) together imply that there exists $\tilde{g} \in \Upsilon_q$ that is deterministic at λ-almost all $v \in V$. Note that $\tilde{g}$ is not necessarily deterministic at all $v \in V$. Furthermore, such $\tilde{g}$ induces the same interim expected allocation probabilities for all $i \in \mathcal{I}$ and λ-almost all $v_i \in V_i$, but not for all $v_i \in V_i$. Step (3) then constructs a deterministic allocation rule $\hat{q}$ that induces the same interim expected the same interim expected allocation probabilities for all $i \in \mathcal{I}$ and $v_i \in V_i$, by modifying $\tilde{g}$ on sets of measure zero. The last step is (conceptually) straightforward.

Theorem 1 is not enough to show the mechanism equivalence result, as it only concerns the equivalence in terms of interim expected allocation probabilities for all agents. We now prove a generalization of Theorem 1, which is then invoked in Theorem 3 to prove the equivalence in terms of interim expected utilities for all agents.

Theorem 2. *Let $h = (h_1, h_2, \ldots, h_N)$ be an integrable function from V to $\mathbb{R}^N$. For any allocation rule q, there exists a deterministic allocation rule $\hat{q}$ such that for all $i \in \mathcal{I}$ and*

[13] Let $\mathbb{E}(f|v_i) := \int_{V_{-i}} f(v_i, v_{-i})\, \lambda_{-i}(\mathrm{d}v_{-i})$ for any integrable function f defined on V.

[14] Manelli and Vincent (2007) use a related technique in the multi-dimensional screening literature. Manelli and Vincent (2007) consider a revenue maximizing multi-product monopolist and study the extreme points of the set of feasible mechanisms. They show that, with multiple goods, extreme points could be stochastic mechanisms. In contrast, we work with the mechanism design setting, study a particular set of interest Υ_q, and show that all extreme points are deterministic. Despite the similarity in the general approach, the technical parts of proofs are dramatically different.

$v_i \in V_i$,

$$\mathbb{E}(\hat{q}h_j|v_i) = \mathbb{E}(qh_j|v_i) \tag{2}$$

for all $1 \le j \le N$.

Without loss of generality, we assume that h is an integrable function from V to $\mathbb{R}^N_{++}$.[15] Theorem 2 can be proved by applying similar arguments as in the proof of Theorem 1 to the following set:

$$\dot{\Upsilon}_q = \{g \in \Upsilon : \mathbb{E}(gh_j|v_i) = \mathbb{E}(qh_j|v_i) \text{ for all } i \in \mathcal{I} \text{ and } \lambda_i\text{-almost all } v_i \in V_i,\ 1 \le j \le N\}.$$

In Appendix C of the Supplemental Material (Chen, He, Li, and Sun (2019)), we detail how to modify the proof of Theorem 1 to prove Theorem 2.

Remark 2. *Theorem 1 and Theorem 2 above are established in the case of independent types. In settings with correlated types, let* ρ *denote the density function of* λ *with respect to* $\otimes_{i\in\mathcal{I}}\lambda_i$*. Then by Theorem 2, for any allocation rule* q*, there exists a deterministic allocation rule* $\hat{q}$ *such that for all* $i \in \mathcal{I}$ *and* $v_i \in V_i$*,*

$$\int_{V_{-i}} \hat{q}(v_i, v_{-i})\, h_j(v_i, v_{-i})\, \rho(v_i, v_{-i})\, \lambda_{-i}(dv_{-i}) = \int_{V_{-i}} q(v_i, v_{-i})\, h_j(v_i, v_{-i})\, \rho(v_i, v_{-i})\, \lambda_{-i}(dv_{-i})$$

for all $1 \le j \le N$*. Theorem 1 for the case of correlated types immediately follows by setting* $h \equiv 1$.

Mechanism equivalence

Next, we present our mechanism equivalence result. For this result, we need additional assumptions of separate payoffs and independent types. We assume that all agents have separable payoffs in the following sense.

Definition 1. *Agent* $i \in \mathcal{I}$ *is said to have separable payoff if for all* $k \in \mathcal{K}$ *and* $v \in V$*, her payoff function can be written as follows:*

$$u_i^k(v) = \sum_{1 \le m \le M} w_{i,m}^k(v_i)\, r_{i,m}^k(v_{-i}),$$

[15] To see this is without loss of generality, let $h_+ = (h_{+1}, h_{+2}, \ldots, h_{+I})$ and $h_- = (h_{-1}, h_{-2}, \ldots, h_{-I})$, where $h_{+j} = \max\{h_j, 0\} + 1$ and $h_{-j} = \min\{h_j, 0\} - 1$ for all $1 \le j \le N$. Theorem 2 can thus be proved by considering h_+ and $-h_-$ respectively, as $h = h_+ + h_-$.

where M is a positive integer, and $w^k_{i,m}$ (resp. $r^k_{i,m}$) is λ_i-integrable (resp. λ_{-i}-integrable) on V_i (resp. on V_{-i}) for $1 \le m \le M$.

In words, the payoff of each agent i is a summation of finite terms, where each term is a product of two components: the first component only depends on agent i's own type, while the second component depends on the other agents' types. Note that this setup is sufficiently general to cover most applications. In particular, it includes the interdependent payoff function as in Jehiel and Moldovanu (2001), and obviously covers the widely adopted private value payoffs as a special case.

Theorem 3. *Suppose that for each agent $i \in \mathcal{I}$, her payoff function is separable. For any BIC and BIR mechanism (q, t), there exists an equivalent deterministic mechanism $(\hat{q}, t)$ that is BIC and BIR. More explicitly,*

(1) q and $\hat{q}$ induce the same interim expected allocation probabilities for all $i \in \mathcal{I}$;

(2) (q, t) and $(\hat{q}, t)$ induce the same interim expected utilities for all $i \in \mathcal{I}$; and

(3) (q, t) and $(\hat{q}, t)$ induce the same ex ante expected social surplus.

Remark 3. Theorem 3 has an interesting implication for the revelation principle. By the standard revelation principle, it is without loss of generality to consider only direct mechanisms that are BIC and BIR. Theorem 3 shows that for each such direct mechanism, there exists an equivalent deterministic mechanism that is BIC and BIR. Thus, it is without loss of generality to consider only direct deterministic mechanisms. For related discussions on deterministic mechanisms and the revelation principle, see Strausz (2003) and Jarman and Meisner (2017).[16]

Remark 4. Our approach is not constructive. While we know that there exists an equivalent deterministic mechanism, we do not know how to construct such an equivalent deterministic mechanism. We discuss the structures of the equivalent deterministic mechanisms in Section 4.2 and Section 4.3. We also provide a recipe for the construction of an approximately equivalent mechanism in Appendix D of the Supplemental Material (Chen, He, Li, and Sun (2019)).

Proof of Theorem 3. Let h be a function from V to $\mathbb{R}^{IKM+1}$ defined as follows. For each $1 \le j \le IKM$, there is a unique vector of integers (i, k, m) where $i \in \mathcal{I}$, $k \in \mathcal{K}$, and $1 \le m \le M$ such that $j = m + (k-1)M + (i-1)KM$, and we let $h_j(v) = r^k_{i,m}(v_{-i})$.

[16] We thank an anonymous referee for suggesting this discussion.

Let $h_{IKM+1}(v) \equiv 1$. By Theorem 2, for any BIC and BIR mechanism (q, t), there exists a deterministic allocation rule $\hat{q}$ such that for any $i \in \mathcal{I}$ and $v_i \in V_i$,

$$\mathbb{E}(\hat{q}|v_i) = \mathbb{E}(q|v_i), \tag{3}$$

$$\mathbb{E}(\hat{q} r^k_{i,m}|v_i) = \mathbb{E}(q r^k_{i,m}|v_i) \tag{4}$$

for all $r^k_{i,m}$, $i \in \mathcal{I}$, $k \in \mathcal{K}$ and $1 \leq m \leq M$.

By Equation (3), q and $\hat{q}$ induce the same interim expected allocation probabilities for all agents. We proceed to verify that (q, t) and $(\hat{q}, t)$ induce the same interim expected utilities for all agents. We calculate agent i's interim expected utility when her true type is v_i and she reports type v'_i as follows:

$$\begin{aligned}
&\int_{V_{-i}} \left[\sum_{k \in \mathcal{K}} u^k_i(v_i, v_{-i})\, \hat{q}^k(v'_i, v_{-i}) - t_i(v'_i, v_{-i}) \right] \lambda_{-i}(\mathrm{d}v_{-i}) \\
&= \sum_{k \in \mathcal{K}} \sum_{1 \leq m \leq M} w^k_{i,m}(v_i) \int_{V_{-i}} r^k_{i,m}(v_{-i})\, \hat{q}^k(v'_i, v_{-i})\, \lambda_{-i}(\mathrm{d}v_{-i}) - \int_{V_{-i}} t_i(v'_i, v_{-i})\, \lambda_{-i}(\mathrm{d}v_{-i}) \\
&= \sum_{k \in \mathcal{K}} \sum_{1 \leq m \leq M} w^k_{i,m}(v_i) \int_{V_{-i}} r^k_{i,m}(v_{-i})\, q^k(v'_i, v_{-i})\, \lambda_{-i}(\mathrm{d}v_{-i}) - \int_{V_{-i}} t_i(v'_i, v_{-i})\, \lambda_{-i}(\mathrm{d}v_{-i}) \\
&= \int_{V_{-i}} \left[\sum_{k \in \mathcal{K}} u^k_i(v_i, v_{-i})\, q^k(v'_i, v_{-i}) - t_i(v'_i, v_{-i}) \right] \lambda_{-i}(\mathrm{d}v_{-i}),
\end{aligned}$$

where the second line follows from the assumption of separable payoffs, the third line follows from Equation (4) and the assumption of independent types. Thus, (q, t) and $(\hat{q}, t)$ deliver the same interim expected utility for each agent, when each agent i has true type v_i and misreports v'_i, for all $i \in \mathcal{I}$ and $v_i \in V_i$. Therefore, if (q, t) is BIC and BIR, then $(\hat{q}, t)$ is BIC and BIR. Since the ex post transfers are kept unchanged, the arguments above also imply that both mechanisms deliver the same ex ante expected social surplus. More explicitly,

$$\begin{aligned}
&\int_V \sum_{i \in \mathcal{I}} \sum_{k \in \mathcal{K}} \left[u^k_i(v)\, \hat{q}^k(v) \right] \lambda(\mathrm{d}v) \\
&= \sum_{i \in \mathcal{I}} \int_{V_i} \int_{V_{-i}} \sum_{k \in \mathcal{K}} \left[u^k_i(v_i, v_{-i})\, \hat{q}^k(v_i, v_{-i}) \right] \lambda_{-i}(\mathrm{d}v_{-i})\, \lambda_i(\mathrm{d}v_i) \\
&= \sum_{i \in \mathcal{I}} \int_{V_i} \int_{V_{-i}} \sum_{k \in \mathcal{K}} \left[u^k_i(v_i, v_{-i})\, q^k(v_i, v_{-i}) \right] \lambda_{-i}(\mathrm{d}v_{-i})\, \lambda_i(\mathrm{d}v_i)
\end{aligned}$$

$$= \int_V \sum_{i \in \mathcal{I}} \sum_{k \in \mathcal{K}} \left[u_i^k(v)\, q^k(v) \right] \lambda(\mathrm{d}v).$$

This completes the proof. □

Remark 5. It is clear from the proof of Theorem 3 that the equivalent deterministic mechanism $(\hat{q}, t)$ guarantees the same ex post monetary transfers and the same expected revenue. This implies our mechanism equivalence result also holds in settings without monetary transfers.

Remark 6. Say that a mechanism satisfies ex post individual rationality (EPIR) if for all $i \in \mathcal{I}$ and $v \in V$, $u_i(v) \geq 0$. For any stochastic BIC and BIR mechanism, if the assumptions of separable payoffs and independent types are satisfied, then there exists an equivalent deterministic mechanism that is BIC and EPIR. For this result, the ex post transfers may need to be adjusted. By Theorem 3, for any BIC and BIR mechanism (q, t), there exists an equivalent deterministic mechanism $(\hat{q}, t)$ that is BIC and BIR. Since (q, t) is BIR, the interim expected utility under (q, t) satisfies

$$U_i(v_i) = \int_{V_{-i}} \left[\sum_{k \in \mathcal{K}} u_i^k(v_i, v_{-i}) q^k(v_i, v_{-i}) - t_i(v_i, v_{-i}) \right] \lambda_{-i}(\mathrm{d}v_{-i}) \geq 0.$$

Define a new transfer scheme $\hat{t}$ as follows:

$$\hat{t}_i(v_i, v_{-i}) = \sum_{k \in \mathcal{K}} u_i^k(v_i, v_{-i})\, q^k(v_i, v_{-i}) - U_i(v_i).$$

It is easy to verity that $(\hat{q}, \hat{t})$ is an equivalent deterministic mechanism, and is BIC and EPIR.

4 Discussions

In this section, we first discuss the assumptions behind our equivalence result in Section 4.1. We then move on to discuss the structures of the equivalent deterministic mechanisms. Here, our discussion is twofold. Section 4.2 illustrates by an example that, even in environments with linear utilities and independent, one-dimensional, private types, we cannot hope for a mechanism equivalence result between the class of BIC mechanisms and the class of DIC and deterministic mechanisms. On a more positive note, Section 4.3 shows that for any mechanism with a symmetric allocation rule, there exists an equivalent deterministic mechanism that preserves symmetry. Finally, Section 4.4 compares our results with the recent literature on the benefit of randomness in settings with multiple agents.

4.1 The limits of the equivalence result

Here, we discuss the assumptions behind our equivalence result, including multiple agents, atomless distribution, separable payoffs, and independent types. The literature of multi-dimensional screening contains abundant examples illustrating that in the case of a single buyer, a multi-product monopolist may find it beneficial to include lotteries as part of the selling mechanism. Atomless distribution is an indispensable requirement for almost all purification results. Though separable payoff is a restriction, the setup is sufficiently general to cover most economic applications; see the discussion immediately after Definition 1 for details. While our result requires independence, it is worth mentioning that we only require independence across agents and we do not make any assumption regarding the correlation of the different coordinates of type v_i. For atomless distribution, separable payoffs, and independent types, we present examples in Appendix A.3 to illustrate that our mechanism equivalence result breaks down if each of these assumptions is violated.

We wish to elaborate further on the requirement of multiple agents. While it is well known that lotteries could strictly improve revenue in the case of a single buyer, our equivalence result implies that the optimal mechanism is deterministic in the case of multiple agents. It is thus important to highlight and explain the differences between the case of a single agent with a multi-dimensional type and the case of multiple agents. In particular, the readers might wonder why, in the case of a single buyer with a two-dimension type (v_1, v_2), the multi-product monopolist cannot use the agent's report along the second dimension to purify the allocation probability of good 1, and vice versa. The problem is that the incentive constraints would break down, since it is the same agent who controls the reports on both dimensions.

4.2 DIC and deterministic mechanisms

In the private value setting, each agent's utility does not depend on the types of the other agents. A mechanism is dominant strategy incentive compatible (DIC) in the private value setting if for all $i \in \mathcal{I}$, for all $v_i, v_i' \in V_i$, and for all $v_{-i} \in V_{-i}$,

$$\sum_{k \in \mathcal{K}} u_i^k(v_i)\, q^k(v_i, v_{-i}) - t_i(v_i, v_{-i}) \geq \sum_{k \in \mathcal{K}} u_i^k(v_i)\, q^k(v_i', v_{-i}) - t_i(v_i', v_{-i}).$$

Existing papers in the literature establish the equivalence of BIC and DIC mechanisms under certain conditions, and our paper proves the equivalence of stochastic and deterministic mechanisms. A natural question to ask is whether there is a class of environments in which, for any BIC mechanism, we can obtain an equivalent mechanism that is both DIC and deterministic. The following example illustrates that even in the single-unit auction setting, this is not the case.

Example 2 (Example 1 revisited)**.** Suppose that there exists an equivalent mechanism $\hat{q} = (\hat{q}^1, \hat{q}^2)$ that is both DIC and deterministic. It follows that for each v_2, there exists a threshold $\bar{v}_1(v_2)$ such that $\hat{q}^1(v_1, v_2) = 1$ for all $v_1 > \bar{v}_1(v_2)$ and $\hat{q}^1(v_1, v_2) = 0$ for all $v_1 < \bar{v}_1(v_2)$. Since the interim expected allocation probability of bidder 2 is $\frac{1}{2}$ for all v_2, it must be that $\bar{v}_1(v_2) \geq \frac{1}{2}$ for all v_2. By the arguments above, we have $\hat{q}^1(v_1, v_2) = 0$ for all $v_1 < \frac{1}{2}$ and v_2. Therefore, the interim expected allocation probability of bidder 1 is 0 for all $v_1 < \frac{1}{2}$. We arrive at a contradiction.[17]

4.3 Application: symmetric auctions

Symmetric auctions received a lot of attention in the mechanism design literature (see, for example, Border (1991) and Deb and Pai (2017)). Applying our results in Section 3, we show that for any mechanism with a symmetric allocation rule, there exists an equivalent deterministic mechanism that preserves symmetry. For simplicity of exposition, we prove this result in the single-unit auction setting.

Consider a single-unit auction with a finite set $\mathcal{I} = \{1, 2, \ldots, I\}$ of risk-neutral bidders $(I \geq 2)$. For all $i \in \mathcal{I}$, agent i's valuation v_i for the object is distributed according to λ_i with support on $V_i = [\underline{v}_i, \bar{v}_i] \subset \mathbb{R}_+$. The agents are ex ante symmetric; that is, $\lambda_1 = \lambda_2 = \ldots = \lambda_I$ and $V_1 = V_2 = \ldots = V_I$.

An allocation rule consists of $I+1$ functions $q = (q^1, q^2, \ldots, q^I, q^{I+1})$, where q^i is the

[17] The allocation rule q in Example 1 satisfies that $q^1(v_1, v_2)$ is nondecreasing in v_1 for all v_2 and that $q^2(v_1, v_2)$ is nondecreasing in v_2 for all v_1. As such, there exists a transfer scheme t such that (q, t) is DIC. Thus, our discussion here also implies that the equivalence of DIC mechanisms and mechanisms that are both DIC and deterministic does not hold, even in the single-unit auction setting.

probability of bidder i getting the object for $i \in \mathcal{I}$ and q^{I+1} is the probability of the seller keeping the object. Let Ψ be the set of all permutations on $\mathcal{I}$. An allocation rule q is said to be symmetric if $q^i(v_1, v_2, \ldots, v_I) = q^{\psi^{-1}(i)}(v_{\psi(1)}, v_{\psi(2)}, \ldots, v_{\psi(I)})$ for all $\psi \in \Psi$, $i \in \mathcal{I}$, and $v \in V$. Proposition 1 below shows that for any symmetric allocation rule, there exists a deterministic and symmetric allocation rule that induces the same interim expected allocation probabilities for all agents. This further implies that there exists an equivalent deterministic mechanism that preserves symmetry.

Proposition 1. *For any symmetric allocation rule q, there exists a deterministic and symmetric allocation rule $\hat{q}$ such that q and $\hat{q}$ induce the same interim expected allocation probabilities for all agents.*

4.4 Benefit of randomness revisited

In a recent contribution, Chawla, Malec, and Sivan (2015) consider a multi-agent setting and focus on the case in which the agents' valuations are independent both across different agents' types and different coordinates of an agent's type. They establish a constant factor upper bound for the benefit of randomness when the agents' values are independent. In the special case of multi-unit multi-item auctions, they show that the revenue of any Bayesian incentive compatible, individually rational randomized mechanism is at most 33.75 times the revenue of the optimal deterministic mechanism. In this paper, we push this result to the extreme and show that the revenue maximizing auction can be deterministically implemented.[18]

5 Conclusion

We show that in a general social choice environment with multiple agents, for any mechanism, there exists an equivalent deterministic mechanism. On the one hand, our result implies that it is without loss of generality to work with stochastic mechanisms, even if the designer does not have access to a randomization device, or cannot fully commit to the outcomes induced by a randomization device. On the other hand, our result implies that the requirement of deterministic mechanisms is not restrictive in itself. Even if one is constrained to use only deterministic mechanisms, there is no loss of revenue or social welfare.

[18] Chawla, Malec, and Sivan (2015, p. 316) remarked that "our bounds on the benefit of randomness are in some cases quite large and we believe they can be improved."

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A Appendix

A.1 Proof of Theorem 1

Step (1) For any allocation rule q, let

$$\Upsilon_q = \{g \in \Upsilon : \mathbb{E}(g|v_i) = \mathbb{E}(q|v_i) \text{ for all } i \in \mathcal{I} \text{ and } \lambda\text{-almost all } v_i \in V_i\}.$$

We first show that Υ_q is nonempty, convex, and weakly compact. Then, by the Krein-Milman Theorem (see Royden and Fitzpatrick (2010, p. 296)), Υ_q admits extreme points.

Lemma 1. *Υ_q is nonempty, convex, and weakly compact.*

Proof. Clearly, Υ_q is nonempty and convex. For weak compactness, it suffices to show that Υ_q is norm closed in $L_1^\lambda(V, \mathbb{R}^K)$, where $L_1^\lambda(V, \mathbb{R}^K)$ is the L_1 space of all integrable mappings from V to $\mathbb{R}^K$ under the probability measure λ. Since Υ_q is convex, by Mazur's Theorem (see Royden and Fitzpatrick (2010, p. 292)), Υ_q is also weakly closed in $L_1^\lambda(V, \mathbb{R}^K)$. Since Υ_q is bounded, it is uniformly integrable. By Theorem 12 in Royden and Fitzpatrick (2010, p. 412), Υ_q is weakly compact in $L_1^\lambda(V, \mathbb{R}^K)$.

In what follows, we show that Υ_q is norm closed in $L_1^\lambda(V, \mathbb{R}^K)$. Consider any sequence $\{g_m\} \subseteq \Upsilon_q$ such that $g_m \to g_0$ in norm in $L_1^\lambda(V, \mathbb{R}^K)$. We show that $g_0 \in \Upsilon_q$. By the Riesz-Fischer Theorem (see Royden and Fitzpatrick (2010, p. 398)), there exists a subsequence $\{g_{m_s}\}$ of $\{g_m\}$

such that $\{g_{m_s}\}$ converges to g_0 λ-almost everywhere. Since $\sum_{k\in\mathcal{K}} g^k_{m_s}(v) = 1$ for λ-almost all v, $\sum_{k\in\mathcal{K}} g^k_0(v) = 1$ for λ-almost all v. Therefore, $g_0 \in \Upsilon$.

Given any $i \in \mathcal{I}$, let $\mathcal{B}(V_i)$ be the Borel σ-algebra of the set V_i. For each $i \in \mathcal{I}$, and for any $\mathcal{B}(V_i) \otimes \big(\bigotimes_{j\neq i}\{V_j, \emptyset\}\big)$-measurable bounded mapping $p\colon V \to \mathbb{R}^K$,

$$\int_V (g_0 \cdot p)\, \lambda(\mathrm{d}v) = \lim_{s\to\infty} \int_V (g_{m_s} \cdot p)\, \lambda(\mathrm{d}v) = \int_V (q \cdot p)\, \lambda(\mathrm{d}v),$$

where the first equality follows from the dominated convergence theorem (see Royden and Fitzpatrick (2010, p. 88)), and the second equality holds since $\{g_{m_s}\} \subseteq \Upsilon_q$. By the arbitrary choice of bounded measurable mapping p, we obtain that $\mathbb{E}(g_0|v_i) = \mathbb{E}(q|v_i)$ for λ-almost all $v_i \in V_i$, which implies that $g_0 \in \Upsilon_q$.[19] □

Step (2) We show that all extremes points of Υ_q are deterministic for λ-almost all $v \in V$. Then, there exists $\tilde{g} \in \Upsilon_q$ that is deterministic for λ-almost all $v \in V$.

Lemma 2. *All extreme points of Υ_q are deterministic for λ-almost all $v \in V$.*

Proof. We prove the proposition by contraposition. We show that if $g' \in \Upsilon_q$ is not deterministic for λ-almost all $v \in V$, then g' is not an extreme point of Υ_q. Suppose that g' is not deterministic for λ-almost all $v \in V$. Then, there exists

(1) $0 < \delta < 1$;
(2) a Borel measurable set $D \subseteq V$ with $\lambda(D) > 0$; and
(3) indices $j_1, j_2 \in \mathcal{K}$

such that for all $v \in D$,

$$\delta \leq g'^{j_1}(v),\ g'^{j_2}(v) \leq 1 - \delta.$$

We proceed to show that there exist distinct $\bar{g}$, $\bar{\bar{g}} \in \Upsilon_q$ such that $g' = \frac{1}{2}(\bar{g} + \bar{\bar{g}})$. This establishes that g' is not an extreme point of Υ_q.

For any $i \in \mathcal{I}$, let D_i be the projection of D on V_i. For any $v_i \in D_i$, let $D_{-i}(v_i) = \{v_{-i} : (v_i, v_{-i}) \in D\}$. Consider the following system of equations where $\alpha \in L^\lambda_\infty(D, \mathbb{R})$ are the unknown:

$$\int_{D_{-i}(v_i)} \alpha(v_i, v_{-i})\, \lambda_{-i}(\mathrm{d}v_{-i}) = 0, \tag{5}$$

for all $i \in \mathcal{I}$ and $v_i \in D_i$.

Since λ_i is atomless for all $i \in \mathcal{I}$, one can show that besides the trivial solution that $\alpha = 0$, the system of equations (5) also has a nontrivial bounded solution α. The proof of this claim is technical and is contained in Appendix B of the Supplemental Material (Chen, He, Li, and Sun (2019)).[20]

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[19] Instead of showing $\mathbb{E}(g_0|v_i) = \mathbb{E}(q|v_i)$ directly, we work with integrals involving bounded measurable mapping p to avoid working with multiple null sets for the sequence of functions $\{g_{m_s}\}$ in Υ_q.

[20] We provide here a summary of the arguments for the existence of a nontrivial bounded solution α. Define $\mathscr{E}$ as follows:

$$\mathscr{E} = \Big\{ \sum_{i\in\mathcal{I}} \psi_i(v_i) : \psi_i \in L^\lambda_\infty(D_i, \mathbb{R}),\ \forall i \in \mathcal{I} \Big\}.$$

Without loss of generality, we assume that $|\alpha| \le \delta$. Since α is defined on D, we extend the domain of α to V by setting $\alpha(v) = 0$ whenever $v \notin D$. We construct $\bar{g}$ and $\bar{\bar{g}}$ as follows: for all $v \in V$,

$$\begin{aligned}\bar{g}(v) &= g'(v) + \alpha(v)\,(e_{j_1} - e_{j_2});\\ \bar{\bar{g}}(v) &= g'(v) + \alpha(v)\,(e_{j_2} - e_{j_1}),\end{aligned}$$

where e_{j_1} and e_{j_2} are the standard basis vectors in $\mathbb{R}^K$.

We proceed to verify that $\bar{g}$, $\bar{\bar{g}} \in \Upsilon_q$. To see that $\bar{g} \in \Upsilon$, note that

(1) $\sum_{k\in\mathcal{K}} \bar{g}^k(v) = \sum_{k\in\mathcal{K}} g'^k(v) = 1$ for λ-almost all $v \in V$;
(2) If $v \in D$, then $\delta \le g'^{j_1}(v)$, $g'^{j_2}(v) \le 1 - \delta$, which implies that $0 \le \bar{g}^{j_1}(v)$, $\bar{g}^{j_2}(v) \le 1$;
(3) If $v \in D$, then $\bar{g}^j(v) = g'^j(v)$ for $j \ne j_1$, j_2; and
(4) If $v \notin D$, then $\bar{g}^j(v) = g'^j(v)$ as $\alpha(v) = 0$.

Next, we show that $\bar{g} \in \Upsilon_q$. Fix any $i \in \mathcal{I}$. We obtain that for λ_i-almost all $v_i \in V_i$,

$$\begin{aligned}\int_{V_{-i}} \bar{g}(v_i, v_{-i})\lambda_{-i}(\mathrm{d}v_{-i}) &= \int_{V_{-i}} g'(v_i, v_{-i})\lambda_{-i}(\mathrm{d}v_{-i}) + \int_{V_{-i}} \alpha(v_i, v_{-i})(e_{j_1} - e_{j_2})\,\lambda_{-i}(\mathrm{d}v_{-i})\\ &= \int_{V_{-i}} q(v_i, v_{-i})\lambda_{-i}(\mathrm{d}v_{-i}) + \left[\int_{V_{-i}} \alpha(v_i, v_{-i})\,\lambda_{-i}(\mathrm{d}v_{-i})\right](e_{j_1} - e_{j_2})\\ &= \int_{V_{-i}} q(v_i, v_{-i})\lambda_{-i}(\mathrm{d}v_{-i}).\end{aligned}$$

By similar reasoning, one can show that $\bar{\bar{g}} \in \Upsilon_q$. Since $\bar{g}$ and $\bar{\bar{g}}$ are distinct and $g' = \frac{1}{2}(\bar{g} + \bar{\bar{g}})$, g' is not an extreme point of Υ_q. □

Step (3) Fix $\tilde{g} \in \Upsilon_q$ that is deterministic for λ-almost all $v \in V$. Note that (1) $\tilde{g}$ is deterministic for λ-almost all $v \in V$, but not for all $v \in V$; and (2) $\tilde{g}$ induces the same interim expected allocation probabilities for all $i \in \mathcal{I}$ and λ_i-almost all $v_i \in V_i$, but not for all $v_i \in V_i$. We now construct a deterministic allocation rule $\hat{g}$ that induces the same interim expected allocation probabilities for all $i \in \mathcal{I}$ and $v_i \in V_i$, by modifying $\tilde{g}$ on sets of measure zero.

Let $D' = \{v \in V : \tilde{g}^k(v) \in (0,1)$ for some k or $\sum_{k\in\mathcal{K}} \tilde{g}^k(v) \ne 1\}$. Since $\tilde{g}$ is deterministic for almost all $v \in V$, $\lambda(D') = 0$. Define a new allocation rule $\tilde{\tilde{g}}$ as follows:

$$\tilde{\tilde{g}}(v) = \begin{cases}(1,\, 0,\, \ldots,\, 0), & \text{if } v \in D';\\ \tilde{g}(v), & \text{otherwise.}\end{cases}$$

Then $\tilde{\tilde{g}}$ is deterministic for all $v \in V$. Note that $\tilde{g}$ and q induce the same interim expected allocation probabilities for all $i \in \mathcal{I}$ and λ_i-almost all $v_i \in V_i$ (the relevant null sets could be very different

Then a bounded measurable function $\alpha \in L_\infty^\lambda(D, \mathbb{R})$ is a solution to the system of equations (5) if and only if $\int_D \alpha(v)\varphi(v)\,\lambda(dv) = 0$ for any $\varphi(v) \in \mathscr{E}$. Thus, our objective is to show that $\mathscr{E}$ is not dense in $L_1^\lambda(D, \mathbb{R})$. Note that the set $\mathscr{E}$ is the collection of all the integrable and additively separable functions, while $L_1^\lambda(D, \mathbb{R})$ is the collection of all the integrable multivariate functions. The key idea of the proof is hence to identify a multivariate bounded measurable function α that cannot be approximated by additively separable functions. If D is a measurable rectangle $\prod_{i\in\mathcal{I}} D_i$, then one can simply take α to be the product $\prod_{i\in\mathcal{I}} \alpha_i$, where the univariate bounded measurable function α_i is nontrivial and has zero integral on D_i. Because D is a general Borel measurable set without any product structure that allows for such a direct separation of variables, the proof for the general case is rather involved.

from the null set D'). Since $\tilde{\tilde{g}}$ is modified from $\tilde{g}$ on the null set D', $\tilde{\tilde{g}}$ and q induce the same interim expected allocation probabilities for all $i \in \mathcal{I}$ and λ_i-almost all $v_i \in V_i$. That is, for all $i \in \mathcal{I}$ and λ_i-almost all $v_i \in V_i$,

$$\int_{V_{-i}} \tilde{\tilde{g}}(v_i, v_{-i})\, \lambda_{-i}(\mathrm{d}v_{-i}) = \int_{V_{-i}} q(v_i, v_{-i})\, \lambda_{-i}(\mathrm{d}v_{-i}). \tag{6}$$

For each $i \in \mathcal{I}$, let D_i'' be the subset of V_i such that Equation (6) does not hold. Then, $\lambda_i(D_i'') = 0$. By Proposition 10.7.6 in Bogachev (2007), for each $i \in \mathcal{I}$, there exists a deterministic allocation rule q_i on $D_i'' \times V_{-i}$ such that for all $v_i \in D_i''$,

$$\int_{V_{-i}} q_i(v_i, v_{-i})\, \lambda_{-i}(\mathrm{d}v_{-i}) = \int_{V_{-i}} q(v_i, v_{-i})\, \lambda_{-i}(\mathrm{d}v_{-i}). \tag{7}$$

Since the sets $\{D_i'' \times V_{-i}\}_{i \in \mathcal{I}}$ may not be disjoint, we construct an allocation rule $\hat{q}$ by taking a suitable decomposition of $\cup_{1 \le i \le I} (D_i'' \times V_{-i})$ as follows:

$$\hat{q}(v) = \begin{cases} q_1(v), & \text{if } v \in (D_1'' \times V_{-1}) \setminus \cup_{2 \le i \le I} (D_i'' \times V_{-i}); \\ q_2(v), & \text{if } v \in (D_2'' \times V_{-2}) \setminus \cup_{3 \le i \le I} (D_i'' \times V_{-i}); \\ \ldots, & \ldots; \\ q_I(v), & \text{if } v \in D_I'' \times V_{-I}; \\ \tilde{\tilde{g}}(v), & v \in V \setminus \cup_{1 \le i \le I} (D_i'' \times V_{-i}). \end{cases}$$

It follows from the construction of $\tilde{\tilde{g}}$ and q_i for all $i \in \mathcal{I}$ that $\hat{q}$ is deterministic for all $v \in V$. We now proceed to verify that $\hat{q}$ and q induce the same interim expected allocation probabilities for all $i \in \mathcal{I}$ and $v_i \in V_i$.

Fix $i \in \mathcal{I}$ and $v_i \in V_i$. If $v_i \in D_i''$, by the definition of $\hat{q}$,

$$\begin{aligned} \int_{V_{-i}} \hat{q}(v_i, v_{-i})\, \lambda_{-i}(\mathrm{d}v_{-i}) &= \int_{\bar{D}_{-i}} \hat{q}(v_i, v_{-i})\, \lambda_{-i}(\mathrm{d}v_{-i}) + \int_{V_{-i} \setminus \bar{D}_{-i}} \hat{q}(v_i, v_{-i})\, \lambda_{-i}(\mathrm{d}v_{-i}) \\ &= 0 + \int_{V_{-i}} q_i(v_i, v_{-i})\, \lambda_{-i}(\mathrm{d}v_{-i}) \\ &= \int_{V_{-i}} q(v_i, v_{-i})\, \lambda_{-i}(\mathrm{d}v_{-i}), \end{aligned}$$

where $\bar{D}_{-i} = \cup_{j \in \mathcal{I}, j \neq i} (D_j'' \times \prod_{k \in \mathcal{I}, k \neq i, j} V_k)$. The second line follows from that $\lambda_j(D_j'') = 0$ for all $j \in \mathcal{I}$, and the third line follows from (7).

If $v_i \notin D_i''$, by the definition of $\hat{q}$,

$$\begin{aligned} \int_{V_{-i}} \hat{q}(v_i, v_{-i})\, \lambda_{-i}(\mathrm{d}v_{-i}) &= \int_{\bar{D}_{-i}} \hat{q}(v_i, v_{-i})\, \lambda_{-i}(\mathrm{d}v_{-i}) + \int_{V_{-i} \setminus \bar{D}_{-i}} \hat{q}(v_i, v_{-i})\, \lambda_{-i}(\mathrm{d}v_{-i}) \\ &= 0 + \int_{V_{-i}} \tilde{\tilde{g}}(v_i, v_{-i})\, \lambda_{-i}(\mathrm{d}v_{-i}) \\ &= \int_{V_{-i}} q(v_i, v_{-i})\, \lambda_{-i}(\mathrm{d}v_{-i}), \end{aligned}$$

where the second line follows from that $\lambda_j(D_j'') = 0$ for all $j \in \mathcal{I}$, and the third line follows from (6).

A.2 Proof of Proposition 1

Let ψ_0 be the identity mapping on $\mathcal{I}$. For each $\psi \in \Psi$, define

$$D_\psi = \{v \in V\colon\ v_{\psi(1)} > v_{\psi(2)} > \ldots > v_{\psi(I)}\}.$$

Clearly,

1. for distinct ψ, $\psi' \in \Psi$, $D_\psi \cap D_{\psi'} = \emptyset$;
2. $\lambda(\cup_{\psi\in\Psi} D_\psi) = 1$;
3. $v \in D_\psi$ for some ψ if and only if $(v_{\psi(1)}, v_{\psi(2)}, \ldots, v_{\psi(I)}) \in D_{\psi_0}$.

The structure of the proof is as follows. For an arbitrary symmetric allocation rule q, Step (1) constructs a deterministic allocation rule $\hat{q}$, Step (2) shows that $\hat{q}$ is symmetric, and Step (3) shows that q and $\hat{q}$ induce the same interim expected allocation probabilities for all agents.

Step (1) By Theorem 2, there exists a deterministic allocation rule $\tilde{q} = (\tilde{q}^1, \tilde{q}^2, \ldots, \tilde{q}^I, \tilde{q}^{I+1})$ such that for all $i, j \in \mathcal{I}$ and $v_j \in V_j$,

$$\int_{V_{-j}} \mathbf{1}_{D_{\psi_0}}(v_j, v_{-j})\tilde{q}^i(v_j, v_{-j})\, \lambda_{-j}(\mathrm{d}v_{-j}) = \int_{V_{-j}} \mathbf{1}_{D_{\psi_0}}(v_j, v_{-j})q^i(v_j, v_{-j})\, \lambda_{-j}(\mathrm{d}v_{-j}). \tag{8}$$

We are going to define a deterministic and symmetric allocation rule $\hat{q}$ so that it is identical with $\tilde{q}$ on D_{ψ_0}. By symmetry, we know that the values of $\hat{q}$ on D_ψ are completely determined by its values on D_{ψ_0} via the permutation ψ. More specifically, the deterministic allocation rule $\hat{q} = (\hat{q}^1, \hat{q}^2, \ldots, \hat{q}^I, \hat{q}^{I+1})$ is defined as follows: for $i \in \mathcal{I}$,

$$\hat{q}^i(v) = \begin{cases} \tilde{q}^{\psi^{-1}(i)}(v_{\psi(1)}, v_{\psi(2)}, \ldots, v_{\psi(I)}) & \text{if } v \in D_\psi \text{ for some } \psi \in \Psi, \\ 0 & \text{otherwise.} \end{cases}$$

Step (2) We show that $\hat{q}$ is symmetric. That is, for all $\psi' \in \Psi$, $i \in \mathcal{I}$, and $v \in V$,

$$\hat{q}^i(v_1, v_2, \ldots, v_I) = \hat{q}^{(\psi')^{-1}(i)}(v_{\psi'(1)}, v_{\psi'(2)}, \ldots, v_{\psi'(I)}).$$

Fix $\psi' \in \Psi$. We first consider the case in which $v \notin D_\psi$ for all $\psi \in \Psi$. Then

$$(v_{\psi'(1)}, v_{\psi'(2)}, \ldots, v_{\psi'(I)}) \notin D_\psi$$

for any $\psi \in \Psi$. It follows from the definition of $\hat{q}$ that for all $i \in \mathcal{I}$,

$$\hat{q}^i(v) = \hat{q}^{(\psi')^{-1}(i)}(v_{\psi'(1)}, v_{\psi'(2)}, \ldots, v_{\psi'(I)}) = 0.$$

Next, suppose that $v \in D_\psi$ for some $\psi \in \Psi$. For notational simplicity, we relabel $v_{\psi'(i')}$ by $\tilde{v}_{i'}$ for all $i' \in \mathcal{I}$. Let $\tilde{\psi} = (\psi')^{-1} \circ \psi$. Since

$$\tilde{v}_{\tilde{\psi}(i')} = \tilde{v}_{(\psi')^{-1}\circ\psi(i')} = v_{\psi'\circ(\psi')^{-1}\circ\psi(i')} = v_{\psi(i')}$$

and $v \in D_\psi$, we have $\tilde{v} \in D_{\tilde{\psi}}$. Therefore, for all $i \in \mathcal{I}$,

$$\begin{aligned} \hat{q}^{(\psi')^{-1}(i)}(v_{\psi'(1)}, v_{\psi'(2)}, \ldots, v_{\psi'(I)}) &= \hat{q}^{(\psi')^{-1}(i)}(\tilde{v}_1, \tilde{v}_2, \ldots, \tilde{v}_I) \\ &= \tilde{q}^{\tilde{\psi}^{-1}\circ(\psi')^{-1}(i)}(\tilde{v}_{\tilde{\psi}(1)}, \tilde{v}_{\tilde{\psi}(2)}, \ldots, \tilde{v}_{\tilde{\psi}(I)}) \end{aligned}$$

$$= \tilde{q}^{\psi^{-1}(i)}(v_{\psi(1)}, v_{\psi(2)}, \ldots, v_{\psi(I)})$$
$$= \hat{q}^i(v_1, v_2, \ldots, v_I),$$

where the first equality holds because of the relabeling, the second equality follows from the definition of $\hat{q}$ and the fact that $\tilde{v} \in D_{\tilde{\psi}}$, the third equality holds because $\psi' \circ \tilde{\psi} = \psi$ and $\tilde{v}_{\tilde{\psi}(i')} = v_{\psi(i')}$ for all $i' \in \mathcal{I}$, and the last equality follows from the definition of $\hat{q}$ and the fact that $v \in D_\psi$.

Step (3) We show that $\hat{q}$ and q induce the same interim expected allocation probabilities for all agents. For all $i, j \in \mathcal{I}$ and $v_j \in V_j$, we have

$$\begin{aligned}
&\int_{V_{-j}} \hat{q}^i(v_1, v_2, \ldots, v_I)\, \lambda_{-j}(\mathrm{d}v_{-j}) \\
&= \sum_{\psi\in\Psi} \int_{V_{-j}} \mathbf{1}_{D_\psi}(v_1, v_2, \ldots, v_I)\hat{q}^i(v_1, v_2, \ldots, v_I)\, \lambda_{-j}(\mathrm{d}v_{-j}) \\
&= \sum_{\psi\in\Psi} \int_{V_{-j}} \mathbf{1}_{D_\psi}(v_1, v_2, \ldots, v_I)\tilde{q}^{\psi^{-1}(i)}(v_{\psi(1)}, v_{\psi(2)}, \ldots, v_{\psi(I)})\, \lambda_{-j}(\mathrm{d}v_{-j}) \\
&= \sum_{\psi\in\Psi} \int_{V_{-j}} \mathbf{1}_{D_{\psi_0}}(v_{\psi(1)}, v_{\psi(2)}, \ldots, v_{\psi(I)})\tilde{q}^{\psi^{-1}(i)}(v_{\psi(1)}, v_{\psi(2)}, \ldots, v_{\psi(I)})\, \lambda_{-j}(\mathrm{d}v_{-j}) \\
&= \sum_{\psi\in\Psi} \int_{V_{-j}} \mathbf{1}_{D_{\psi_0}}(v_{\psi(1)}, v_{\psi(2)}, \ldots, v_{\psi(I)})q^{\psi^{-1}(i)}(v_{\psi(1)}, v_{\psi(2)}, \ldots, v_{\psi(I)})\, \lambda_{-j}(\mathrm{d}v_{-j}) \\
&= \sum_{\psi\in\Psi} \int_{V_{-j}} \mathbf{1}_{D_{\psi_0}}(v_{\psi(1)}, v_{\psi(2)}, \ldots, v_{\psi(I)})q^i(v_1, v_2, \ldots, v_I)\, \lambda_{-j}(\mathrm{d}v_{-j}) \\
&= \sum_{\psi\in\Psi} \int_{V_{-j}} \mathbf{1}_{D_\psi}(v_1, v_2, \ldots, v_I)q^i(v_1, v_2, \ldots, v_I)\, \lambda_{-j}(\mathrm{d}v_{-j}) \\
&= \int_{V_{-j}} q^i(v_1, v_2, \ldots, v_I)\, \lambda_{-j}(\mathrm{d}v_{-j}).
\end{aligned}$$

The first and seventh equalities hold because $\lambda(\cup_{\psi\in\Psi} D_\psi) = 1$. The second equality follows from the definition of $\hat{q}$. The third and sixth equalities hold since $v \in D_\psi$ if and only if $(v_{\psi(1)}, v_{\psi(2)}, \ldots, v_{\psi(I)}) \in D_{\psi_0}$. The fourth equality follows from (8). The fifth equality follows from the symmetry of q.

A.3 The limits of the equivalence result

For atomless distribution, separable payoffs, and independent types, we present examples here to illustrate that our mechanism equivalence result breaks down if each of these assumptions is violated. Recall that our approach of proving the existence of equivalent deterministic mechanism keeps the ex post transfers unchanged. In the same vein, we also require that the transfers be kept unchanged in the examples on separable payoffs and independent types.

To prove that there exists a deterministic allocation rule that delivers the same interim expected allocation probabilities for all agents, we do not require the assumption of independent types. We invoke additional assumptions of separable payoffs and independent types to establish that the equivalent deterministic mechanism is BIC. When types are correlated, our approach no longer works. However, under certain conditions, one can invoke the Crémer-Mclean type arguments to approximately satisfy the incentive constraints by adjusting transfers. Indeed, by combining the results in McAfee and Reny (1992) and Miller, Pratt, Zeckhauser, and Johnson (2007), one can show that under the conditions in their papers, there exists a deterministic mechanism that is approximately equivalent.

Example 3 (Atomless distribution)**.** Consider a setting with two agents. Suppose that λ_1 has an atom $d \in V_1$ with $\lambda_1(d) > 0$. Note that the assumption of atomless distribution is violated. Consider the following mapping $q = (q^1, q^2, \ldots, q^K)$:

$$q(v_1, v_2) = \begin{cases} (\frac{1}{2}, \frac{1}{2}, 0, \ldots, 0) & \text{if } v_1 = d; \\ (0, 1, 0, \ldots, 0) & \text{otherwise.} \end{cases}$$

Since the outcome only depends on the report of agent 1, for any $v_2 \in V_2$,

$$\int_{V_1} q^1(v_1, v_2)\, \lambda_1(\mathrm{d}v_1) = \int_{V_1 \setminus \{d\}} q^1(v_1, v_2)\, \lambda_1(\mathrm{d}v_1) + \lambda_1(d)\, q^1(d, v_2) = \frac{1}{2}\, \lambda_1(d). \tag{9}$$

We claim that there does not exist a deterministic allocation rule that delivers the same interim expected allocation probabilities for all agents. Suppose to the contrary, such a deterministic allocation rule $\hat{q}$ exists. For all $v_1 \neq d$,

$$\int_{V_2} \hat{q}^1(v_1, v_2)\, \lambda_2(\mathrm{d}v_2) = \int_{V_2} q^1(v_1, v_2)\, \lambda_2(\mathrm{d}v_2) = 0,$$

which implies that $\hat{q}^1(v_1, v_2) = 0$ for λ_2-almost all $v_2 \in V_2$.

By Fubini's Theorem (see Royden and Fitzpatrick (2010, p. 416)), for λ_2-almost all $v_2 \in V_2$, $\hat{q}^1(v_1, v_2) = 0$ for λ_1-almost all $v_1 \in V_1 \setminus \{d\}$. Therefore, for λ_2-almost all $v_2 \in V_2$,

$$\begin{aligned} &\int_{V_1} q^1(v_1, v_2)\, \lambda_1(\mathrm{d}v_1) \\ &= \int_{V_1} \hat{q}^1(v_1, v_2)\, \lambda_1(\mathrm{d}v_1) \\ &= \int_{V_1 \setminus \{d\}} \hat{q}^1(v)\, \lambda_1(\mathrm{d}v_1) + \lambda_1(d)\, \hat{q}^1(d, v_2) \\ &= \lambda_1(d)\, \hat{q}^1(d, v_2). \end{aligned} \tag{10}$$

It follows from (9) and (10) that $\hat{q}^1(d, v_2) = \frac{1}{2}$ for λ_2-almost all $v_2 \in V_2$, which contradicts the assumption that $\hat{q}(d, v_2)$ is either 0 or 1.

Example 4 (Separable payoff)**.** Consider a single-unit common value auction with two bidders. The bidders' valuations for the object $v = (v_1, v_2)$ are uniformly distributed on the square $[0, 1]^2$. Let λ denote the uniform distribution on the square $[0, 1]^2$. Each agent's payoff is 1 if she gets the object and $v_1 + v_2 \geq 1$, and 0 otherwise. More succinctly, the payoff function of bidder i is $\mathbf{1}_{[1-v_i,1]}(v_j)$ if she gets the object and 0 otherwise. Note that the assumption of separable payoffs is violated.

Consider the allocation rule $q = (q^1, q^2)$ with $q^1(v) = q^2(v) = 1/2$ for all v, where q^i is the probability of bidder i getting the object for $i \in \{1, 2\}$. We claim that there does not exist a BIC and deterministic allocation rule that delivers the same interim expected utilities for all bidders. Suppose to the contrary, such a BIC and deterministic allocation rule $\hat{q} = (\hat{q}^1, \hat{q}^2)$ exists. Then the payoff of bidder 1 for any v_1 is

$$\int_{V_2} \mathbf{1}_{[1-v_1,1]}(v_2)\, \hat{q}^1(v_1, v_2)\; \mathrm{d}v_2$$

$$
\begin{aligned}
&= \int_{V_2} \mathbf{1}_{[1-v_1,1]}(v_2)\, q^1(v_1,v_2)\; \mathrm{d}v_2 \\
&= \frac{1}{2}\int_{V_2} \mathbf{1}_{[1-v_1,1]}(v_2)\; \mathrm{d}v_2 \\
&= \frac{v_1}{2}.
\end{aligned}
$$

Since $\hat{q}$ is BIC, for $v_1 < 1$, agent 1 has an incentive to truthfully report her type than to misreport $v_1+\epsilon$ (where ϵ is sufficiently small such that $v_1+\epsilon \le 1$). That is,

$$
\begin{aligned}
\frac{v_1}{2} &\ge \int_{V_2} \mathbf{1}_{[1-v_1,1]}(v_2)\, \hat{q}^1(v_1+\epsilon, v_2)\; \mathrm{d}v_2 \\
&= \int_{1-v_1}^{1} \hat{q}^1(v_1+\epsilon, v_2)\; \mathrm{d}v_2 \\
&= \int_{1-v_1-\epsilon}^{1} \hat{q}^1(v_1+\epsilon, v_2)\; \mathrm{d}v_2 - \int_{1-v_1-\epsilon}^{1-v_1} \hat{q}^1(v_1+\epsilon, v_2)\; \mathrm{d}v_2 \\
&= \frac{v_1+\epsilon}{2} - \int_{1-v_1-\epsilon}^{1-v_1} \hat{q}^1(v_1+\epsilon, v_2)\; \mathrm{d}v_2
\end{aligned}
$$

Rearrange the inequality, we have

$$
\int_{1-v_1}^{1-v_1+\epsilon} \hat{q}^1(v_1,v_2)\; \mathrm{d}v_2 \ge \frac{\epsilon}{2}.
$$

Fix any $\epsilon' > 0$, consider the region

$$
D = \{(v_1,v_2) :\, 1 \le v_1+v_2 \le 1+\epsilon',\, 0 \le v_1 \le 1,\, 0 \le v_2 \le 1\}.
$$

We have

$$
\int_D \hat{q}^1(v_1,v_2)\; \mathrm{d}\lambda(v_1,v_2) = \int_0^1 \int_{1-v_1}^{\min\{1,1-v_1+\epsilon'\}} \hat{q}^1(v_1,v_2)\; \mathrm{d}v_2\; \mathrm{d}v_1 \ge \frac{1}{2}\,\lambda(D). \tag{11}
$$

Since the model and the allocation rule are symmetric with respect to agents 1 and 2, we can switch indices 1 and 2 in Equation (11) to obtain that

$$
\int_D \hat{q}^2(v_1,v_2)\; \mathrm{d}\lambda(v_1,v_2) = \int_0^1 \int_{1-v_2}^{\min\{1,1-v_2+\epsilon'\}} \hat{q}^2(v_1,v_2)\; \mathrm{d}v_1\; \mathrm{d}v_2 \ge \frac{1}{2}\,\lambda(D). \tag{12}
$$

By the feasibility constraint that $\hat{q}^1(v) + \hat{q}^2(v) \le 1$ for all v, we have

$$
\int_D \hat{q}^1(v_1,v_2)\; \mathrm{d}\lambda(v_1,v_2) + \int_D \hat{q}^2(v_1,v_2)\; \mathrm{d}\lambda(v_1,v_2) \le \int_D 1\; \mathrm{d}\lambda(v_1,v_2) = \lambda(D), \tag{13}
$$

It then follows from (11) - (13) that

$$
\int_D \hat{q}^1(v_1,v_2)\; \mathrm{d}\lambda(v_1,v_2) = \int_D \hat{q}^2(v_1,v_2)\; \mathrm{d}\lambda(v_1,v_2) = \frac{1}{2}\,\lambda(D).
$$

This implies that for almost all v_1, for all ϵ such that $\epsilon \le v_1$,

$$
\int_{1-v_1}^{1-v_1+\epsilon} \hat{q}^1(v_1,v_2)\; \mathrm{d}v_2 = \frac{\epsilon}{2}.
$$

That is, for almost all v_1, for any $1 - v_1 \leq a \leq b \leq 1$,

$$\int_a^b \hat{q}^1(v_1, v_2)\, \mathrm{d}v_2 = \frac{b-a}{2}.$$

Therefore, for almost all v_1, $\hat{q}^1(v_1, v_2) = \frac{1}{2}$ for almost all $v_2 \geq 1 - v_1$. We arrive at contradiction.

Example 5 (Independent types)**.** Consider a single-unit auction with two bidders. Let $V_1 = V_2 = [0, 1]$ be endowed with the joint distribution λ, which has density $\rho(v_1, v_2) = 2$ if $v_1 + v_2 \geq 1$ and 0 otherwise. The payoff function of bidder i is 1 if she gets the good and 0 otherwise. Note that the assumption of independent types is violated.

Consider the allocation rule $q = (q^1, q^2)$ with $q^1(v) = q^2(v) = 1/2$ for all v, where q^i is the probability of bidder i getting the object for $i \in \{1, 2\}$. We claim that there does not exist a BIC and deterministic allocation rule that delivers the same interim expected utilities for all bidders. Suppose to the contrary, such a BIC and deterministic allocation rule $\hat{q} = (\hat{q}^1, \hat{q}^2)$ exists. Then the payoff of bidder 1 for any $v_1 > 0$ is

$$\int_{1-v_1}^1 \hat{q}^1(v_1, v_2)\, \frac{1}{v_1}\, \mathrm{d}v_2 = \int_{1-v_1}^1 q^1(v_1, v_2)\, \frac{1}{v_1}\, \mathrm{d}v_2 = \frac{1}{2} \int_{1-v_1}^1 \frac{1}{v_1}\, \mathrm{d}v_2 = \frac{1}{2}.$$

Since $\hat{q}$ is BIC, for any $0 < v_1 < 1$, agent 1 has an incentive to truthfully report her type than to misreport $v_1' > v_1$. We have

$$\begin{aligned}
\frac{1}{2} &\geq \int_{1-v_1}^1 \hat{q}^1(v_1', v_2)\, \frac{1}{v_1}\, \mathrm{d}v_2 \\
&= \int_{1-v_1'}^1 \hat{q}^1(v_1', v_2)\, \frac{1}{v_1}\, \mathrm{d}v_2 - \int_{1-v_1'}^{1-v_1} \hat{q}^1(v_1', v_2)\, \frac{1}{v_1}\, \mathrm{d}v_2 \\
&= \frac{v_1'}{v_1} \int_{1-v_1'}^1 \hat{q}^1(v_1', v_2)\, \frac{1}{v_1'}\, \mathrm{d}v_2 - \frac{1}{v_1} \int_{1-v_1'}^{1-v_1} \hat{q}^1(v_1', v_2)\, \mathrm{d}v_2 \\
&= \frac{v_1'}{2v_1} - \frac{1}{v_1} \int_{1-v_1'}^{1-v_1} \hat{q}^1(v_1', v_2)\, \mathrm{d}v_2.
\end{aligned}$$

Rearranging the inequality, we have

$$\int_{1-v_1'}^{1-v_1} \hat{q}^1(v_1', v_2)\, \mathrm{d}v_2 \geq \frac{v_1' - v_1}{2}.$$

Using similar arguments as in Example 4, we can show that for almost all v_1, $\hat{q}^1(v_1, v_2) = \frac{1}{2}$ for almost all $v_2 \geq 1 - v_1$. We arrive at a contradiction.