Equivalence of Stochastic and Deterministic Mechanisms*

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Abstract

We consider a general social choice environment that has multiple agents, a finite set of alternatives, and independent and dispersed information. We show that for any Bayesian incentive compatible mechanism, there exists an equivalent deterministic mechanism that (1) is Bayesian incentive compatible; (2) delivers the same interim expected allocation probabilities and the same interim expected utilities for all agents; and (3) delivers the same ex ante expected social surplus. This result holds in settings with a rich class of utility functions, multi-dimensional types, interdependent valuations, and in settings without monetary transfers. To prove our result, we develop a novel methodology of mutual purification, and establish its link with the mechanism design literature.

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1 Introduction

Myerson (1981) provides the framework that has become the paradigm for the study of optimal auction design. Under a regularity condition, the optimal auction allocates the object to the bidder with the highest virtual value, provided that this virtual value is above the seller's opportunity cost. In other words, the optimal auction in Myerson's setting is deterministic.[1]

A natural conjecture is that the optimality of deterministic mechanisms generalizes beyond Myerson's setting. McAfee and McMillan (1988) claim that under a general regularity condition on consumers' demand, stochastic delivery is not optimal for a multi-product monopolist. However, this result has been proven to be incorrect in settings with a single agent. Several papers have shown that a multi-product monopolist may find it beneficial to include lotteries as part of the selling mechanism; see, for example, Thanassoulis (2004), Manelli and Vincent (2006, 2007), Pycia (2006), Pavlov (2011), and more recently, Hart and Reny (2015) and Rochet and Thanassoulis (2017).[2] In this paper, we prove a mechanism equivalence result that implies the optimality of deterministic mechanisms in remarkably general environments with multiple agents.

We consider a general social choice environment that has multiple agents, a finite set of alternatives, and independent and dispersed information. We show that for any Bayesian incentive compatible mechanism, there exists an equivalent deterministic mechanism that (1) is Bayesian incentive compatible; (2) delivers the same interim expected allocation probabilities and the same interim expected utilities for all agents; and (3) delivers the same ex ante expected social surplus. In addition to the standard social choice environments with linear utilities and one-dimensional, private types, our result holds in settings with a rich class of utility functions, multi-dimensional types, interdependent valuations, and in settings without monetary transfers.

[1] Also see Riley and Zeckhauser (1983) who consider a one-good monopolist selling to consumers with unit demand and show that lotteries do not help the one-good monopolist.

[2] In environments in which different types are associated with different risk attitudes, it is known that stochastic mechanisms may perform better; see, for example, Laffont and Martimort (2002, p. 67) and Strausz (2006). Gauthier and Laroque (2014) propose a new technique in solving optimization problems, and apply this technique to study when a deterministic local optimum can be locally improved upon by a stochastic deviation in adverse selection and moral hazard problems. Aggarwal, Fiat, Goldberg, Hartline, Immorlica, and Sudan (2011) also study derandomization of auctions. They focus on prior-free auctions, rather than the Bayesian setting.

Our result implies that any mechanism, including the optimal mechanism (whether in terms of revenue or efficiency), can be implemented using a deterministic mechanism and nothing can be gained from designing more intricate mechanisms with possibly more complex randomization in the allocation rule. As pointed out in Hart and Reny (2015, p. 912), Aumann commented that it is surprising that randomization cannot increase revenue when there is only one good. Indeed, aforementioned papers in the screening literature establish that randomization helps when there are multiple goods. Nevertheless, we show that in general social choice environment with multiple agents, the revenue maximizing mechanism can always be deterministically implemented. This is in sharp contrast with the results in the screening literature.

Our result has important implications beyond the revenue contrast. The mechanism design literature essentially builds on the assumption that a mechanism designer can credibly commit to any outcome of a mechanism. This requirement implies that any outcome of the mechanism must be verifiable before it can be employed. In this vein, a stochastic mechanism demands not only that a randomization device be available to the mechanism designer, but also that the outcome of the randomization device be objectively verified. As noted in Laffont and Martimort (2002, p. 67),

> Ensuring this verifiability is a more difficult problem than ensuring that a deterministic mechanism is enforced, because any deviation away from a given randomization can only be statistically detected once a sufficient number of realizations of the contracts have been observed. ... The enforcement of such stochastic mechanisms in a bilateral one-shot relationship is thus particularly problematic. This has led scholars to give up those random mechanisms or, at least, to focus on economic settings where they are not optimal.[3]

Our result implies that every mechanism can in fact be deterministically implemented, and thereby irons out the conceptual difficulties associated with stochastic mechanisms.[4]

[3] Also see Bester and Strausz (2001) and Strausz (2003).

[4] There are other ways to circumvent the problem that the designer is not able to commit to outcomes induced by randomization devices. For example, probabilities in the selling mechanism can be considered as the discount factor from a temporal interpretation (e.g. as in Salant (1989)), and the designer is committing to a delay rather than committing to randomizing. Our contribution in this paper is to show that we do not have to think about any changes in the model, and that the randomization in the allocation rule can be fully absorbed using the agents' private information.

Along the lines of implementation, our paper is related to the literature of reduced form implementation (see, for example, Border (1991) for the case of the single-unit auction and Cai, Daskalakis, and Weinberg (2018) for the case of multi-item auction). In many applications of mechanism design, it is convenient to work with interim expected allocation probabilities. The reduced form implementation literature asks what interim expected allocation probabilities can be implemented. Our result implies that whatever interim expected allocation probabilities that can be implemented can actually be implemented in a deterministic manner. As such, even if the mechanism designer does not have access to randomization devices or cannot commit to the outcomes induced by randomization devices, we can rest assured working with interim expected allocation probabilities.

This paper joins the strand of literature that studies mechanism equivalence. Though motivations vary, these results show that it is without loss of generality to consider the various subclasses of mechanisms. As in the case of dominant-strategy mechanisms (see Manelli and Vincent (2010) and Gershkov, Goeree, Kushnir, Moldovanu, and Shi (2013)) and symmetric auctions (see Deb and Pai (2017)), our findings imply that the requirement of deterministic mechanisms is not restrictive in itself.[5]

To prove the existence of an equivalent deterministic mechanism, we develop a new methodology of mutual purification and establish its link with the literature of mechanism design.[6] The notion of mutual purification is both conceptually and technically different from the usual purification principle in the literature related to Bayesian games. We clarify these two different notions of purification in the next three paragraphs.

It follows from the general purification principle in Dvoretzky, Wald, and Wolfowitz (1950) that any behavioral-strategy Nash equilibrium in a finite-action Bayesian game with independent and dispersed information corresponds to some pure-strategy Bayesian Nash

[5] Manelli and Vincent (2010) show that for any Bayesian incentive compatible auction, there exists an equivalent dominant-strategy incentive compatible auction that yields the same interim expected utilities for all agents. Gershkov, Goeree, Kushnir, Moldovanu, and Shi (2013) extend this equivalence result to social choice environments with linear utilities and independent, one-dimensional, private types; also see Footnote 11 for related discussion. Deb and Pai (2017) show that restricting the seller to using a symmetric auction imposes virtually no restriction on her ability to achieve discriminatory outcomes. Others papers that study mechanism equivalence include Border (1991), Eso and Futo (1999), Börgers and Norman (2009).

[6] Some of our technical results extend the corresponding mathematical results in Arkin and Levin (1972); see the Supplemental Material (Chen, He, Li, and Sun (2018)) for a detailed discussion.

equilibrium with the same payoff.[7] In particular, independent and dispersed information allows the agents to replace their behavioral strategies by some equivalent pure strategies one-by-one.[8] The point is that under the independent information assumption, any agent who has dispersed information could purify her own behavioral strategy regardless whether other agents have dispersed information. Example 6 in the Supplemental Material (Chen, He, Li, and Sun (2018)) illustrates this idea of self purification. Given a behavioral-strategy Nash equilibrium in a 2-agent Bayesian game with independent information, there is an equivalent pure strategy for the agent with dispersed information, while the other agent with an atom in her type space could not purify her behavioral strategy.

In contrast, the purification result of this paper is based on the dispersed information associated with the other agents. Example 7 in the Supplemental Material (Chen, He, Li, and Sun (2018)) partially illustrates this idea of mutual purification. For a given randomized mechanism in a 2-agent setting with independent information, the agent with an atom in her type space can achieve the same interim payoff by some deterministic mechanism, while there does not exist such a deterministic mechanism for the other agent with dispersed information. In other words, our result becomes possible because each agent relies on the dispersed information of the other agents rather than her own. This also explains why a similar result does not hold in the one-agent setting since there is no dispersed information from other agents for such a single agent to purify the relevant randomized mechanism. In addition, we emphasize that in the settings with multiple agents, the notion of mutual purification requires not only that each agent obtain the same interim payoff under some deterministic mechanism, but also that a single deterministic mechanism deliver the same interim payoffs for all the agents simultaneously.

From a methodological point of view, the general purification principle in Dvoretzky, Wald, and Wolfowitz (1950) is simply a version of the classical Lyapunov Theorem about the convex range of an atomless finite-dimensional vector measure. Our purification result is technically different. First, the problem we consider is infinite-dimensional because we require the same interim expected allocation probabilities/ utilities for the equivalent mechanism

[7]See Radner and Rosenthal (1982), Milgrom and Weber (1985) and Khan, Rath, and Sun (2006). Furthermore, by applying the purification idea to a sequence of Bayesian games, Harsanyi (1973) provided an interpretation of mixed-strategy equilibrium in complete information games; see Govindan, Reny, and Robson (2003) and Morris (2008) for more discussion.

[8]See the proof of Theorem 1 in Khan, Rath, and Sun (2006).

at the interim level with a continuum of types. Note that Lyapunov's Theorem fails in an infinite-dimensional setting.[9] Second, it is clearly impossible to obtain a purified deterministic mechanism that delivers the same interim expected allocation probabilities as the original stochastic mechanism, conditioned on the joint types of all the agents.[10] However, our result on mutual purification shows that such an equivalence becomes possible when the conditioning operation is imposed on the individual types of every agent simultaneously, although the combination of the individual types of every agent is the joint types of all the agents. To the best of our knowledge, this paper is the first to consider the purification of a randomized decision rule that retains the same expected payoffs conditioned on the individual types of every agent in an economic model.

Our paper contributes to the Bayesian mechanism design literature in relying on specific aspects of agents' private information. These information aspects are often crucial in pinning down different properties of the optimal mechanism. For instance, agents with independent types retain information rents (see Myerson (1981)), whereas the mechanism designer can fully extract the surplus when the agents' types are correlated (see Crémer and McLean (1988)). Our result builds on the assumption that the agents' private information is independent and dispersed. This assumption facilitates the development of the novel methodology of mutual purification, which lies at the core of our arguments.

The rest of the paper is organized as follows. Section 2 introduces the model. Section 3 presents the mechanism equivalence result. Section 4 discusses the assumptions behind our equivalence result, the structures of the equivalence deterministic mechanisms, and the recent literature on the benefit of randomness in settings with multiple agents. Section 5 concludes. The appendices contain proofs and other technical results omitted from the main body of the paper, and examples delineating the limits of our mechanism equivalence result.

[9] See, for example, Diestel and Uhl (1977, p. 261).

[10] Since the joint types of all the agents carry the full information, the expected allocation probability of a stochastic mechanism conditioned on the joint types is simply the stochastic mechanism itself.

2 Model

2.1 Notation

We consider an environment with a finite set $\mathcal{I} = \{1, 2, \dots, I\}$ of risk-neutral agents ($I \geq 2$) and a finite set $\mathcal{K} = \{1, 2, \dots, K\}$ of social alternatives. The set of possible types V_i of agent i is a closed subset of finite dimensional Euclidean space $\mathbb{R}^l$ with generic element v_i. The set of possible type profiles is $V \equiv V_1 \times V_2 \times \cdots \times V_I$ with generic element $v = (v_1, v_2, \dots, v_I)$. We write v_{-i} for a type profile of agent i's opponents; that is, $v_{-i} \in V_{-i} = \Pi_{j \neq i} V_j$. The type profile v is distributed according to a probability distribution λ. For each agent $i \in \mathcal{I}$, λ_i is the marginal distribution of λ on V_i and is assumed to be atomless. Types are assumed to be independent. If $(Y, \mathcal{Y})$ is a measurable space, then ΔY is the set of all probability measures on $(Y, \mathcal{Y})$. If Y is a metric space, then we treat it as a measurable space with its Borel σ-algebra.

2.2 Mechanism

By the revelation principle, it is without loss of generality to consider only direct mechanisms characterized by $K + I$ functions, $\{q^k(v)\}_{k \in \mathcal{K}}$ and $\{t_i(v)\}_{i \in \mathcal{I}}$, where v is the profile of reports, $q^k(v) \geq 0$ is the probability that alternative k is implemented with $\sum_{k \in \mathcal{K}} q^k(v) = 1$, and $t_i(v)$ is the monetary transfer that agent i makes to the mechanism designer. We write $u_i^k(v)$ for agent i's gross utility in alternative k. For notational ease, we only define the following objects under the assumption of truthful reporting. We write

$$u_i(v) = \sum_{k \in \mathcal{K}} u_i^k(v)\, q^k(v) - t_i(v).$$

for agent i's utility. We write

$$Q_i^k(v_i) = \int_{V_{-i}} q^k(v_i, v_{-i}) \lambda_{-i}(\mathrm{d}v_{-i})$$

for the interim expected allocation probability (from agent i's perspective) that alternative k is implemented. Agent i's interim expected utility is

$$U_i(v_i) = \int_{V_{-i}} \left[\sum_{k \in \mathcal{K}} u_i^k(v_i, v_{-i}) q^k(v_i, v_{-i}) - t_i(v_i, v_{-i}) \right] \lambda_{-i}(\mathrm{d}v_{-i}).$$

The ex ante expected social surplus is

$$\int_V \sum_{i\in\mathcal{I}} \sum_{k\in\mathcal{K}} \left[u_i^k(v)\, q^k(v) \right] \lambda(\mathrm{d}v).$$

Definition 1. *A mechanism is Bayesian incentive compatible (BIC) if for all $i \in \mathcal{I}$ and $v_i \in V_i$,*

$$U_i(v_i) \geq \int_{V_{-i}} \left[\sum_{k\in\mathcal{K}} u_i^k(v_i, v_{-i})\, q^k(v_i', v_{-i}) - t_i(v_i', v_{-i}) \right] \lambda_{-i}(\mathrm{d}v_{-i})$$

for any alternative type $v_i' \in V_i$.

A mechanism satisfies Bayesian individual rationality (BIR) if for all $i \in \mathcal{I}$ and $v_i \in V_i$,

$$U_i(v_i) \geq 0.$$

Definition 2. *A mechanism (q, t) is said to be deterministic at v if the mechanism (q, t) implements some alternative $k \in \mathcal{K}$ for sure at v. That is, $q^k(v) = 1$ for some $k \in \mathcal{K}$. A mechanism (q, t) is deterministic if the mechanism is deterministic at all $v \in V$.*

2.3 Mechanism equivalence

We employ the following notion of mechanism equivalence in this paper.

Definition 3. *Two mechanisms (q, t) and $(\tilde{q}, \tilde{t})$ are equivalent if and only if they deliver the same interim expected allocation probabilities and the same interim expected utilities for all agents, and the same ex ante expected social surplus.*

Remark 1. Our equivalence notion is stronger than the prevailing mechanism equivalence notions used in the literature. For example, Manelli and Vincent (2010) and Gershkov, Goeree, Kushnir, Moldovanu, and Shi (2013) define two mechanisms to be equivalent if they deliver the same interim expected utilities for all agents and the same ex ante expected social surplus.[11]

[11] Gershkov, Goeree, Kushnir, Moldovanu, and Shi (2013, Section 4.1) show that their BIC-DIC equivalence result breaks down when requiring the same interim expected allocation probabilities. They also note that "this notion (of interim expected allocation probabilities) becomes relevant when, for instance, the designer is not utilitarian or when preferences of agents outside the mechanism play a role."

To illustrate our notion of mechanism equivalence, it is best to consider an example. The example is deliberately kept simple. Our result is far more general and the proof is much more complex.

Example 1. Consider a single-unit auction with two bidders. The bidders' valuations are independently distributed, and each bidder i's valuation v_i is uniformly distributed on the $[0,1]$ interval. Consider the following stochastic allocation rule $q = (q_1, q_2)$ with

$$q_1(v_1, v_2) = v_1 \text{ and } q_2(v_1, v_2) = 1 - q_1(v_1, v_2),$$

where q_i is the probability of agent i getting the object for $i \in \{1, 2\}$. In the construction of the equivalent deterministic mechanism below, the transfers are kept unchanged. Thus, we do not specify here the transfer scheme t in the mechanism (q, t). The readers may think of t as an arbitrary transfer scheme such that (q, t) is BIC.

The interim expected probability of bidder 1 getting the object is

$$\int_0^1 q_1(v_1, v_2)\, \mathrm{d}v_2 = \int_0^1 v_1\, \mathrm{d}v_2 = v_1$$

for all v_1, and the interim expected probability of bidder 2 getting the object is

$$\int_0^1 q_2(v_1, v_2)\, \mathrm{d}v_1 = \int_0^1 (1 - v_1)\, \mathrm{d}v_1 = \frac{1}{2}$$

for all v_2.

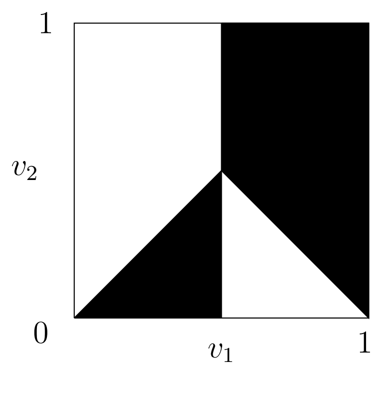

Figure 1: Bidder 1 is allocated the object in the shaded region, and bidder 2 is allocated the object in the unshaded region.

It is easy to verify that the following deterministic mechanism $(\hat{q}, t)$ is equivalent in terms of interim expected allocation probabilities (Figure 1 provides a graphical illustration of the mechanism $(\hat{q}, t)$). Since the transfers are kept unchanged, the deterministic mechanism $(\hat{q}, t)$ is also equivalent in terms of interim expected utilities for all agents and the ex ante expected social surplus.

$$\hat{q}_1(v_1, v_2) = \begin{cases} 1, & v_2 \leq v_1 \leq 1/2, \\ 1, & \max\{1 - v_2, \frac{1}{2}\} \leq v_1, \\ 0, & \text{otherwise.} \end{cases} \quad \hat{q}_2(v_1, v_2) = 1 - \hat{q}_1(v_1, v_2).$$

In Section 3, we show that for whatever stochastic mechanism that the mechanism designer may choose to use, however complicated, there always exists an equivalent mechanism that is deterministic.

3 Equivalence result

This section presents our mechanism equivalence result. To make the logic of our arguments and the roles played by the various assumptions clear, we break down our analysis into two steps. In the first step, we show that for any allocation rule, there exists a deterministic allocation rule that delivers the same interim expected allocation probabilities for all agents. This step only requires the assumption of atomless distribution. While the assumption of independent types is not needed, for simplicity of exposition, we present this result in settings with independent types (see Remark 2 below for a detailed discussion). In the second step, under additional assumptions of independent types and separable payoffs, we show that for any BIC and BIR mechanism, there exists a deterministic mechanism that is BIC and BIR, delivers the same interim expected allocation probabilities and the same interim expected utilities for all agents, and delivers the same ex ante expected social surplus.

Interim expected allocation probabilities

We first show that for any allocation rule, there exists a deterministic allocation rule that delivers the same interim expected allocation probabilities for all agents. For this result, we only require that λ_i be atomless for all $i \in \mathcal{I}$. Let

$$\Upsilon = \{q : \sum_{k \in \mathcal{K}} q^k(v) = 1 \text{ for } \lambda\text{-almost all } v \in V\}.$$

Theorem 1. *For any allocation rule q, there exists a deterministic allocation rule $\hat{q}$ such that q and $\hat{q}$ induce the same interim expected allocation probabilities for all agents. That is, for all $i \in \mathcal{I}$ and $v_i \in V_i$,*

$$\mathbb{E}(\hat{q}|v_i) = \mathbb{E}(q|v_i).^{12} \tag{1}$$

The proof of Theorem 1 is relegated to the appendix. Here, we provide a sketch of the proof. For any $q \in \Upsilon$, let

$$\Upsilon_q = \{q' \in \Upsilon : \mathbb{E}(q'|v_i) = \mathbb{E}(q|v_i) \text{ for all } i \in \mathcal{I} \text{ and } \lambda_i\text{-almost all } v_i \in V_i\}.$$

Step (1) shows that the set Υ_q is nonempty, convex, and weakly compact. Therefore, the set Υ_q admits extreme points. Step (2) proceeds to show that all extreme points of Υ_q are allocation rules that are deterministic at λ-almost all $v \in V$.[13] Indeed, if q' is not deterministic at λ-almost all $v \in V$, then there exist distinct $\bar{q}$, $\bar{\bar{q}} \in \Upsilon_q$ such that $q' = \frac{1}{2}(\bar{q} + \bar{\bar{q}})$. Thus, q' is not an extreme point of Υ_q. The existence of $\bar{q}$ and $\bar{\bar{q}}$ relies on the assumption that λ_i is atomless for all $i \in I$. Step (1) and Step (2) together imply that there exists $\tilde{q} \in \Upsilon_q$ that is deterministic at λ-almost all $v \in V$. Note that $\tilde{q}$ is not necessarily deterministic at all $v \in V$. Furthermore, such $\tilde{q}$ induces the same interim expected allocation probabilities for all $i \in \mathcal{I}$ and λ-almost all $v_i \in V_i$, but not for all $v_i \in V_i$. Step (3) then constructs a deterministic allocation rule $\hat{q}$ that induces the same interim expected the same interim expected allocation probabilities for all $i \in \mathcal{I}$ and all $v_i \in V_i$, by modifying $\tilde{q}$ on sets of measure zero. The last step is (conceptually) straightforward.

Theorem 1 can be substantially generalized.

Theorem 2. *Let h be an integrable function from V to $\mathbb{R}^N$ for some positive integer N. For any allocation rule q, there exists a deterministic allocation rule $\hat{q}$ such that for all $i \in \mathcal{I}$ and $v_i \in V_i$,*

$$\mathbb{E}(\hat{q}h_j|v_i) = \mathbb{E}(qh_j|v_i) \tag{2}$$

for all $1 \le j \le N$.

[12] For any integrable function f defined on V, $\mathbb{E}(f|v_i) = \int_{V_{-i}} f(v_i, v_{-i})\, \lambda_{-i}(\mathrm{d}v_{-i})$.

[13] Manelli and Vincent (2007) use a related technique in the multi-dimensional screening literature. Manelli and Vincent (2007) consider a revenue maximizing multi-product monopolist and study the extreme points of the set of feasible mechanisms. They show that, with multiple goods, extreme points could be stochastic mechanisms. In contrast, we work with the mechanism design setting, study a particular set of interest Υ_q, and show that all extreme points are deterministic. Apart from this general approach, the technical parts of the proofs are dramatically different.

Without loss of generality, we assume that h is an integrable function from V to $\mathbb{R}^N_{++}$.[14] Theorem 2 can be proved by applying similar arguments as in the proof of Theorem 1 to the following set:

$$\dot{\Upsilon}_q = \{q' \in \Upsilon : \mathbb{E}(q'h_j|v_i) = \mathbb{E}(qh_j|v_i) \text{ for all } i \in \mathcal{I} \text{ and } \lambda_i\text{-almost all } v_i \in V_i,\ 1 \le j \le N\}.$$

In Appendix C of the Supplemental Material (Chen, He, Li, and Sun (2018)), we detail how to modify the proof of Theorem 1 to prove Theorem 2.

Remark 2. *Theorem 1 and Theorem 2 above are established in the case of independent types. In settings with correlated types, let ρ denote the density function. Then by Theorem 2, for any allocation rule q, there exists a deterministic allocation rule $\hat{q}$ such that for all $i \in \mathcal{I}$ and $v_i \in V_i$,*

$$\int_{V_{-i}} \hat{q}(v_i, v_{-i})\, h_j\, \rho(v_i, v_{-i})\, \lambda_{-i}(dv_{-i}) = \int_{V_{-i}} q(v_i, v_{-i})\, h_j\, \rho(v_i, v_{-i})\, \lambda_{-i}(dv_{-i})$$

for all $1 \le j \le N$. Theorem 1 for the case of correlated types immediately follows by setting $h \equiv 1$.

Mechanism equivalence

Next, we present our mechanism equivalence result. For this result, we need additional assumptions of separate payoffs and independent types. We assume that all agents have separable payoffs in the following sense.

Definition 1. *Agent $i \in \mathcal{I}$ is said to have separable payoff if for all $k \in \mathcal{K}$ and $v \in V$, her payoff function can be written as follows:*

$$u_i^k(v) = \sum_{1 \le m \le M} w_{i,m}^k(v_i)\, r_{i,m}^k(v_{-i}),$$

where M is a positive integer, and $w_{i,m}^k$ (resp. $r_{i,m}^k$) is λ_i-integrable (resp. λ_{-i}-integrable) on V_i (resp. on V_{-i}) for $1 \le m \le M$.

In words, the payoff of each agent i is a summation of finite terms, where each term is a product of two components: the first component only depends on agent i's own type,

[14] To see this is without loss of generality, let $h_+ = (h_{+1}, h_{+2}, \ldots, h_{+I})$ and $h_- = (h_{-1}, h_{-2}, \ldots, h_{-I})$, where $h_{+j} = \max\{h_j, 0\} + 1$ and $h_{-j} = \min\{h_j, 0\} - 1$ for all $1 \le j \le N$. Theorem 2 can thus be proved by considering h_+ and h_- respectively, as $h = h_+ + h_-$.

while the second component depends on the other agents' types. Note that this setup is sufficiently general to cover most applications. In particular, it includes the interdependent payoff function as in Jehiel and Moldovanu (2001), and obviously covers the widely adopted private value payoffs as a special case.

Theorem 3. *Suppose that for each agent $i \in \mathcal{I}$, her payoff function is separable. For any BIC and BIR mechanism (q, t), there exists an equivalent deterministic mechanism $(\hat{q}, t)$ that is BIC and BIR. More explicitly,*

(1) q and $\hat{q}$ induce the same interim expected allocation probabilities for all $i \in \mathcal{I}$;

(2) (q, t) and $(\hat{q}, t)$ induce the same interim expected utilities for all $i \in \mathcal{I}$; and

(3) (q, t) and $(\hat{q}, t)$ induce the same ex ante expected social surplus.

Remark 3. Note that our approach is not constructive. While we know that there exists an equivalent deterministic mechanism, we do not know how to construct such an equivalent deterministic mechanism. We discuss the structures of the equivalent deterministic mechanisms in Section 4.2 and Section 4.3. We also provide a recipe for the construction of an approximately equivalent mechanism in Appendix D of the Supplemental Material (Chen, He, Li, and Sun (2018)).

Proof of Theorem 3. For any BIC and BIR mechanism (q, t), by Theorem 2, there exists a deterministic allocation rule $\hat{q}$ such that for any $i \in \mathcal{I}$ and $v_i \in V_i$,

$$\mathbb{E}(\hat{q}|v_i) = \mathbb{E}(q|v_i), \tag{3}$$

$$\mathbb{E}(\hat{q} r^k_{i,m}|v_i) = \mathbb{E}(q r^k_{i,m}|v_i) \tag{4}$$

for all $r^k_{i,m}$, $i \in \mathcal{I}$, $k \in \mathcal{K}$ and $1 \leq m \leq M$.

By Equation (3), q and $\hat{q}$ induce the same interim expected allocation probabilities for all agents. We proceed to verify that (q, t) and $(\hat{q}, t)$ induce the same interim expected utilities for all agents. We calculate agent i's interim expected utility if she reports type v'_i when her true type is v_i as follows:

$$\begin{aligned}
&\int_{V_{-i}} \left[\sum_{k \in \mathcal{K}} u^k_i(v_i, v_{-i})\, \hat{q}^k(v'_i, v_{-i}) - t_i(v'_i, v_{-i}) \right] \lambda_{-i}(\mathrm{d}v_{-i}) \\
&= \sum_{k \in \mathcal{K}} \sum_{1 \leq m \leq M} w^k_{i,m}(v_i) \int_{V_{-i}} r^k_{i,m}(v_{-i})\, \hat{q}^k(v'_i, v_{-i})\, \lambda_{-i}(\mathrm{d}v_{-i}) - \int_{V_{-i}} t_i(v'_i, v_{-i})\, \lambda_{-i}(\mathrm{d}v_{-i})
\end{aligned}$$

$$= \sum_{k \in \mathcal{K}} \sum_{1 \le m \le M} w_{i,m}^k(v_i) \int_{V_{-i}} r_{i,m}^k(v_{-i})\, q^k(v_i', v_{-i})\, \lambda_{-i}(\mathrm{d}v_{-i}) - \int_{V_{-i}} t_i(v_i', v_{-i})\, \lambda_{-i}(\mathrm{d}v_{-i})$$

$$= \int_{V_{-i}} \left[\sum_{k \in \mathcal{K}} u_i^k(v_i, v_{-i})\, q^k(v_i', v_{-i}) - t_i(v_i', v_{-i}) \right] \lambda_{-i}(\mathrm{d}v_{-i}),$$

where the second line follows from the assumption of separable payoffs, the third line follows from Equation (4) and the assumption of independent types. Thus, (q, t) and $(\hat{q}, t)$ deliver the same interim expected utility for each agent, when each agent i has true type v_i and misreports v_i', for all $i \in \mathcal{I}$ and $v_i \in V_i$. Therefore, if (q, t) is BIC and BIR, then $(\hat{q}, t)$ is BIC and BIR. Since the ex post transfers are kept unchanged, the arguments above also imply that both mechanisms deliver the same ex ante expected social surplus. More explicitly,

$$\begin{aligned}
&\int_V \sum_{i \in \mathcal{I}} \sum_{k \in \mathcal{K}} \left[u_i^k(v)\, \hat{q}^k(v) \right] \lambda(\mathrm{d}v) \\
&= \sum_{i \in \mathcal{I}} \int_{V_i} \int_{V_{-i}} \sum_{k \in \mathcal{K}} \left[u_i^k(v_i, v_{-i})\, \hat{q}^k(v_i, v_{-i}) \right] \lambda_{-i}(\mathrm{d}v_{-i})\, \lambda_i(\mathrm{d}v_i) \\
&= \sum_{i \in \mathcal{I}} \int_{V_i} \int_{V_{-i}} \sum_{k \in \mathcal{K}} \left[u_i^k(v_i, v_{-i})\, q^k(v_i, v_{-i}) \right] \lambda_{-i}(\mathrm{d}v_{-i})\, \lambda_i(\mathrm{d}v_i) \\
&= \int_V \sum_{i \in \mathcal{I}} \sum_{k \in \mathcal{K}} \left[u_i^k(v)\, q^k(v) \right] \lambda(\mathrm{d}v).
\end{aligned}$$

This completes the proof. □

Remark 4. It is clear from the proof of Theorem 3 that the equivalent deterministic mechanism $(\hat{q}, t)$ guarantees the same ex post monetary transfers and the same expected revenue. This implies our mechanism equivalence result also holds in settings without monetary transfers.

Remark 5. Say that a mechanism satisfies ex post individual rationality (EPIR) if for all $i \in \mathcal{I}$ and $v \in V$, $u_i(v) \ge 0$. For any stochastic BIC and BIR mechanism, if the assumptions of separable payoffs and independent types are satisfied, then there exists an equivalent deterministic mechanism that is BIC and EPIR. For this result, the ex post transfers may need to be adjusted. By Theorem 3, for any BIC and BIR mechanism (q, t), there exists an equivalent deterministic mechanism $(\hat{q}, t)$ that is BIC and BIR. Since (p, t) is BIR, $U_i(v_i) \ge 0$.

Define a new transfer scheme $\hat{t}$ as follows:

$$\hat{t}_i(v_i, v_{-i}) = \sum_{k \in \mathcal{K}} u_i^k(v_i, v_{-i})\, p^k(v_i, v_{-i}) - U_i(v_i).$$

It is easy to verity that $(\hat{q}, \hat{t})$ is an equivalent deterministic mechanism, and is BIC. For all $v \in V$, we have

$$u_i(v) = \sum_{k \in \mathcal{K}} u_i^k(v_i, v_{-i})\, p^k(v_i, v_{-i}) - \hat{t}_i(v_i, v_{-i}) = U_i(v_i) \geq 0.$$

Thus, $(\hat{q}, \hat{t})$ is also EPIR.

4 Discussions

In this section, we first discuss the assumptions behind our equivalence result in Section 4.1. We then move on to discuss the structures of the equivalent deterministic mechanisms. Here, our discussion is twofold. Section 4.2 illustrates by an example that, even in environments with linear utilities and independent, one-dimensional, private types, we cannot hope for a mechanism equivalence result between the class of BIC mechanisms and the class of DIC and deterministic mechanisms. On a more positive note, Section 4.3 shows that for any symmetric allocation rule, there exists an equivalent deterministic mechanism that preserves symmetry. Finally, Section 4.4 compares our results with the recent literature on the benefit of randomness in settings with multiple agents.

4.1 The limits of the equivalence result

Here, we discuss the assumptions behind our equivalence result, including multiple agents, atomless distribution, separable payoffs, and independent types. The literature of multi-dimensional screening contains abundant examples illustrating that in the case of a single buyer, a multi-product monopolist may find it beneficial to include lotteries as part of the selling mechanism. Atomless distribution is an indispensable requirement for almost all purification results. Though separable payoff is a restriction, the setup is sufficiently general to cover most economic applications; see the discussion immediately after Definition 1 for details. While our result requires independence, it is worth mentioning that we only require independence across agents and we do not make any assumption regarding the correlation of the different coordinates of type v_i. For atomless distribution, separable payoffs, and independent types, we present examples in Appendix A.3 to illustrate that our mechanism equivalence result breaks down if each of these assumptions is violated.

We wish to elaborate further on the requirement of multiple agents. While it is well known that lotteries could strictly improve revenue in the case of a single buyer, our equivalence result implies that the optimal mechanism is deterministic in the case of multiple agents. It is thus important to highlight and explain the differences between the case of a single agent with a multi-dimensional type and the case of multiple agents. In particular, the readers might wonder why, in the case of a single buyer with a two-dimension type (v_1, v_2), the multi-product monopolist cannot use the agent's report along the second dimension to purify the allocation probability of good 1, and vice versa. The problem is that the incentive constraints would break down, since it is the same agent who controls the reports on both dimensions.

4.2 DIC and deterministic mechanisms

In the private value setting, each agent's utility does not depend on the types of her opponents. A mechanism is dominant strategy incentive compatible (DIC) in the private value setting if for all $i \in \mathcal{I}$ and $v_i \in V_i$,

$$\sum_{k\in\mathcal{K}} u_i^k(v_i)\, q^k(v_i, v_{-i}) - t_i(v_i, v_{-i}) \geq \sum_{k\in\mathcal{K}} u_i^k(v_i)\, q^k(v_i', v_{-i}) - t_i(v_i', v_{-i})$$

for any alternative type $v_i' \in V_i$. Existing papers in the literature establish the equivalence of BIC and DIC mechanisms under certain conditions, and our paper proves the equivalence of stochastic and deterministic mechanisms. A natural question to ask is whether there is a class of environments in which, for any BIC mechanism, we can obtain an equivalent mechanism that is both DIC and deterministic. The following example illustrates that even in the single-unit auction setting, this is not the case.

Example 2 (Example 1 revisited)**.** Suppose that there exists an equivalent mechanism $\hat{q} = (\hat{q}_1, \hat{q}_2)$ that is both DIC and deterministic. Then, for each $v_2 \in V_2$, there exists a threshold $\bar{v}_1(v_2)$ such that $\hat{q}_2(v_1, v_2) = 1$ for all $v_1 > \bar{v}_1(v_2)$ and 0 for all $v_1 < \bar{v}_1(v_2)$. Since the interim expected allocation probability of bidder 2 is $\frac{1}{2}$ for all $v_2 \in V_2$, it must be that

$\bar{v}_1(v_2) = \frac{1}{2}$ for all $v_2 \in V_2$. By the arguments above, $\hat{q}_2(v_1, v_2) = 1$ and $\hat{q}_1(v_1, v_2) = 0$ for all $v_1 > \frac{1}{2}$ and $v_2 \in V_2$. Therefore, the interim expected allocation probability of bidder 1 is 0 for all $v_1 > \frac{1}{2}$. We arrive at a contradiction.

4.3 Application: symmetric auctions

Symmetric auctions received a lot of attention in the mechanism design literature (see, for example, Border (1991) and Deb and Pai (2017)). Applying our results in Section 3, we show that for any symmetric allocation rule, there exists an equivalent deterministic mechanism that preserves symmetry. For simplicity of exposition, we prove this result in the single-unit auction setting.

Consider a single-unit auction with a finite set $\mathcal{I} = \{1, 2, \ldots, I\}$ of risk-neutral bidders ($I \geq 2$). For all $i \in \mathcal{I}$, agent i's valuation v_i for the object is distributed according to λ_i with support on $V_i = [\underline{v}_i, \bar{v}_i] \subset \mathbb{R}_+$. The agents are ex ante symmetric; that is, $\lambda_1 = \lambda_2 = \ldots = \lambda_I$ and $V_1 = V_2 = \ldots = V_I$.

An allocation rule consists of I functions $q = (q_1, \ldots, q_I)$ with $\sum_{i \in \mathcal{I}} q_i(v) \leq 1$, where q_i is the probability that bidder i gets the object. Let Ψ be the set of all permutations on $\mathcal{I}$. An allocation rule q is said to be symmetric if $q_i(v_1, v_2, \ldots, v_I) = q_{\psi^{-1}(i)}(v_{\psi(1)}, \ldots, v_{\psi(I)})$ for all $i \in \mathcal{I}$, $v \in V$, and $\psi \in \Psi$. Proposition 1 below shows that for any symmetric allocation rule, there exists a deterministic and symmetric allocation rule that induces the same interim expected allocation probability for all agents. This further implies that there exists an equivalent deterministic mechanism that preserves symmetry.

Proposition 1. *For any symmetric allocation rule q, there exists a deterministic and symmetric allocation rule $\hat{q}$ such that q and $\hat{q}$ induce the same interim expected allocation probability for all agents.*

4.4 Benefit of randomness revisited

In a recent contribution, Chawla, Malec, and Sivan (2015) consider a multi-agent setting and focus on the case in which the agents' valuations are independent both across different agents' types and different coordinates of an agent's type. They establish a constant factor upper bound for the benefit of randomness when the agents' values are independent. In the special case of multi-unit multi-item auctions, they show that the revenue of any Bayesian incentive compatible, individually rational randomized mechanism is at most 33.75 times the revenue of the optimal deterministic mechanism. In this paper, we push this result to the extreme and show that the revenue maximizing auction can be deterministically implemented.[15]

5 Conclusion

We prove the following mechanism equivalence result: in a general social choice environment with multiple agents, for any mechanism, there exists an equivalent deterministic mechanism. On the one hand, our result implies that it is without loss of generality to work with stochastic mechanisms, even if the mechanism designer does not have access to a randomization device, or cannot fully commit to the outcomes induced by a randomization device. On the other hand, our result implies that the requirement of deterministic mechanisms is not restrictive in itself. Even if one is constrained to employ only deterministic mechanisms, there is no loss of revenue or social welfare.

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[15] Chawla, Malec, and Sivan (2015, p. 316) remarked that "our bounds on the benefit of randomness are in some cases quite large and we believe they can be improved."

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A Appendix

A.1 Proof of Theorem 1

Step (1) For any $q \in \Upsilon$, let

$$\Upsilon_q = \{q' \in \Upsilon : \mathbb{E}(q'|v_i) = \mathbb{E}(q|v_i) \text{ for all } i \in \mathcal{I} \text{ and } \lambda\text{-almost all } v_i \in V_i\}.$$

We first show that Υ_q is nonempty, convex, and weakly compact. Then, by the Krein-Milman Theorem (see Royden and Fitzpatrick (2010, p. 296)), Υ_q admits extreme points.

Lemma 1. *Υ_q is nonempty, convex, and weakly compact.*

Proof. Clearly, Υ_q is nonempty and convex. For weak compactness, it suffices to show that Υ_q is norm closed in $L_1^\lambda(V, \mathbb{R}^K)$, where $L_1^\lambda(V, \mathbb{R}^K)$ is the L_1 space of all integrable mappings from V to $\mathbb{R}^K$ under the probability measure λ. Since Υ_q is convex, by Mazur's Theorem (see Royden and Fitzpatrick (2010, p. 292)), Υ_q is also weakly closed in $L_1^\lambda(V, \mathbb{R}^K)$. Since Υ_q is bounded, it is uniformly integrable. By Theorem 12 in Royden and Fitzpatrick (2010, p. 412), Υ_q is weakly compact in $L_1^\lambda(V, \mathbb{R}^K)$.

In what follows, we show that Υ_q is norm closed in $L_1^\lambda(V, \mathbb{R}^K)$. Consider any sequence $\{q_m\} \subseteq \Upsilon_q$ such that $q_m \to q_0$ in norm in $L_1^\lambda(V, \mathbb{R}^K)$. We show that $q_0 \in \Upsilon_q$. By the Riesz-Fischer Theorem (see Royden and Fitzpatrick (2010, p. 398)), there exists a subsequence $\{q_{m_s}\}$ of $\{q_m\}$ such that $\{q_{m_s}\}$ converges to q_0 λ-almost everywhere. Since $\sum_{k \in \mathcal{K}} q_{m_s}^k(v) = 1$ for λ-almost all v, $\sum_{k \in \mathcal{K}} q_0^k(v) = 1$ for λ-almost all v. Therefore, $q_0 \in \Upsilon$.

Given any $i \in \mathcal{I}$, let $\mathcal{B}(V_i)$ be the Borel σ-algebra of the set V_i. For all $i \in \mathcal{I}$ and $\mathcal{B}(V_i) \otimes (\bigotimes_{j \neq i} \{V_j, \emptyset\})$-measurable bounded mapping $p \colon V \to \mathbb{R}^K$,

$$\int_V (q_0 \cdot p)\, \lambda(\mathrm{d}v) = \lim_{s \to \infty} \int_V (q_{m_s} \cdot p)\, \lambda(\mathrm{d}v) = \int_V (q \cdot p)\, \lambda(\mathrm{d}v),$$

where the first equality follows from the dominated convergence theorem (see Royden and Fitzpatrick (2010, p. 88)), and the second equality holds since $\{q_{m_s}\} \subseteq \Upsilon_q$. Thus, $q_0 \in \Upsilon_q$. □

Step (2) We show that all extremes points of Υ_q are deterministic for λ-almost all $v \in V$. Then, there exists $\tilde{q} \in \Upsilon_q$ that is deterministic for λ-almost all $v \in V$.

Lemma 2. *All extreme points of Υ_q are deterministic for λ-almost all $v \in V$.*

Proof. We prove the proposition by contraposition. We show that if $q' \in \Upsilon_q$ is not deterministic for λ-almost all $v \in V$, then q' is not an extreme point of Υ_q. Suppose that q' is not deterministic for λ-almost all $v \in V$. Then, there exists

(1) $0 < \delta < 1$;

(2) a Borel measurable set $D \subseteq V$ with $\lambda(D) > 0$; and

(3) indices j_1, j_2

such that for all $v \in D$,

$$\delta \leq q'^{j_1}(v),\ q'^{j_2}(v) \leq 1 - \delta.$$

We proceed to show that there exist distinct $\bar{q}$, $\bar{\bar{q}} \in \Upsilon_q$ such that $\tilde{q} = \frac{1}{2}(\bar{q} + \bar{\bar{q}})$. This establishes that $\tilde{q}$ is not an extreme point of Υ_q.

For any $i \in \mathcal{I}$, let D_i be the projection of D on V_i. For any $v_i \in D_i$, let $D_{-i}(v_i) = \{v_{-i} : (v_i, v_{-i}) \in D\}$. Consider the following system of equations where $\alpha \in L_\infty^\lambda(D, \mathbb{R})$ are the unknown:

$$\int_{D_{-i}(v_i)} \alpha(v_i, v_{-i})\, \lambda_{-i}(\mathrm{d}v_{-i}) = 0, \tag{5}$$

for all $i \in \mathcal{I}$ and $v_i \in D_i$.

Since λ_i is atomless for all $i \in \mathcal{I}$, one can show that besides the trivial solution that $\alpha = 0$, the system of equations (5) also has a nontrivial bounded solution α. The proof of this claim is technical and is contained in Appendix B of the Supplemental Material (Chen, He, Li, and Sun (2018)).

Without loss of generality, we assume that $|\alpha| \le \delta$. Since α is defined on D, we extend the domain of α to V by setting $\alpha(v) = 0$ whenever $v \notin D$. We construct $\bar{q}$ and $\bar{\bar{q}}$ as follows: for all $v \in V$,

$$\begin{aligned} \bar{q}(v) &= q'(v) + \alpha(v)\, (e_{j_1} - e_{j_2}); \\ \bar{\bar{q}}(v) &= q'(v) + \alpha(v)\, (e_{j_2} - e_{j_1}), \end{aligned}$$

where e_{j_1} and e_{j_2} are the standard basis vectors in $\mathbb{R}^K$.

We proceed to verify that $\bar{q}$, $\bar{\bar{q}} \in \Upsilon_q$. To see that $\bar{q} \in \Upsilon$, note that

(1) For all $v \in V$, $\sum_{k \in \mathcal{K}} \bar{q}^k(v) = \sum_{k \in \mathcal{K}} q'^k(v) = 1$;
(2) If $v \in D$, then $\delta \le q'^{j_1}(v)$, $q'^{j_2}(v) \le 1 - \delta$, which implies that $0 \le \bar{q}^{j_1}(v)$, $\bar{q}^{j_2}(v) \le 1$;
(3) If $v \in D$, then $\bar{q}^j(v) = q'^j(v)$ for $j \ne j_1$, j_2; and
(4) If $v \notin D$, then $\bar{q}^j(v) = q'^j(v)$ as $\alpha(v) = 0$.

To see that $\bar{q} \in \Upsilon_q$, note that for all $i \in \mathcal{I}$ and $\mathcal{B}(V_i) \otimes \left(\bigotimes_{j \ne i}\{V_j, \emptyset\}\right)$-bounded measurable mapping $p \in L_\infty^\lambda(V, \mathbb{R}^K)$, we have

$$\begin{aligned} \int_V (\bar{q} \cdot p)\, \lambda(\mathrm{d}v) &= \int_V (q' \cdot p)\, \lambda(\mathrm{d}v) + \int_V \alpha(v)((e_{j_1} - e_{j_2}) \cdot p(v))\, \lambda(\mathrm{d}v) \\ &= \int_V (q' \cdot p)\, \lambda(\mathrm{d}v) + \int_{V_i} \int_{V_{-i}} \alpha(v)((e_{j_1} - e_{j_2}) \cdot p(v))\, \lambda_{-i}(\mathrm{d}v_{-i})\, \lambda_i(\mathrm{d}v_i) \\ &= \int_V (q' \cdot p)\, \lambda(\mathrm{d}v) + \int_{V_i} (e_{j_1} - e_{j_2}) \cdot p(v) \int_{V_{-i}} \alpha(v)\, \lambda_{-i}(\mathrm{d}v_{-i}) \lambda_i(\mathrm{d}v_i) \\ &= \int_V (q' \cdot p)\, \lambda(\mathrm{d}v). \end{aligned}$$

where the first line follows from the construction of $\bar{q}$, the third line follows from the measurability of p, and the fourth line follows from the property of α.

By similar reasoning, one can show that $\bar{\bar{q}} \in \Upsilon_q$. Since $\bar{q}$ and $\bar{\bar{q}}$ are distinct and $q' = \frac{1}{2}(\bar{q} + \bar{\bar{q}})$, q' is not an extreme point of Υ_q. □

Step (3) Fix $\tilde{q} \in \Upsilon_q$ that is deterministic for λ-almost all $v \in V$. Note that (1) $\tilde{q}$ is deterministic for λ-almost all $v \in V$, but not for all $v_i \in V_i$; and (2) $\tilde{q}$ induces the same interim expected allocation probabilities for all $i \in \mathcal{I}$ and λ_i-almost all $v_i \in V_i$, but not for all $v_i \in V_i$. We now construct a deterministic allocation rule $\hat{q}$ that induces the same interim expected allocation probabilities for all $i \in \mathcal{I}$ and $v_i \in V_i$, by modifying $\tilde{q}$ on sets of measure zero.

Let $D' = \{v \in V : \tilde{q}^k(v) \in (0,1) \text{ for some } k\}$. Since $\tilde{q}$ is deterministic for almost all $v \in V$, $\lambda(D') = 0$. Define a new allocation rule $\tilde{\tilde{q}}$ as follows:

$$\tilde{\tilde{q}}(v) = \begin{cases} (1,\, 0,\, \dots,\, 0), & \text{if } v \in D'; \\ \tilde{q}(v), & \text{otherwise.} \end{cases}$$

Then $\tilde{\tilde{q}}$ is deterministic for all $v \in V$, and induces the same interim expected allocation probabilities for all $i \in \mathcal{I}$ and λ_i-almost all $v_i \in V_i$. That is, for all $i \in \mathcal{I}$ and λ_i-almost all $v_i \in V_i$,

$$\int_{V_{-i}} \tilde{\tilde{q}}(v_i, v_{-i})\, \lambda_{-i}(\mathrm{d}v_{-i}) = \int_{V_{-i}} q(v_i, v_{-i})\, \lambda_{-i}(\mathrm{d}v_{-i}). \tag{6}$$

For each $i \in \mathcal{I}$, let D_i'' be the subset of V_i such that Equation (6) does not hold. Then, $\lambda_i(D_i'') = 0$. By Proposition 10.7.6 in Bogachev (2007), for each $i \in I$, there exists a deterministic allocation rule q_i on $D_i'' \times V_{-i}$ such that for all $v_i \in D_i''$,

$$\int_{V_{-i}} q_i(v_i, v_{-i})\, \lambda_{-i}(\mathrm{d}v_{-i}) = \int_{V_{-i}} q(v_i, v_{-i})\, \lambda_{-i}(\mathrm{d}v_{-i}). \tag{7}$$

Finally, we construct an allocation rule $\hat{q}$ as follows:

$$\hat{q}(v) = \begin{cases} q_1(v), & \text{if } v \in (D_1'' \times V_{-1}) \setminus \cup_{2 \le i \le I} (D_i'' \times V_{-i}); \\ q_2(v), & \text{if } v \in (D_2'' \times V_{-2}) \setminus \cup_{3 \le i \le I} (D_i'' \times V_{-i}); \\ \dots, & \dots; \\ q_I(v), & \text{if } v \in D_I'' \times V_{-I}; \\ \tilde{\tilde{q}}(v), & \text{otherwise.} \end{cases}$$

It follows from the construction of $\tilde{\tilde{q}}$ and q_i for all $i \in \mathcal{I}$ that $\hat{q}$ is deterministic for all $v \in V$. We now proceed to verify that $\hat{q}$ and q induce the same interim expected allocation probabilities for all $i \in \mathcal{I}$ and $v_i \in V_i$.

Fix $i \in \mathcal{I}$ and $v_i \in V_i$. If $v_i \in D_i''$, by the definition of $\hat{q}$,

$$\begin{aligned} \int_{V_{-i}} \hat{q}(v_i, v_{-i})\, \lambda_{-i}(\mathrm{d}v_{-i}) &= \int_{\bar{D}_{-i}} \hat{q}(v_i, v_{-i})\, \lambda_{-i}(\mathrm{d}v_{-i}) + \int_{V_{-i} \setminus \bar{D}_{-i}} \hat{q}(v_i, v_{-i})\, \lambda_{-i}(\mathrm{d}v_{-i}) \\ &= 0 + \int_{V_{-i}} q_i(v_i, v_{-i})\, \lambda_{-i}(\mathrm{d}v_{-i}) \\ &= \int_{V_{-i}} q(v_i, v_{-i})\, \lambda_{-i}(\mathrm{d}v_{-i}), \end{aligned}$$

where $\bar{D}_{-i} = \cup_{j \in \mathcal{I},\, j \neq i} \left(D_j'' \times \prod_{k \in \mathcal{I},\, k \neq i,\, j} V_s\right)$. The second line follows from that $\lambda_j(D_j'') = 0$ for all $j \in \mathcal{I}$, and the third line follows from (7).

If $v_i \notin D_i'$, by the definition of $\hat{q}$,

$$\begin{aligned} \int_{V_{-i}} \hat{q}(v_i, v_{-i})\, \lambda_{-i}(\mathrm{d}v_{-i}) &= \int_{\bar{D}_{-i}} \hat{q}(v_i, v_{-i})\, \lambda_{-i}(\mathrm{d}v_{-i}) + \int_{V_{-i} \setminus \bar{D}_{-i}} \hat{q}(v_i, v_{-i})\, \lambda_{-i}(\mathrm{d}v_{-i}) \\ &= 0 + \int_{V_{-i}} \tilde{\tilde{q}}(v_i, v_{-i})\, \lambda_{-i}(\mathrm{d}v_{-i}) \end{aligned}$$

$$= \int_{V_{-i}} q(v_i, v_{-i})\, \lambda_{-i}(\mathrm{d}v_{-i}),$$

where the second line follows from that $\lambda_j(D''_j) = 0$ for all $j \in \mathcal{I}$, and the third line follows from (6).

A.2 Proof of Proposition 1

Let ψ_0 be the identity mapping on $\mathcal{I}$. For each $\psi \in \Psi$, define

$$D_\psi = \{v \in V : \ v_{\psi(1)} > v_{\psi(2)} > \ldots > v_{\psi(I)}\}.$$

Clearly,

1. for distinct ψ, $\psi' \in \Psi$, $D_\psi \cap D_{\psi'} = \emptyset$;
2. $\lambda(\cup_{\psi \in \Psi} D_\psi) = 1$;
3. $v \in D_\psi$ for some ψ if and only if $(v_{\psi(1)}, v_{\psi(2)}, \ldots, v_{\psi(I)}) \in D_{\psi_0}$.

The structure of the proof is as follows. For an arbitrary symmetric allocation rule q, Step (1) constructs a deterministic allocation rule $\hat{q}$, Step (2) shows that $\hat{q}$ is symmetric, and Step (3) shows that q and $\hat{q}$ induce the same interim expected allocation probabilities for all agents.

Step (1) By Theorem 2, there exists a deterministic allocation rule $\tilde{q} = (\tilde{q}_1, \ldots, \tilde{q}_I)$ such that for all $i, j \in \mathcal{I}$ and $v_j \in V_j$,

$$\int_{V_{-j}} \mathbf{1}_{D_{\psi_0}}(v_j, v_{-j}) \tilde{q}_i(v_j, v_{-j})\, \lambda_{-j}(\mathrm{d}v_{-j}) = \int_{V_{-j}} \mathbf{1}_{D_{\psi_0}}(v_j, v_{-j}) q_i(v_j, v_{-j})\, \lambda_{-j}(\mathrm{d}v_{-j}). \tag{8}$$

Consider the following deterministic allocation rule $\hat{q} = (\hat{q}_1, \hat{q}_2, \ldots, \hat{q}_I)$:

$$\hat{q}_i(v) = \begin{cases} \tilde{q}_{\psi^{-1}(i)}(v_{\psi(1)}, v_{\psi(2)}, \ldots, v_{\psi(I)}) & \text{if } v \in D_\psi \text{ for some } \psi \in \Psi, \\ 0 & \text{otherwise.} \end{cases}$$

Step (2) We show that $\hat{q}$ is symmetric. That is, for all $i \in \mathcal{I}$, $v \in V$, and $\psi' \in \Psi$,

$$\hat{q}_i(v_1, v_2, \ldots, v_I) = \hat{q}_{(\psi')^{-1}(i)}(v_{\psi'(1)}, v_{\psi'(2)}, \ldots, v_{\psi'(I)}).$$

Fix $\psi' \in \Psi$. Consider first the case in which $v \notin D_\psi$ for all $\psi \in \Psi$. Then

$$(v_{\psi'(1)}, v_{\psi'(2)}, \ldots, v_{\psi'(I)}) \notin D_\psi$$

for any $\psi \in \Psi$. It follows from the definition of $\hat{q}$ that

$$\hat{q}_i(v) = \hat{q}_{(\psi')^{-1}(i)}(v_{\psi'(1)}, v_{\psi'(2)}, \ldots, v_{\psi'(I)}) = 0.$$

Next, suppose that $v \in D_\psi$ for some $\psi \in \Psi$. For notational simplicity, we relabel $v_{\psi'(i')}$ by $\tilde{v}_{i'}$ for each $i' \in \mathcal{I}$. Let $\tilde{\psi} = (\psi')^{-1} \circ \psi$. Since

$$\tilde{v}_{\tilde{\psi}(i')} = \tilde{v}_{(\psi')^{-1} \circ \psi(i')} = v_{\psi' \circ (\psi')^{-1} \circ \psi(i')} = v_{\psi(i')}$$

and $v \in D_\psi$, we have $\tilde{v} \in D_{\tilde{\psi}}$. Therefore,

$$\hat{q}_{(\psi')^{-1}(i)}(v_{\psi'(1)}, v_{\psi'(2)}, \ldots, v_{\psi'(I)}) = \hat{q}_{\psi_1^{-1}(i)}(\tilde{v}_1, \tilde{v}_2, \ldots, \tilde{v}_I)$$

$$
\begin{aligned}
&= \tilde{q}_{\tilde{\psi}^{-1}\circ\psi_1^{-1}(i)}(\tilde{v}_{\tilde{\psi}(1)}, \tilde{v}_{\tilde{\psi}(2)}, \ldots, \tilde{v}_{\tilde{\psi}(I)}) \\
&= \tilde{q}_{\psi^{-1}(i)}(v_{\psi(1)}, v_{\psi(2)}, \ldots, v_{\psi(I)}) \\
&= \hat{q}_i(v_1, v_2, \ldots, v_I),
\end{aligned}
$$

where the first equality holds because of the relabeling, the second equality follows from the definition of $\hat{q}$ and the fact that $\tilde{v} \in D_{\tilde{\psi}}$, the third equality holds because $\psi' \circ \tilde{\psi} = \psi$ and $\tilde{v}_{\tilde{\psi}(i')} = v_{\psi(i')}$ for all $i' \in \mathcal{I}$, and the last equality follows from the definition of $\hat{q}$ and the fact that $v \in D_\psi$.

Step (3) We show that $\hat{q}$ and q induce the same interim expected allocation probabilities for all agents. For all $i \in \mathcal{I}$ and $v_i \in V_i$, we have

$$
\begin{aligned}
&\int_{V_{-i}} \hat{q}_i(v_1, v_2, \ldots, v_I)\, \lambda_{-i}(\mathrm{d}v_{-i}) \\
&= \sum_{\psi\in\Psi} \int_{V_{-i}} \mathbf{1}_{D_\psi}(v_1, v_2, \ldots, v_I)\hat{q}_i(v_1, v_2, \ldots, v_I)\, \lambda_{-i}(\mathrm{d}v_{-i}) \\
&= \sum_{\psi\in\Psi} \int_{V_{-i}} \mathbf{1}_{D_\psi}(v_1, v_2, \ldots, v_I)\tilde{q}_{\psi^{-1}(i)}(v_{\psi(1)}, v_{\psi(2)}, \ldots, v_{\psi(I)})\, \lambda_{-i}(\mathrm{d}v_{-i}) \\
&= \sum_{\psi\in\Psi} \int_{V_{-i}} \mathbf{1}_{D_{\psi_0}}(v_{\psi(1)}, v_{\psi(2)}, \ldots, v_{\psi(I)})\tilde{q}_{\psi^{-1}(i)}(v_{\psi(1)}, v_{\psi(2)}, \ldots, v_{\psi(I)})\, \lambda_{-i}(\mathrm{d}v_{-i}) \\
&= \sum_{\psi\in\Psi} \int_{V_{-i}} \mathbf{1}_{D_{\psi_0}}(v_{\psi(1)}, v_{\psi(2)}, \ldots, v_{\psi(I)})q_{\psi^{-1}(i)}(v_{\psi(1)}, v_{\psi(2)}, \ldots, v_{\psi(I)})\, \lambda_{-i}(\mathrm{d}v_{-i}) \\
&= \sum_{\psi\in\Psi} \int_{V_{-i}} \mathbf{1}_{D_{\psi_0}}(v_{\psi(1)}, v_{\psi(2)}, \ldots, v_{\psi(I)})q_i(v_1, v_2, \ldots, v_I)\, \lambda_{-i}(\mathrm{d}v_{-i}) \\
&= \sum_{\psi\in\Psi} \int_{V_{-i}} \mathbf{1}_{D_\psi}(v_1, v_2, \ldots, v_I)q_i(v_1, v_2, \ldots, v_I)\, \lambda_{-i}(\mathrm{d}v_{-i}) \\
&= \int_{V_{-i}} q_i(v_1, v_2, \ldots, v_I)\, \lambda_{-i}(\mathrm{d}v_{-i}).
\end{aligned}
$$

The first and seventh equalities hold since $\lambda(\cup_{\psi\in\Psi} D_\psi) = 1$. The second equality follows from the definition of $\hat{q}$. The third and sixth equalities hold since $v \in D_\psi$ if and only if $(v_{\psi(1)}, \ldots, v_{\psi(I)}) \in D_{\psi_0}$. The fourth equality follows from (8). The fifth equality follows from the symmetry of q.

A.3 The limits of the equivalence result

The literature of multi-dimensional screening contains abundant examples illustrating that in the case of a single buyer, a multi-product monopolist may find it beneficial to include lotteries as part of the selling mechanism. For atomless distribution, separable payoffs, and independent types, we present examples here to illustrate that our mechanism equivalence result breaks down if each of these assumptions is violated. Recall that our approach of proving the existence of equivalent deterministic mechanism keeps the ex post transfers unchanged. For the examples on separable payoffs and independent types, when showing the equivalence result breaks down, we impose the restriction that the transfers are kept unchanged.

To prove that there exists a deterministic allocation rule that delivers the same interim expected allocation probabilities for all agents, we do not require the assumption of independent types. We invoke additional assumptions of separable payoffs and independent types to establish that the equivalent deterministic mechanism is BIC. When types are correlated, our approach no longer works. However, in certain cases, one can invoke the Crémer-Mclean type arguments to get around the incentive constraints by adjusting transfers. Indeed, by combining the results in McAfee and Reny (1992) and Miller, Pratt, Zeckhauser, and Johnson (2007), one can show that under the conditions in their papers, there exists a deterministic mechanism that is approximately equivalent.

Example 3 (Atomless distribution)**.** Consider a setting with two agents. Suppose that λ_1 has an atom $d \in V_1$ with $\lambda_1(d) > 0$. Note that the assumption of atomless distribution is violated. Consider the following mapping $q = (q^1, q^2, \dots, q^K)$:

$$q(v_1, v_2) = \begin{cases} (\frac{1}{2}, \frac{1}{2}, 0, \dots, 0) & \text{if } v_1 = d; \\ (0, 1, 0, \dots, 0) & \text{otherwise.} \end{cases}$$

Since the outcome only depends on the report of agent 1, for any $v_2 \in V_2$,

$$\int_{V_1} q^1(v_1, v_2)\, \lambda_1(\mathrm{d}v_1) = \int_{V_1 \setminus \{d\}} q^1(v_1, v_2)\, \lambda_1(\mathrm{d}v_1) + \lambda_1(d)\, q^1(d, v_2) = \frac{1}{2}\, \lambda_1(d). \tag{9}$$

We claim that there does not exist a deterministic allocation rule that delivers the same interim expected allocation probabilities for all agents. Suppose to the contrary, such a deterministic allocation rule $\hat{q}$ exists. For all $v_1 \neq d$,

$$\int_{V_2} \hat{q}^1(v_1, v_2)\, \lambda_2(\mathrm{d}v_2) = \int_{V_2} q^1(v_1, v_2)\, \lambda_2(\mathrm{d}v_2) = 0,$$

which implies that $\hat{q}^1(v_1, v_2) = 0$ for λ_2-almost all $v_2 \in V_2$.

By Fubini's Theorem (see Royden and Fitzpatrick (2010, p. 416)), for λ_2-almost all $v_2 \in V_2$, $\hat{q}^1(v_1, v_2) = 0$ for λ_1-almost all $v_1 \in V_1 \setminus \{d\}$. Therefore, for λ_2-almost all $v_2 \in V_2$,

$$\begin{aligned} & \int_{V_1} q^1(v_1, v_2)\, \lambda_1(\mathrm{d}v_1) \\ = & \int_{V_1} \hat{q}^1(v_1, v_2)\, \lambda_1(\mathrm{d}v_1) \\ = & \int_{V_1 \setminus \{d\}} \hat{q}^1(v)\, \lambda_1(\mathrm{d}v_1) + \lambda_1(d)\, \hat{q}^1(d, v_2) \\ = & \lambda_1(d)\, \hat{q}^1(d, v_2). \end{aligned} \tag{10}$$

It follows from (9) and (10) that $\hat{q}^1(d, v_2) = \frac{1}{2}$ for λ_2-almost all $v_2 \in V_2$. We arrive at a contradiction.

Example 4 (Separable payoff)**.** Consider a single-unit common value auction with two bidders. The bidders' valuations for the object (v_1, v_2) are uniformly distributed on the square $[0, 1]^2$. Let λ denote the uniform distribution on the square $[0, 1]^2$. Each agent's payoff is 1 if she gets the object and $v_1 + v_2 \geq 1$, and 0 otherwise. More succinctly, the payoff function of bidder i is $\mathbf{1}_{[1-v_i,1]}(v_j)$ if she gets the object and 0 otherwise. Note that the assumption of separable payoffs is violated.

Consider the stochastic allocation rule $q = (q_1, q_2)$ with $q_1(v) = q_2(v) = 1/2$ for all $v \in V$, where q_i is the probability of agent i getting the object for $i \in \{1, 2\}$. We claim that there does not exists a deterministic and BIC allocation rule that delivers the same interim expected utilities for all bidders. Suppose to the contrary, such a deterministic and BIC allocation rule $\hat{q} = (\hat{q}_1, \hat{q}_2)$ exists.

Then the payoff of bidder 1 for any v_1 is

$$\begin{aligned}
&\int_{V_2} \mathbf{1}_{[1-v_1,1]}(v_2)\, \hat{q}_1(v_1, v_2)\ \mathrm{d}v_2 \\
&= \int_{V_2} \mathbf{1}_{[1-v_1,1]}(v_2)\, q_1(v_1, v_2)\ \mathrm{d}v_2 \\
&= \frac{1}{2} \int_{V_2} \mathbf{1}_{[1-v_1,1]}(v_2)\ \mathrm{d}v_2 \\
&= \frac{1}{2} \int_{1-v_1}^{1} 1\ \mathrm{d}v_2 \\
&= \frac{v_1}{2}.
\end{aligned}$$

Since $\hat{q}$ is BIC, for $v_1 < 1$, agent 1 has an incentive to truthfully report her type than to misreport $v_1 + \epsilon$ (where ϵ is sufficiently small such that $v_1 + \epsilon \leq 1$). That is,

$$\begin{aligned}
\frac{v_1}{2} &\geq \int_{V_2} \mathbf{1}_{[1-v_1,1]}(v_2)\, \hat{q}_1(v_1 + \epsilon, v_2)\ \mathrm{d}v_2 \\
&= \int_{1-v_1}^{1} \hat{q}_1(v_1 + \epsilon, v_2)\ \mathrm{d}v_2 \\
&= \int_{1-v_1-\epsilon}^{1} \hat{q}_1(v_1 + \epsilon, v_2)\ \mathrm{d}v_2 - \int_{1-v_1-\epsilon}^{1-v_1} \hat{q}_1(v_1 + \epsilon, v_2)\ \mathrm{d}v_2 \\
&= \frac{v_1 + \epsilon}{2} - \int_{1-v_1-\epsilon}^{1-v_1} \hat{q}_1(v_1 + \epsilon, v_2)\ \mathrm{d}v_2
\end{aligned}$$

Rearrange the inequality, we have

$$\int_{1-v_1}^{1-v_1+\epsilon} \hat{q}_1(v_1, v_2)\ \mathrm{d}v_2 \geq \frac{\epsilon}{2}.$$

Fix any $\epsilon' > 0$, consider the region

$$D = \{(v_1, v_2) :\ 1 \leq v_1 + v_2 \leq 1 + \epsilon',\ 0 \leq v_1 \leq 1,\ 0 \leq v_2 \leq 1\}.$$

We have

$$\int_D \hat{q}_1(v_1, v_2)\ \mathrm{d}\lambda(v_1, v_2) = \int_0^1 \int_{1-v_1}^{\min\{1, 1-v_1+\epsilon'\}} \hat{q}_1(v_1, v_2)\ \mathrm{d}v_2\ \mathrm{d}v_1 \geq \frac{1}{2}\, \lambda(D). \tag{11}$$

By symmetry, for agent 2, we have

$$\int_D \hat{q}_2(v_1, v_2)\ \mathrm{d}\lambda(v_1, v_2) = \int_0^1 \int_{1-v_2}^{\min\{1, 1-v_2+\epsilon'\}} \hat{q}_2(v_1, v_2)\ \mathrm{d}v_1\ \mathrm{d}v_2 \geq \frac{1}{2}\, \lambda(D). \tag{12}$$

By the feasibility constraint that $\hat{q}_1(v) + \hat{q}_2(v) \leq 1$ for all v, we have

$$\int_D \hat{q}_1(v_1, v_2)\ \mathrm{d}\eta(v_1, v_2) + \int_D \hat{q}_2(v_1, v_2)\ \mathrm{d}\lambda(v_1, v_2) \leq \int_D 1\ \mathrm{d}\eta(v_1, v_2) = \lambda(D), \tag{13}$$

It then follows from (11) - (13) that

$$\int_D \hat{q}_1(v_1, v_2)\ \mathrm{d}\lambda(v_1, v_2) = \int_D \hat{q}_2(v_1, v_2)\ \mathrm{d}\lambda(v_1, v_2) = \frac{1}{2}\,\lambda(D).$$

This implies that for almost all v_1, for all ϵ such that $\epsilon \leq v_1$,

$$\int_{1-v_1}^{1-v_1+\epsilon} \hat{q}_1(v_1, v_2)\ \mathrm{d}v_2 = \frac{\epsilon}{2}.$$

That is, for almost all v_1, for any $1 - v_1 \leq a \leq b \leq 1$,

$$\int_a^b \hat{q}_1(v_1, v_2)\ \mathrm{d}v_2 = \frac{b-a}{2}.$$

Therefore, for almost all v_1, $\hat{q}_1(v_1, v_2) = \frac{1}{2}$ for almost all $v_2 \geq 1 - v_1$. We arrive at contradiction.

Example 5 (Independent types)**.** Consider a single-unit auction with two bidders. Let $V_1 = V_2 = [0, 1]$ endowed with the joint distribution λ, which has the density $\rho(v_1, v_2) = 2$ if $v_1 + v_2 \geq 1$ and 0 otherwise. The payoff function of bidder 1 is 1 if she gets the good and 0 otherwise. Note that the assumption of independent types is violated.

Consider the stochastic allocation rule $q = (q_1, q_2)$ with $q_1(v) = q_2(v) = 1/2$ for all $v \in V$, where q_i is the probability of agent i getting the object for $i \in \{1, 2\}$. We claim that there does not exists a deterministic and BIC allocation rule that delivers the same interim expected utilities for all bidders. Suppose to the contrary, such a deterministic and BIC allocation rule $\hat{q} = (\hat{q}_1, \hat{q}_2)$ exists. Then the payoff of bidder 1 for any $v_1 > 0$ is

$$\int_{1-v_1}^1 \hat{q}_1(v_1, v_2)\,\frac{1}{v_1}\ \mathrm{d}v_2 = \int_{1-v_1}^1 q_1(v_1, v_2)\,\frac{1}{v_1}\ \mathrm{d}v_2 = \frac{1}{2}\int_{1-v_1}^1 \frac{1}{v_1}\ \mathrm{d}v_2 = \frac{1}{2}.$$

Since $\hat{q}$ is BIC, for any $0 < v_1 < 1$, agent 1 has an incentive to truthfully report her type than to misreport $v_1' > v_1$. We have

$$\begin{aligned}
\frac{1}{2} &\geq \int_{1-v_1}^1 \hat{q}_1(v_1', v_2)\,\frac{1}{v_1}\ \mathrm{d}v_2 \\
&= \int_{1-v_1'}^1 \hat{q}_1(v_1', v_2)\,\frac{1}{v_1}\ \mathrm{d}v_2 - \int_{1-v_1'}^{1-v_1} \hat{q}_1(v_1', v_2)\,\frac{1}{v_1}\ \mathrm{d}v_2 \\
&= \frac{v_1'}{v_1}\int_{1-v_1'}^1 \hat{q}_1(v_1', v_2)\,\frac{1}{v_1'}\ \mathrm{d}v_2 - \frac{1}{v_1}\int_{1-v_1'}^{1-v_1} \hat{q}_1(v_1', v_2)\ \mathrm{d}v_2 \\
&= \frac{v_1'}{2v_1} - \frac{1}{v_1}\int_{1-v_1'}^{1-v_1} \hat{q}_1(v_1', v_2)\ \mathrm{d}v_2.
\end{aligned}$$

Rearranging the inequality, we have

$$\int_{1-v_1'}^{1-v_1} \hat{q}_1(v_1', v_2)\,\mathrm{d}v_2 \geq \frac{v_1' - v_1}{2}.$$

Using similar arguments as in Example 4, we can show that for almost all v_1, $\hat{q}_1(v_1, v_2) = \frac{1}{2}$ for almost all $v_2 \geq 1 - v_1$. We arrive at a contradiction.