EFR and backward induction under general preferences*

Yi-Chun Chen[a], Xiao Luo[a], Xuewen Qian[b†]

[a] *Department of Economics, National University of Singapore, Singapore 117570*

[b] *School of Economics, University of Nottingham Ningbo China, Ningbo, 315100, China*

Abstract

We study the behavioral implications of Pearce's (1984) notion of extensive-form rationalizability (EFR) in a broad range of constantly monotone preferences (which require that, conditioning on every contingent event, a constant act that attains a higher payoff will be preferred over another constant act that attains a lower payoff). In a generic class of perfect-information games without relevant ties, we formulate and show that EFR under various preference models yields the unique backward induction (BI) outcome regardless of the elimination order of EFR, even though the EFR strategy profile and the BI strategy profile might be largely distinct. Our result implies that EFR strategic behavior in a variety of preference models is observationally outcome-indistinguishable from the one in a subjective expected utility (SEU) model in a generic game. In a model consisting of all constantly monotone preferences, we show that EFR gives rise to the BI plan of action for all players in generic games without relevant ties in the sense of Heifetz and Perea (2015). *JEL Classification:* C70, C72

Keywords: EFR; backward induction; generic games; general preferences

*We thank Geir Asheim, Yossi Greenberg, Takashi Kunimoto, Shravan Luckraz, Andrés Perea, Yongchuan Qiao, Chen Qu, Xiang Sun, Yang Sun, Yifei Sun, Satoru Takahashi, Chih-Chun Yang, and seminar participants at the National University of Singapore and Nanjing Audit University. An earlier version of the paper was presented at the 18th SAET Conference in Taipei, Taiwan. Financial support from the National University of Singapore and the University of Nottingham Ningbo China is gratefully acknowledged.

†Corresponding author. *E-mail:* ecsycc@nus.edu.sg (Y. Chen), ecslx@nus.edu.sg (X. Luo), Xuewen.Qian@nottingham.edu.cn (X. Qian).

1 Introduction

Pearce (1984) puts forward the notion of extensive form rationalizability (EFR henceforth), which embodies a form of forward induction reasoning: "A player should use all information she acquired about her opponents' past behavior in order to improve her prediction of their future, simultaneous, and past (unobserved) behavior, relying on the assumption that they are rational" (Battigalli 1997, p. 41). EFR and backward induction (BI henceforth) are conceptually distinct and often have different strategic implications. Somewhat surprisingly, as shown by Reny (1992) and Battigalli (1997), EFR and BI must yield the same terminal node in generic games. That is, EFR and BI are outcome-equivalent in a generic class of "perfect-information games without relevant ties"; cf. also Heifetz and Perea (2015); Perea (2019).

Pearce's EFR is defined within a Bayesian framework, in which players are implicitly assumed to maximize the expected utility given a probabilistic belief about opponents' strategy choices. Though subjective expected utility maximization is undoubtedly the dominant model in economics, many economists would probably view axioms such as "transitivity" or "monotonicity" as more basic tenets of rationality than the sure-thing principle and other components of the Savage (1954) model. The Ellsberg paradox and related experimental evidence demonstrate that a decision maker may display an aversion to uncertainty or ambiguity, and thereby motivate generalizations of the subjective expected utility model; see, e.g., Camerer and Weber (1992); Etner et al. (2012); and Gilboa and Marinacci (2013) for surveys on recent developments. Moreover, Bayesian updating is closely related to the dynamic consistency property; violations of dynamic consistency are to be expected if preferences violate the sure-thing principle and such preferences are employed to analyze dynamic-choice problems (cf. Epstein and Le Breton (1993); Ghirardato (2002); and Siniscalchi (2012) for more discussion). The notion of "rationality" should, therefore, be extended to accommodate various modes of behavior with dynamically (in)consistent preferences. The main purpose of this paper is to extend the aforementioned outcome-indistinguishability result of EFR to a variety of preference models, including many important models that arise in economic applications.

We take a preference-based approach to EFR by using the notion of a model of preference. A model of preference is a collection of "conditional preference families" (CPF) adopted by players in a game; a CPF for a player specifies the player's preference conditioning in every contingency. In particular, we consider a class of admissible preferences called "constantly monotone" (CM) preferences, which require only that one strategy be preferred to another strategy if both strategies generate constant payoffs and the former constant payoff is higher than the latter (Definition 1).

The property of constant monotonicity appears to be a fairly weak and innocuous assumption on preferences, because many preference models discussed in the literature satisfy the property. Notably, a CM preference may have no utility function representation, and may not even be complete or transitive. Our analytical framework is also flexible to accommodate dynamically (in)consistent preferences; it is applicable to a wide variety of preference models discussed in the literature—e.g., the subjective expected utility (SEU) model (Savage 1954), the ordinal expected utility model (Börgers 1993), the probabilistic sophistication model (Machina and Schmeidler 1992), the maxmin expected utility model (Gilboa and Schmeidler 1989), the Choquet expected utility model (Schmeidler 1989), the regular preference model (Epstein and Wang 1996), the lexicographic preference model (Blume et al. 1991), the smooth ambiguity model (Klibanoff et al. 2005), and the obviously dominant preference model (Li 2017).

We define the notion of EFR for a preference model by an iterative elimination of "inferior" strategies (Definition 2). We show that the outcome equivalence between BI and EFR holds for a wide range of modes of behavior. More specifically, we show that for a variety of preference models, the EFR strategy profiles result in a unique BI outcome in generic games without relevant ties in the sense of Battigalli (1997); moreover, this outcome equivalence holds regardless of different elimination orders of EFR (Theorem 1). This result extends Battigalli's Theorem 4 to general preferences. For the CM preference model, we show that in a generic game without relevant ties in the sense of Heifetz and Perea (2015), EFR gives rise to the BI plan of action—that is, EFR and BI are indistinguishable in terms of plan of action (Theorem 2).[1]

It is worth noting that EFR is nonmonotonic with respect to preference models (e.g., the EFR solution set under the SEU model fails to be a subset of the one under the CM model); cf. the illustrative example in Section 2. The lack of monotonicity is related to the "order-dependent" aspect of EFR: Different elimination orders may lead to different solution sets; cf. also Catonini (2020) for more discussion. Although EFR strategic behavior under different preference models might be distinct, our main result shows that EFR under a broad class of preference models is observationally indistinguishable from the BI outcome in a generic game.

Our study has a number of implications. First, from an outside observer's point of view, the preferences and strategy—i.e., a complete plan of action—of a player are unlikely to be observed in dynamic games. Theorem 1 offers an indistinguishability result in terms of realization outcome:

[1] A "plan of action" or "reduced strategy" for a player is part of his strategy that specifies the player's actions only at decision nodes that are not precluded by his strategy. Rubinstein (1991) points out that if a strategy of a player is interpreted as a plan of action in a literal manner, it should not need to specify actions after histories that are impossible if the player carries out his strategy.

In generic games, EFR under general preferences yields the same outcome of EFR in the Bayesian framework. In the same vein, Shimoji and Watson (1998) show that Bayesian consistency in Pearce's EFR is behaviorally irrelevant. Our indistinguishability result holds true not only for Bayesian models, but also for non-Bayesian models (e.g., the maxmin expected utility, the Choquet expected utility, and the regular preferences).

Second, EFR captures a forward-induction argument. Theorem 1 implies that forward and backward induction reasoning generically lead to the same outcome, even if players are allowed to adopt different modes of behavior that arise from different models of preference. This outcome equivalence is irrespective of the elimination order of EFR. In the Bayesian framework, Chen and Micali (2013) show the order independence of EFR outcomes in extensive games. Our outcome-order independence is applicable to a wide range of preference models. Moreover, this result holds for small perturbations on payoffs (Corollary 1). Consequently, the BI outcome can be viewed as a robust implication of EFR under a broad class of preferences in generic games.

Third, in the case of generic perfect-information games, Aumann (1995) shows that "common knowledge of rationality" yields a unique BI strategy profile in a generic game; Perea (2014, Theorem 6) shows that "common belief in future rationality" implies the unique BI strategy profile. Theorem 2 asserts that EFR must lead to the BI plan of action for every player in such a generic game, given that players have constantly monotone preferences. Theorem 2 thus offers a novel rationale for BI: By allowing for more admissible preferences in generic games, BI can be supported by EFR through the lens of a plan-of-action strategy, even though the latter embodies a form of forward-induction reasoning. In contrast, the notions of Aumann's rationality and Perea's future rationality rely critically upon the assumption that players are completely forward looking—i.e., they only reason about opponents' behavior in the future of the game, and take opponents' past choices for granted without drawing any conclusions from their past behavior. A distinct feature of our justification for BI is that players can make both kinds of forward/explanatory and backward/predictive inferences in a broad domain of preferences.

The rest of the paper is organized as follows. Section 2 offers an illustrative example, Section 3 introduces our analytical framework, and Section 4 presents the main results. Section 5 includes a number of notable models of preference, and Section 6 concludes. To facilitate reading, all proofs are relegated to the Appendix.

2 An illustrative example

The following example, due to Battigalli (1997), demonstrates the main result of this paper.

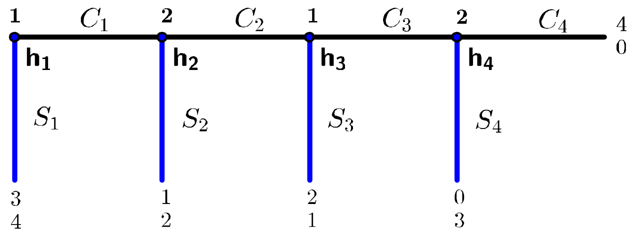


Fig. 1. A generic game.

Apparently, BI yields a unique strategy profile (S_1S_3, S_2S_4). EFR yields the set of strategy profiles that survive iteratively eliminating never "sequential best replies" as follows. In the first round, C_1S_3 and C_2C_4 are eliminated because S_1S_3 is better than C_1S_3 conditioning on h_1; C_2S_4 is better than C_2C_4 conditioning on h_4. In the second round, player 2 thinks that upon reaching h_2, player 1 should play C_1C_3 to survive the first round of elimination; thus, C_2S_4 is player 2's only rational play. In the meantime, player 1 thinks that upon reaching h_3, player 2 should play C_2S_4 to survive the first round of elimination; thus, C_1C_3 is no longer a best response conditioning on h_3. Therefore, C_1C_3, S_2S_4, and S_2C_4 are eliminated in this round and the elimination stops. That is, EFR defines an elimination procedure as follows:

$$\mathbb{BR}^0 = \{S_1S_3, S_1C_3, C_1S_3, C_1C_3\} \times \{S_2S_4, S_2C_4, C_2S_4, C_2C_4\};$$

$$\mathbb{BR}^1 = \{S_1S_3, S_1C_3, C_1C_3\} \times \{S_2S_4, S_2C_4, C_2S_4\};$$

$$\mathbb{BR}^2 = \{S_1S_3, S_1C_3\} \times \{C_2S_4\}.$$

Hence, EFR yields a solution set $\{S_1S_3, S_1C_3\} \times \{C_2S_4\}$ in which every strategy profile gives rise to the same BI outcome: the terminal node through playing S_1. That is, although EFR and BI can prescribe very different strategies—namely, C_2S_4 and S_2S_4 for player 2—they lead to the same outcome in this generic game (in which no same payoff is assigned to two distinct terminal nodes for any player). Note that the elimination procedure involves only a dominance relation between pure strategies conditioning on reachable information sets. We show that in a broad domain of admissible "constantly monotone" preferences, EFR yields the same outcome as EFR under the SEU model in perfect-information games without relevant ties (see Theorem 1).

Now consider the CM model that consists of all constantly monotone preferences. The constant monotonicity of preferences is a rather weak form of monotonicity: It requires that one strategy be preferred to another strategy if (i) the two strategies give rise to constant payoffs and (ii) the former payoff is higher than the latter. Note that C_1S_3 is not a constant-payoff strategy because it may yield different payoffs, 1 or 2, for player 1. The strategy C_1S_3 is not constant-strategy dominated by S_1S_3; thus it cannot be eliminated in the first round. However, C_2C_4, C_1C_3 and C_1S_3 can be consecutively eliminated under constantly monotone preferences (e.g., C_2C_4 is constant-strategy dominated by C_2S_4 conditioning on h_4). That is, EFR under the CM model coincides with a backward iterated dominance procedure. Consequently, under such a rich model of preference, EFR yields the set $\{S_1S_3, S_1C_3\} \times \{S_2S_4, S_2C_4\}$, which is consistent with BI in terms of "plan of action." We show that in the CM model, EFR gives rise to the unique BI plan of action for all players in a perfect-information game without ties in the sense of Heifetz and Perea (2015) (see Theorem 2). This example also shows that the "obviously dominant" preference model of Li (2017) is not rich enough for the "strategic" equivalence between EFR and BI. Under such a preference model, EFR yields the same solution set as the SEU model; hence EFR and BI prescribe different strategies for player 2 in this example.

3 Analytical framework

Let Ω be a (finite) set of states and let $\mathcal{F}(\Omega)$ denote the set of all acts $f : \Omega \to \mathbb{R}$. A preference $\succeq$ on Ω is a binary relation over $\mathcal{F}(\Omega)$. For acts f and g in $\mathcal{F}(\Omega)$, $f \succeq g$ means f is weakly preferred to g; $f \succ g$ means f is preferred to g. An event is a subset of states; let

$$\Sigma_\Omega \equiv \{E : \emptyset \neq E \subseteq \Omega\}$$

denote the collection of all nonempty events.[2] Given any event $E \in \Sigma_\Omega$, let $\mathfrak{P}(\Omega|E)$ denote the set of conditional preferences for which the complement of E is null in the sense of Savage (1954)—i.e., any two acts that yield the same outcomes for each state in E are ranked as being indifferent. That is, $\mathfrak{P}(\Omega|E)$ is the set of preferences that "believe" E (cf., e.g., Morris (1996); Epstein and Wang (1996); and Epstein (1997)); it thus satisfies consequentialism, which requires that conditional preferences should not depend on an event's not occurring.

[2] To deal with very general preferences, we consider a comprehensive analytical framework in which $\wp \in \mathcal{P}_i(\cdot)$ is defined on conditioning events in Σ_Ω; this framework is in the spirit of Myerson's (1986) definition of "conditional probability systems" on finite set Ω. In a measure-theoretic framework, the analytical framework can be simplified by defining $\wp = \left(\succeq_{S_{-i}(h)}\right)_{h \in H_i}$ exclusively for a sigma-algebra that represents player i's information structure as a natural class of observable events that represents player i's information structure in game Γ.

3.1 (Conditional) constantly monotone preferences: CPF

Consider an event $E \in \Sigma_\Omega$. Let $\mathbf{r}_E \in \mathcal{F}(\Omega)$ denote the constant act conditioning on E such that $\mathbf{r}_E(\omega) = r\ \forall \omega \in E$ (where $r \in \mathbb{R}$).

Definition 1. *A preference* $\succeq$ *in* $\mathfrak{P}(\Omega|E)$ *is* constantly monotone (CM) *if for all acts* $\mathbf{r}_E, \mathbf{r}'_E \in \mathcal{F}(\Omega)$, $\mathbf{r}_E \succ \mathbf{r}'_E$ *whenever* $r > r'$.

The constant monotonicity condition is rather weak: It only requires that the (conditional) preference orderings over (conditional) constant acts be consistent with the natural order on real numbers. Obviously, a CM preference is not required to be complete and transitive; it might have no utility function representation. Many preferences discussed in the literature satisfy this condition, including but not limited to subjective expected utility, ordinal expected utility, maxmin expected utility, Choquet expected utility, regular preferences, and the smooth ambiguity decision model. Throughout the paper, we restrict attention to the domain of constantly monotone preferences. Let

$$\mathcal{P}^{CM}(\cdot) \equiv \times_{E \in \Sigma_\Omega} \mathcal{P}^{CM}(\Omega|E),$$

where $\mathcal{P}^{CM}(\Omega|E)$ denotes the admissible set of all CM preferences. We call $\wp = \left(\succeq^{\wp}_E\right)_{E \in \Sigma_\Omega}$ in $\mathcal{P}^{CM}(\cdot)$ a *conditional preference family (CPF).*

Remark 1. A CPF specifies a family of conditional preferences in every hypothetical event. Because we do not impose the requirement of dynamic consistency in our analytical framework, the notion of a CPF can be used to represent any arbitrary family of dynamically (in)consistent preferences (cf. Subsection 4.3). The notion of CPF can be viewed as a natural generalization of a "conditional probability system" (CPS) (Myerson (1986)) or "lexicographic conditional probability system" (LCPS) (Blume et al. (1991)). In the Bayesian framework in which players' payoffs or vNM indexes in games are fixed and common knowledge, a CPF can be used to represent an array of conditional probability measures defined on all conditioning events; for instance,

1. a plain and simple CPF is just a CPF belief with no restriction of an updating rule;

2. a fully Bayesian consistent CPF/CPS belief is required to satisfy Bayes' rule for all conditioning events (see, e.g., Myerson (1986)).[3]

[3] A Bayesian-consistent CPF belief is required to satisfy Bayes' rule only for all non-savage-null conditioning events (see, e.g., Ghirardato (2002)).

3.2 Extensive games with perfect information

We focus on extensive games with perfect information and without chance moves. Consider a (finite) extensive game with perfect information:

$$\Gamma = (N, H, Z, \{A_h\}_{h\in H}, \{u_i\}_{i\in N}),$$

where

- $N = \{1, \ldots, n\}$ is a finite set of players (with typical player $i \in N$);
- H is a finite set of nodes (with an initial node $h^0 \in H$). Let $H_i \subseteq H$ denote the set of decision nodes at which player i must make a choice;
- $Z \subseteq H$ is the set of terminal nodes;
- A_h is a finite set of choices available at decision node $h \in H \backslash Z$;
- $u_i : Z \to \mathbb{R}$ is player i's payoff function that assigns a payoff $u_i(z)$ to each terminal node $z \in Z$.

As usual, assume Γ are common knowledge. In what follows, we also assume players' payoffs as vNM indices are common knowledge, unless explicitly stated otherwise (e.g., Section 5.1(i)).

A (pure) strategy of player i is defined as a mapping $s_i : H_i \to \cup_{h\in H_i} A_h$ such that $s_i(h) \in A_h$ for all decision nodes $h \in H_i$. Denote by S_i the set of player i's strategies. Let $S_{-i} \equiv \times_{j\neq i} S_j$ and $S \equiv S_i \times S_{-i}$. For each strategy profile $s \in S$, a strategic-form payoff to player i is given by $u_i(\mathbf{z}(s))$, where $\mathbf{z}(s)$ denotes the terminal node induced by s. The expected payoff for a mixed-strategy profile is defined in the usual way.

We say game Γ is *without relevant ties* (Battigalli 1997) if for all $i \in N$, $s_i, s_i' \in S_i$ and $s_{-i} \in S_{-i}$,

$$\mathbf{z}(s_i, s_{-i}) \neq \mathbf{z}(s_i', s_{-i}) \Rightarrow u_i(\mathbf{z}(s_i, s_{-i})) \neq u_i\left(\mathbf{z}(s_i', s_{-i})\right),$$

that is, given opponents' choice of strategy profile, if player i's two strategies lead to different terminal nodes, then player i must have different payoffs. In other words, for every player i's decision node $h \in H_i$ and for every two distinct terminal nodes $z, z' \in Z$ following h (induced by player i's different strategies given the opponents' choice of strategy profile), we have that $u_i(z) \neq u_i(z')$. Apparently, a game without relevant ties has a unique BI strategy profile. Throughout the paper, we restrict attention to the generic class of perfect-information games without relevant ties.

From a decision-theoretic perspective, each player i makes a decision in the face of opponents' strategic uncertainty in $\Omega \equiv S_{-i}$; each strategy $s_i \in S_i$ induces an act in $\mathcal{F}(S_{-i})$ that satisfies $s_i(s_{-i}) = u_i(\mathbf{z}(s_i, s_{-i}))$ for all $s_{-i} \in S_{-i}$.

Example 1 (The CM model)**.** *For all $i \in N$, nonempty $E_{-i} \subseteq S_{-i}$, player i's preference $\succeq_{E_{-i}}$ conditioning on E_{-i}, we require that $\succeq_{E_{-i}}$ satisfy the constant monotonicity property:*

$$\forall s_i, s_i' \in S_i,\ s_i \succ_{E_{-i}} s_i' \text{ if } u_i\left(\mathbf{z}(s_i, s_{-i})\right) = r > r' = u_i\left(\mathbf{z}(s_i', s_{-i})\right), \forall s_{-i} \in E_{-i}.$$

In words, s_i constant-strategy dominates s_i' conditioning on E_{-i} if strategies s_i and s_i' induce the conditionally constant acts $\mathbf{r}_{E_{-i}}$ and $\mathbf{r}'_{E_{-i}}$ $(r > r')$, respectively. Let $\mathcal{P}_i^{\mathrm{CM}}(S_{-i}|E_{-i})$ denote the set of player i's CM preferences conditioning on E_{-i} and let

$$\mathcal{P}_i^{\mathrm{CM}}(\cdot) \equiv \times_{E_{-i}\in\Sigma_{S_{-i}}} \mathcal{P}_i^{\mathrm{CM}}(S_{-i}|E_{-i}).$$

Define the CM preference model $\mathcal{P}^{\mathrm{CM}}(\cdot) = \times_{i\in N}\mathcal{P}_i^{\mathrm{CM}}(\cdot)$.

Example 2 (The SEU Model)**.** *In the Bayesian framework, players are assumed to have a subjective probabilistic belief over every uncertain prospect. Let*

$$\Delta^{\otimes}(S_{-i}) \equiv \times_{E_{-i}\in\Sigma_{S_{-i}}} \Delta(S_{-i}|E_{-i}),$$

where $\Delta(S_{-i}|E_{-i})$ is the set of probability distributions on S_{-i} conditioning on E_{-i}, i.e., $\mu(E_{-i}|E_{-i}) = 1, \forall\mu(\cdot|E_{-i}) \in \Delta(S_{-i}|E_{-i})$. For all $i \in N$, $s_i \in S_i$ and $E_{-i} \in \Sigma_{S_{-i}}$, denote conditional expectations by

$$\mathbb{E}^{\mu}\left(u_i\left(\mathbf{z}\left(s_i, \cdot\right)|E_{-i}\right)\right) \equiv \textstyle\sum_{s_{-i}\in S_{-i}} u_i\left(\mathbf{z}\left(s_i, s_{-i}\right)\right)\mu\left(s_{-i}|E_{-i}\right).$$

Let $u_i^{\mu}(s_i) \equiv \left(\mathbb{E}^{\mu}\left(u_i\left(\mathbf{z}\left(s_i,\cdot\right)|E_{-i}\right)\right)\right)_{E_{-i}\in\Sigma_{S_{-i}}}$ denote the family of conditional expectations of s_i under μ. We have two kinds of SEU models for game Γ as follows.

1. *The SEU model (with a prior belief), denoted by $\mathcal{P}^{\mathrm{SEU}}(\cdot)$:*

   $$\mathcal{P}_i^{\mathrm{SEU}}(\cdot) = \left\{u_i^{\mu} : \mu \in \Delta^{\otimes}(S_{-i})\right\},\ \forall i \in N,$$

   where $\mu \in \Delta^{\otimes}(S_{-i})$ represents a CPF belief with no restriction of an updating rule.

2. *The SEU model (with a CPS belief; cf., e.g., Myerson (1986)), denoted by $\mathcal{P}^{\mathrm{SEU}^*}(\cdot)$:*

   $$\mathcal{P}_i^{\mathrm{SEU}^*}(\cdot) = \left\{u_i^{\mu} : \mu \in \Delta^{*}(S_{-i})\right\},\ \forall i \in N,$$

where $\mu \in \Delta^* (S_{-i})$ is a CPS belief on S_{-i}.

A *model of preference (for game Γ)* is defined by

$$\mathcal{P}(\cdot) = \{\mathcal{P}_i(\cdot)\}_{i \in N},$$

where $\emptyset \neq \mathcal{P}_i(\cdot) \subseteq \mathcal{P}_i^{CM}(\cdot)$, $\forall i \in N$. The set $\mathcal{P}_i(\cdot)$ is interpreted as the set of admissible CPFs adopted by player i in game Γ. Note that $\wp \in \mathcal{P}_i(\cdot)$ must be an array of conditional CM preferences in $\mathcal{P}_i^{CM}(\cdot)$.[4]

Let $h \in H$ and $E \equiv E_i \times E_{-i} \subseteq S$. Let $E(h)$ denote the set of strategy profiles in E that reach h; let

$$E_{-i}(h) = \{s_{-i} \in E_{-i} : (s_i, s_{-i}) \text{ reaches } h \text{ for some } s_i \in S_i\},$$

that is, $s_{-i} \in E_{-i}(h)$ represents a strategy profile in E_{-i} that can reach h (through a strategy $s_i \in S_i$). We say "s_i can reach h via E_{-i}" if s_i can reach h through a strategy profile $s_{-i} \in E_{-i}$. For decision node $h \in H_i$, $S_{-i}(h)$ can be viewed as a strategic-form representation of player i's information structure at h.

4 EFR and BI under general preferences

4.1 EFR under general preferences: Definition

In the Bayesian framework, the behavioral assumption for EFR is that each player forms a consistent probabilistic conjecture about opponents' behavior and then chooses a sequential best response subject to this conjecture; cf. Battigalli (1997). We extend this idea to general preferences. Consider an arbitrary preference model $\mathcal{P}(\cdot)$ for game Γ. Let $E = E_i \times E_{-i} \subseteq S$. For player $i \in N$, a strategy $s_i \in E_i$ is a *$\mathcal{P}_i$-best reply* on E if there exists a CPF $\wp \in \mathcal{P}_i(\cdot)$ such that for all $h \in H_i$, if s_i can reach h via E_{-i}, then $s_i' \not\succ^{\wp}_{E_{-i}(h)} s_i$ for all $s_i' \in E_i$ that can reach h via E_{-i}. Let

$$\mathbb{BR}(\mathcal{P}, E) = \times_{i \in N} \mathbb{BR}_i(\mathcal{P}_i, E),$$

where $\mathbb{BR}_i(\mathcal{P}_i, E)$ denotes the set of all $\mathcal{P}_i$-best replies on E for player i. We extend Pearce's (1984) notion of EFR to general preferences. The notion is defined by an iterative elimination of "inferior"

[4] Our approach is consistent with Epstein's (1997, pp. 6-7) notion of a model of preference on a finite space Ω: PREF3 in Epstein (1997) ensures that any preference on a nonempty $E \subseteq \Omega$ can be regarded as a conditional preference on E.

strategies—i.e., never $\mathcal{P}_i$-best replies for each player i in game Γ.

Definition 2. *The* $\mathcal{P}$-EFR procedure *is defined as an elimination sequence* $\{\mathbb{BR}^k(\mathcal{P}, S)\}_{k\geq 0}$ *such that* $\mathbb{BR}^0(\mathcal{P}, S) \equiv S$ *and* $\mathbb{BR}^{k+1}(\mathcal{P}, S) = \mathbb{BR}(\mathcal{P}, \mathbb{BR}^k(\mathcal{P}, S))$ *for all* $k \geq 0$. *Let* $\mathbb{BR}^\infty(\mathcal{P}, S) \equiv \cap_{k\geq 0}\mathbb{BR}^k(\mathcal{P}, S)$ *denote the solution set of* $\mathcal{P}$*-EFR strategy profiles.*

In the Bayesian framework in which a player's belief is represented by a probabilistic belief about opponents' strategy profiles and satisfies the Bayesian updating rule in every contingency, the definition of $\mathcal{P}^{SEU^*}$-EFR delivers a (correlated) version of Pearce's EFR; in two-player games, $\mathcal{P}^{SEU^*}$-EFR is consistent with Pearce's EFR (using independent beliefs). Shimoji and Watson (1998) show that Pearce's notion of EFR is behaviorally irrelevant with respect to whether players update their beliefs according to Bayes' rule; thus $\mathcal{P}^{SEU}$-EFR and $\mathcal{P}^{SEU^*}$-EFR yield the same solution set. For the sake of simplicity, we focus on an unconstrained SEU model $\mathcal{P}^{SEU}(\cdot)$ in which players are not restricted to independent beliefs about opponents' strategies and the beliefs are not required to satisfy the Bayesian updating rule upon arrival of new information—that is, each player has a CPF belief in a game situation.

Remark 2. Epstein (1997) and Chen et al. (2016) study normal-form rationalizability in strategic games with various modes of behavior; Definition 2 offers a definition of EFR in dynamic perfect-information games under general preferences. Observe that the definition of $\mathbb{BR}_i(\mathcal{P}_i, E_{-i})$ requires the adoption of an updated preference $\succeq^{\wp}_{E_{-i}(h)}$ (according to $\wp \in \mathcal{P}_i(\cdot)$) at decision node $h \in H_i$ whenever $E_{-i}(h) = E_{-i} \cap S_{-i}(h) \neq \emptyset$. In other words, if E_{-i} is not falsified at decision node h, then player i must hold a "compatible" preference $\succeq^{\wp}_{E_{-i}(h)}$ that believes E_{-i}. This compatibility property is related to the probabilistic notion of "strong belief" in Battigalli and Siniscalchi (2002).

4.2 Main results

We can now present the central result of this paper—i.e., an outcome equivalence between BI and EFR under general preferences, regardless of the elimination order of $\mathcal{P}$-EFR.[5]

Theorem 1. *Consider an arbitrary preference model* $\mathcal{P}(\cdot)$ *for a (finite) perfect-information game without relevant ties in Battigalli (1997). If* $\mathcal{P}^{SEU}(\cdot) \subseteq \mathcal{P}(\cdot)$*, then* $\mathcal{P}$*-EFR yields a unique BI*

[5] An elimination order of $\mathcal{P}$-EFR is a more flexible form of elimination procedure that allows players to eliminate some, but not necessarily all, "inferior" strategies in each round of elimination (see Definition 4 in the Appendix). The $\mathcal{P}$-EFR procedure in Definition 2 is a "fast" $\mathcal{P}$-EFR elimination order.

outcome (in terms of induced terminal node). Moreover, every elimination order of $\mathcal{P}$-EFR yields the same BI outcome.

Theorem 1 generalizes Battigalli's (1997, Theorem 4) outcome equivalence for the SEU preference model to a wide range of preference models. Our result asserts that in any arbitrary preference model $\mathcal{P}(\cdot) \supseteq \mathcal{P}^{SEU}(\cdot)$, $\mathcal{P}$-EFR must yield the unique BI outcome in a perfect-information game without relevant ties. Theorem 1 thus offers an outcome-indistinguishability result for EFR: The relaxation of the SEU model does not change the observed EFR play, and in that sense has no empirical significance—e.g., by weakening knowledge of cardinal preferences in the SEU model, EFR under the ordinal expected utility model (Börgers 1993) cannot affect the BI outcome in generic games. The outline of the proof of Theorem 1 goes as follows.

(1) For any preference model $\mathcal{P}(\cdot) \supseteq \mathcal{P}^{SEU}(\cdot)$, every never $\mathcal{P}_i$-best reply is a never $\mathcal{P}_i^{SEU}$-best reply. Therefore, any elimination order of $\mathcal{P}$-EFR can be regarded as the foremost segment of an elimination order of $\mathcal{P}^{SEU}$-EFR—that is, the first few steps of an elimination order of $\mathcal{P}^{SEU}$-EFR. Because (i) Chen and Micali (2013) show that every elimination order of $\mathcal{P}^{SEU}$-EFR leads to the same outcome set of terminal nodes and (ii) the BI procedure is a possible elimination order of $\mathcal{P}^{SEU}$-EFR in a generic game, every elimination order of $\mathcal{P}$-EFR retains the BI outcome in a generic game.

(2) Any elimination order of $\mathcal{P}$-EFR yields a unique outcome in a generic game (see Lemma 2 in the Appendix). Suppose, on the contrary, that an elimination order of $\mathcal{P}$-EFR yields different terminal nodes. Then, there exists a "last" player such that no following players can induce different terminal nodes by their $\mathcal{P}$-EFR strategies. Hence, by the constant monotonicity property of $\mathcal{P}(\cdot)$, the last player should unambiguously single out his optimal (constant) action under the preference model $\mathcal{P}(\cdot)$ and could not induce different terminal nodes in a generic game. By (1), every elimination order of $\mathcal{P}$-EFR must yield the unique BI outcome in a generic game.

The example in Figure 2 shows that in a nongeneric game with relevant ties, EFR might be outcome-distinguishable under general preferences.[6]

[6] Battigalli (1997) uses the same extensive-form structure, without relevant ties, to demonstrate that iterated deletion of inferior strategies does not coincide with iterated deletion of dominated strategies. We here use this extensive-form structure with relevant ties to show that different preference models might generate different observed outcomes for the EFR solution concept, although it never happens in a generic case.

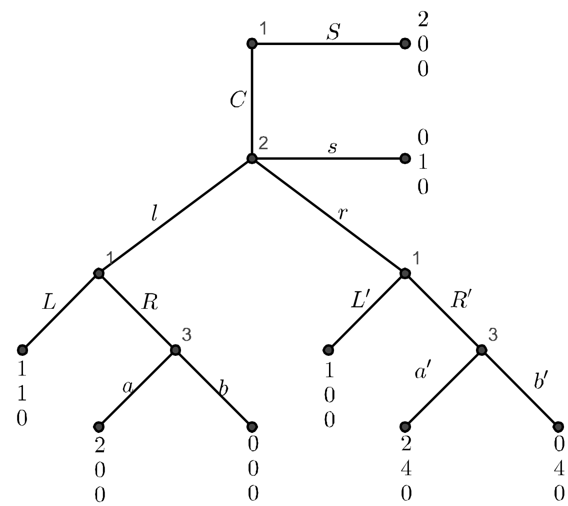


Fig. 2. EFR is outcome-distinguishable in a nongeneric game with relevant ties.

In this game, player 3 has relevant ties at his two decision nodes; players 1 and 2 have no relevant ties. The $\mathcal{P}^{SEU}$-EFR procedure coincides with iterated elimination of (conditionally) strictly dominated strategies in which player 1's strategy CLL' is eliminated in the first round; player 2's strategy l is eliminated in the second round. Note that player 1's strategy CLL' is strictly dominated by a pure strategy S; player 2's strategy l is strictly dominated only by a mixed strategy (e.g., $0.5s+0.5r$), conditioning on player 2's decision node, in the reduced game after performing the first round of elimination. Consider a preference model $\mathcal{P}^{SD}(\cdot)$ in which $\mathcal{P}_i^{SD}(\cdot)$ contains all "strong dominance" preferences, which are a strong form of strict dominance, by using a pure-strategy dominator, rather than a mixed-strategy dominator (cf. Subsection 5.3). The $\mathcal{P}^{SD}$-EFR procedure stops after the one-round elimination of strict dominated strategies because player 2's strategy l is not (conditionally) strongly dominated by any pure strategy. Consequently, $\mathcal{P}^{SD}$-EFR is outcome-distinguishable from $\mathcal{P}^{SEU}$-EFR in this nongeneric game with relevant ties. (See Appendix^{+} for more discussion of EFR under non-Bayesian beliefs preferences in this example.)

Theorem 1 shows an outcome equivalence in the case of generic games. As demonstrated by the illustrative example in Section 2, a $\mathcal{P}^{SEU}$-EFR strategy profile (S_1C_3, C_2S_4) could be quite different from the unique BI strategy profile (S_1S_3, S_2S_4); thus, $\mathcal{P}$-EFR is strategic distinguishable from $\mathcal{P}^{SEU}$-EFR in terms of complete plan of action. Notably, for player 2, $\mathcal{P}^{SEU}$-EFR strategy C_2S_4 is different from the BI strategy S_2S_4 according to the notion of "plan of action," because they specify distinct actions at player 2's first decision node. (A plan of action for a player is part of his strategy that specifies the player's actions only at decision nodes that are not precluded by his plan; see Rubinstein (1991).) However, if we consider the rich model $\mathcal{P}^{CM}(\cdot)$, this example shows that $\mathcal{P}^{CM}$-EFR must yield the same BI plan of action for all players. More specifically, in this example $\mathcal{P}^{CM}$-EFR defines a (unique) backward iterated dominance procedure:

$$\mathbb{BR}^0(\mathcal{P}^{CM}, S) = \{S_1S_3, S_1C_3, C_1S_3, C_1C_3\} \times \{S_2S_4, S_2C_4, C_2S_4, C_2C_4\};$$

$$\mathbb{BR}^1(\mathcal{P}^{CM}, S) = \{S_1S_3, S_1C_3, C_1S_3, C_1C_3\} \times \{S_2S_4, S_2C_4, C_2S_4\};$$

$$\mathbb{BR}^2(\mathcal{P}^{CM}, S) = \{S_1S_3, S_1C_3, C_1S_3\} \times \{S_2S_4, S_2C_4, C_2S_4\};$$

$$\mathbb{BR}^3(\mathcal{P}^{CM}, S) = \{S_1S_3, S_1C_3, C_1S_3\} \times \{S_2S_4, S_2C_4\};$$

$$\mathbb{BR}^4(\mathcal{P}^{CM}, S) = \{S_1S_3, S_1C_3\} \times \{S_2S_4, S_2C_4\} = \mathbb{BR}^\infty(\mathcal{P}^{CM}, S).$$

Consequently, for every player $i = 1, 2$, $\mathcal{P}^{CM}$-EFR strategies generate a unique BI plan of action (S_i). Theorem 2 below shows that $\mathcal{P}^{CM}$-EFR gives rise to a unique BI plan of action for all players in generic games without relevant ties in the sense of Heifetz and Perea (2015).

Theorem 2. *Suppose Γ is a (finite) perfect-information game without relevant ties in the sense of Heifetz and Perea (2015)—i.e., for every player i's decision node $h \in H_i$ and for every two distinct terminal nodes $z, z' \in Z$ following h, we have $u_i(z) \neq u_i(z')$. Then $\mathcal{P}^{CM}$-EFR yields the unique BI plan of action for all players. Moreover, $\mathcal{P}^{CM}$-EFR is an order-independent elimination procedure in terms of plan of action.*

Theorem 2 asserts that $\mathcal{P}^{CM}$-EFR must generically lead to the BI plan of action for all players. That is, by allowing for more admissible preferences in generic games, the notion of EFR provides a justification for playing BI strategies through the lens of a plan of action. The concept of EFR captures the spirit of forward-induction reasoning: Players try to draw inferences about future play from past strategic behavior; they should not simply regard surprise events as "mistakes" or "trembles." In the literature, Aumann (1995) shows that "common knowledge of rationality" yields the unique BI strategy profile in a generic game; Perea (2014, Theorem 6) shows that "common belief in future rationality" implies the unique BI strategy profile.

It is worth noting that the notions of Aumann's rationality and Perea's future rationality rely crucially on the assumption that players are completely forward looking—i.e., they only reason about opponents' behavior in the future of the game, and take opponents' past choices for granted without drawing any conclusion from their past behavior; cf. Stalnaker (1998) and Asheim (2002) for related discussion. The crux of their characterization results for BI is that their definitions of "rationality" rule out the forward-induction pattern of strategic reasoning in game situations. To avoid grappling with the conceptual "conundrum" of such an approach, Arieli and Aumann (2015) make use of the notion of "strong belief" that captures a form of forward inferences. Arieli and Aumann show that the logic of BI applies only to "simple" generic games in which each player moves just once; moreover, "common strong belief of rationality" implies the unique BI strategy profile in a generic game, given that a player's agents act independently, rendering forward inferences invalid. Remarkably, Reny (1992, Proposition 3) and Battigalli (1997, Theorem 4) show that forward-induction and backward-induction reasoning can be reconciled in generic games, in terms of the outcome of play. Theorem 2 extends this line of outcome equivalence to a plan-of-action equivalence by allowing for more flexible preferences. In this respect, our equivalence result offers an additional justification for BI in generic games, in which players can make both kinds of forward/explanatory and backward/predictive inferences. The typologies of players' preferences can provide an interesting explanation of similarities and differences between forward-induction and backward-induction strategic behavior.

Perea (2014, Theorem 6.1) shows that the order-independent "backward dominance procedure" yields the BI strategies for every player in an arbitrary perfect-information game without relevant ties. Our Theorem 2 implies that such an equivalence result still holds true by accommodating forward-induction reasoning in the rich domain of CM preferences. That is, forward induction has no bite for the CM preference model.

Remark 3. Battigalli (1997) introduces the notion of "no relevant ties." This notion reflects the idea that whenever a player's choice can affect the outcome (given opponents' strategies), this also affects the player's own payoff. According to Battigalli's definition of no relevant ties, if opponents' strategies can affect the outcome (given a player's choice), then the player's payoff ties are deemed "irrelevant"—that is, they could not affect $\mathcal{P}^{SEU}$-EFR strategic behavior under this circumstance. However, such payoff ties can be "relevant" to $\mathcal{P}^{CM}$-EFR strategic behavior when players are allowed to have constantly monotone preferences. We thereby adopt a stronger form of "no relevant ties" in Heifetz and Perea (2015). That is, whenever a player's choice can affect the outcome (possibly by means of opponents' strategies), it also affect the player's own payoffs—i.e., players are required to have no relevant payoff ties in this sense. Obviously, if a game is without relevant ties in the sense of Heifetz and Perea, then the game is also without relevant ties in the sense of Battigalli. The reverse is not necessarily true: As demonstrated by the example in Figure 3, a game without relevant ties is in the sense of Battigalli, but not in the sense of Heifetz and Perea.

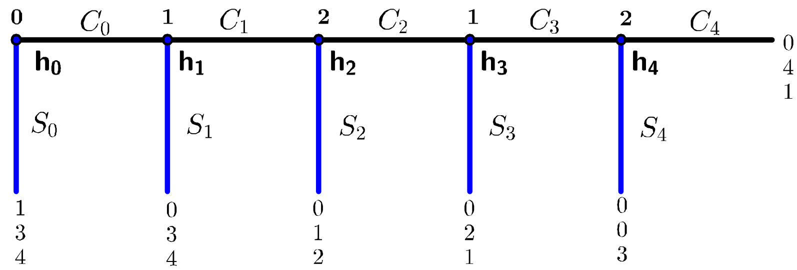


Fig. 3. A variation of the game in Figure 1.

In this game, player 0's payoff ties at the last five terminal nodes are "irrelevant" in the sense of Battigalli (1997) because these terminal nodes are not essentially affected by player 0's own choice (although they are affected by opponents' strategies). However, such payoff ties do matter for $\mathcal{P}^{CM}$-EFR strategic behavior. In this game, S_0 constant-dominates C_0. Therefore, h_1 can never be reached after removing C_0 in the first round of the $\mathcal{P}^{CM}$-EFR elimination procedure. Consequently, C_1C_3 becomes a $\mathcal{P}^{CM}$-EFR strategy with such payoff ties. In contrast, if there are no such payoff ties through perturbing player 0's payoffs, then C_0 is not constant-dominated and thus it cannot be removed in the first round of the $\mathcal{P}^{CM}$-EFR elimination procedure. As a consequence, C_1C_3 is no longer a $\mathcal{P}^{CM}$-EFR strategy without such payoff ties. Heifetz and Perea's (2015) definition nicely captures this type of "relevant" ties for $\mathcal{P}^{CM}$-EFR strategic behavior.[7] This example shows that in a perfect-information game without relevant ties in the sense of Heifetz and Perea, $\mathcal{P}^{CM}$-EFR defines a (unique) backward iterated dominance procedure; $\mathcal{P}^{CM}$-EFR is consistent with BI in terms of plan of action.

How robust are our equivalence results in Theorems 1 and 2 if players' payoffs are allowed to be slightly perturbed? While the notion of "no relevant ties" is preserved under payoff perturbations, the EFR solution set could be sensitive to small payoff perturbations in the broad domain of constantly monotone preferences (cf. also Footnote 7). Corollary 1 below asserts that Theorems 1 and 2 are robust to small perturbations of payoffs. We say Γ^{ε} is an *ε-perturbation of game* Γ if,

[7] This example also shows that $\mathcal{P}^{CM}$-EFR could be sensitive to small payoff perturbations in a perfect-information game without relevant ties in the sense of Battigalli: By appropriately perturbing player 0's payoffs in Figure 3, C_1C_3 might no longer be a $\mathcal{P}^{CM}$-EFR strategy. Heifetz and Perea's (2015) definition of "relevant" ties rules out such an undesirable incident (see Corollary 1).

for every player $i \in N$, payoff function u_i in Γ is replaced by a slightly perturbed payoff function u_i^ε—i.e., sup norm $\|\mathbf{u}_i^\varepsilon - \mathbf{u}_i\|_\infty \leq \varepsilon$, where $\mathbf{u}_i = (u_i(z))_{z \in Z}$ denotes i's payoff vector and $\varepsilon > 0$ is sufficiently small.

Corollary 1. *(i) Suppose that Γ is a (finite) perfect-information game without relevant ties in the sense of Battigalli (1997); Γ^ε is an ε-perturbation of Γ; and $\mathcal{P}(\cdot)$ is an arbitrary preference model for Γ^ε such that $\mathcal{P}(\cdot) \supseteq \mathcal{P}^{SEU}(\cdot)$. Then, $\mathcal{P}$-EFR for Γ^ε yields the unique BI outcome in Γ, regardless of the elimination order of $\mathcal{P}$-EFR. (ii) Suppose that Γ is a (finite) perfect-information game without relevant ties in the sense of Heifetz and Perea (2015); and Γ^ε is an ε-perturbation of Γ. Then, $\mathcal{P}^{CM}$-EFR for Γ^ε yields the unique BI plan of action for all players in Γ.*

5 Preference models: Examples

In this section, we present a number of notable models of preference for a (finite) extensive game Γ with perfect information. We consider three categories of preference models: (1) preference models (with Bayesian beliefs), (2) preference models (with non-Bayesian beliefs), and (3) dominance models (with no beliefs). It is easy to see that these preference models reside in the domain of CM preferences (i.e., $\mathcal{P}(\cdot) \subseteq \mathcal{P}^{CM}(\cdot)$).

5.1 Preference models (with Bayesian beliefs)

(i) The Ordinal Expected Utility (OEU) Model. In the SEU model, we implicitly assume that players' payoffs as vNM indices in Γ are common knowledge. Börgers (1993) relaxes this assumption by assuming that only preferences, rather than vNM indices, over outcomes of the game are common knowledge. By introducing (strictly increasing) vNM index $v : \mathbb{R} \to \mathbb{R}$, we can obtain OEU models for game Γ in a similar way. For any $\mu \in \Delta^\otimes(S_{-i})$ (where $\Delta^\otimes(S_{-i})$ is the set of CPF beliefs on S_{-i}), let

$$u_i^{\mu,v}(s_i) \equiv \left(\mathbb{E}^{\mu,v}\left(u_i\left(\mathbf{z}(s_i,\cdot)\,|E_{-i}\right)\right)\right)_{E_{-i}\in\Sigma_{S_{-i}}},\ \forall s_i \in S_i$$

where $\mathbb{E}^{\mu,v}\left(u_i\left(\mathbf{z}(s_i,\cdot)\,|E_{-i}\right)\right) \equiv \sum_{s_{-i}\in S_{-i}} v\left(u_i\left(\mathbf{z}(s_i,s_{-i})\right)\right)\mu(s_{-i}|E_{-i})$. The OEU model $\mathcal{P}^{OEU}(\cdot)$ can be defined as:

$$\mathcal{P}_i^{OEU}(\cdot) = \left\{u_i^{\mu,v} : \mu \in \Delta^\otimes(S_{-i}) \text{ and } v \text{ is a (strictly increasing) vN-M index}\right\},\ \forall i \in N.$$

Börgers shows that the $\mathcal{P}^{OEU}$-best reply can be characterized by a notion of "pure-strategy dominance."

(ii) The Probability Sophisticated (PS) Model. By dropping the assumption of an expected utility functional form but keeping the assumption that players' preferences are based on probabilistic beliefs, Machina and Schmeidler (1992) introduce the probability sophisticated model. Let $V_i(\cdot)$ denote Machina and Schmeidler's (non-)expected utility preference functional on distributions on the payoff-outcome set $u_i(Z)$. Let

$$S_{-i}(u_i(z)) \equiv \{s_{-i} \in S_{-i} : u_i(\mathbf{z}(s_i, s_{-i})) = u_i(z)\}, \ \forall z \in Z.$$

For any $\mu \in \Delta^{\otimes}(S_{-i})$, let

$$V_i^{\mu}(s_i) \equiv \left(V_i\left(p_{\mu(s_{-i}|E_{-i})}\right)\right)_{E_{-i} \in \Sigma_{S_{-i}}}, \ \forall s_i \in S_i,$$

where $p_{\mu(s_{-i}|E_{-i})}(u_i(z)) = \mu(S_{-i}(u_i(z))|E_{-i}) \ \forall z \in Z$. We can define the PS model $\mathcal{P}^{PS}(\cdot)$ as follows:

$$\mathcal{P}_i^{PS}(\cdot) = \left\{V_i^{\mu} : \mu \in \Delta^{\otimes}(S_{-i})\right\}, \ \forall i \in N.$$

5.2 Preference models (with non-Bayesian beliefs)

The Ellsberg paradox and related experimental evidence demonstrate that a decision maker may display an aversion to uncertainty or ambiguity, and thereby motivate generalizations of the Bayesian models. Unlike in the case of Bayesian models, players may not have a subjective probabilistic belief over every uncertain prospect. Here we consider three special classes of preference models: Machina and Schmeidler's (1992) maxmin expected utility with multiple-priors beliefs, Epstein and Wang's (1996) regular preferences, and Klibanoff et al.'s (2005) smooth ambiguity model by adopting a second-order probabilistic belief over uncertain prior beliefs.

(i) The Maxmin Expected Utility (MEU) Model. We can define MEU models with a multiple prior set of beliefs under different updating rules for game Γ as follows. For (nonempty) compact subset $\Delta \subseteq \Delta^{\otimes}(S_{-i})$, let

$$u_i^{\Delta}(s_i) \equiv \left(\min_{\mu \in \Delta} \mathbb{E}^{\mu}\left(u_i(\mathbf{z}(s_i, \cdot)|E_{-i})\right)\right)_{E_{-i} \in \Sigma_{S_{-i}}}, \forall s_i \in S_i.$$

1. The MEU model (with multiple prior beliefs), denoted by $\mathcal{P}^{MEU}(\cdot)$:

$$\mathcal{P}_i^{MEU}(\cdot) = \left\{u_i^{\Delta} : \text{nonempty, convex and compact } \Delta \subseteq \Delta^{\otimes}(S_{-i})\right\}, \ \forall i \in N.$$

2. The MEU model (with multiple prior CPS beliefs in Ahn (2016)), denoted by $\mathcal{P}^{MEU^*}(\cdot)$:

$$\mathcal{P}_i^{MEU^*}(\cdot) = \left\{u_i^{\Delta} : \text{nonempty, convex and compact } \Delta \subseteq \Delta^*(S_{-i})\right\}, \ \forall i \in N,$$

where $\Delta^*(S_{-i}) \subseteq \Delta^{\otimes}(S_{-i})$ is the set of CPS beliefs on S_{-i}.

(ii) The Regular Preference (RP) Model. Epstein and Wang (1996) defined a class of "regular" preferences that can be represented by utility functions; see also Epstein (1997). Regular preferences accommodate nonexpected utility models, e.g., Choquet expected utility. Let $U^R(\cdot|E_{-i})$ be the set of regular preferences on S_{-i} that know E_{-i}. In particular, $u^R(\cdot|E_{-i}) \in U^R(\cdot|E_{-i})$ satisfies:[8]

1. (Conditional) Certainty Equivalence: $u^R(s_i|E_{-i}) = r$ if $u_i(\mathbf{z}(s_i, s_{-i})) = r$, $\forall s_{-i} \in E_{-i}$.
2. (Conditional) Weak Monotonicity: $u^R(s_i|E_{-i}) \geq u^R(s_i'|E_{-i})$ if $u_i(\mathbf{z}(s_i, s_{-i})) \geq u_i(\mathbf{z}(s_i', s_{-i}))$, $\forall s_{-i} \in E_{-i}$.

We can define a regular preference model $\mathcal{P}^{RP}(\cdot)$ for game Γ as follows:

$$\mathcal{P}_i^{RP}(\cdot) = \times_{E_{-i} \in \Sigma_{S_{-i}}} U^R(\cdot|E_{-i}), \forall i \in N.$$

(iii) The Smooth Ambiguity (SA) Model. According to Klibanoff et al. (2005), players have a second-order probability over their possible prior beliefs about opponents' strategies. Let $\phi : \mathbb{R} \to \mathbb{R}$ be an increasing function that captures ambiguity attitudes, and let μ be a second-order probabilistic belief over $\Delta^{\otimes}(S_{-i})$. The conditional utility on nonempty subset E_{-i} is defined as

$$u_i^{\phi,\mu}(s_i|E_{-i}) = \int_{\Delta^{\otimes}(S_{-i})} \phi\left[\sum_{s_{-i} \in S_{-i}} u_i(\mathbf{z}(s_i, s_{-i}))\,\pi(s_{-i})\right] d\mu_{E_{-i}}, \ \forall s_i \in S_i,$$

where $\mu_{E_{-i}}$ is the marginal probability measure over $\Delta(S_{-i}|E_{-i})$. We can define an SA model $\mathcal{P}^{SA}(\cdot)$ as follows:

$$\mathcal{P}_i^{SA}(\cdot) = \left\{u_i^{\phi,\mu} : \text{increasing function } \phi : \mathbb{R} \to \mathbb{R} \text{ and } \mu \in \Delta\left(\Delta^{\otimes}(S_{-i})\right)\right\}, \ \forall i \in N,$$

where $u_i^{\phi,\mu}(s_i) \equiv \left(u_i^{\phi,\mu}(s_i|E_{-i})\right)_{E_{-i} \in \Sigma_{S_{-i}}}$, $\forall s_i \in S_i$.

[8] Inner/outer regularity in Epstein and Wang (1996) are both satisfied, since S_{-i} is finite.

5.3 Dominance models (with no beliefs)

In the CM model $\mathcal{P}^{CM}(\cdot)$, preferences are defined by a straightforward (conditional) constant-strategy dominance without being referred to beliefs. We note several noteworthy models of preference by simple (conditional) dominance relations. Consider a game Γ. Let $i \in N$, and let $E_{-i} = \times_{j \neq i} E_j \subseteq S_{-i}$.

We say $s_i \in E_i$ is *conditionally (strictly) dominated* on E_{-i} if there exists $\sigma_i \in \Delta(E_i)$ such that

$$u_i(\mathbf{z}(s_i, s_{-i})) < u_i(\mathbf{z}(\sigma_i, s_{-i})), \forall s_{-i} \in E_{-i};$$

that is, mixed strategy σ_i strictly dominates s_i conditioning on E_{-i}. Define a *(strict) dominance model* $\mathcal{P}^{D}(\cdot)$: For each player $i \in N$, $\mathcal{P}_i^{D}(\cdot)$ is the set of player i's preferences defined by conditional dominance. Similarly, we can easily define a *weak dominance (WD) model* $\mathcal{P}^{WD}(\cdot)$ by replacing it with a weak version of conditional dominance: That is, $s_i \in E_i$ is *conditionally weakly dominated* on E_{-i} if there exists $\sigma_i \in \Delta(E_i)$ such that

$$u_i(\mathbf{z}(s_i, s_{-i})) \leq u_i(\mathbf{z}(\sigma_i, s_{-i})), \forall s_{-i} \in E_{-i}$$

and the inequality is strict for some $s_{-i} \in E_{-i}$; that is, mixed strategy σ_i weakly dominates s_i conditioning on E_{-i}.

The definition of conditional dominance allows the use of mixed strategies as dominators. That is, it is possible that a strategy is conditionally dominated only by a mixed strategy. Based on this consideration, we have several variants of conditional dominance models as follows. We say $s_i \in E_i$ is *conditionally strongly dominated* on E_{-i} if there exists $s_i' \in E_i$ such that

$$u_i(\mathbf{z}(s_i, s_{-i})) < u_i(\mathbf{z}(s_i', s_{-i})), \forall s_{-i} \in E_{-i};$$

that is, strategy s_i' strictly dominates s_i conditioning on E_{-i}. Define a *strong dominance (SD) model* $\mathcal{P}^{SD}(\cdot)$: For each player $i \in N$, $\mathcal{P}_i^{SD}(\cdot)$ is the set of player i's preferences defined by the conditional strong dominance. The SD model is related to Epstein's (1997) monotonic utility model and Chen and Luo's (2012) strongly monotonic reference model.

We say $s_i \in E_i$ is *conditionally pure-strategy dominated* on E_{-i} in the sense of Börgers (1993) if for every $F_{-i} \subseteq E_{-i}$, there exists $s_i' \in E_i$ such that

$$u_i(\mathbf{z}(s_i, s_{-i})) \leq u_i(\mathbf{z}(s_i', s_{-i})), \forall s_{-i} \in F_{-i},$$

and the inequality is strict for some $s_{-i} \in F_{-i}$; that is, s_i' weakly dominates s_i conditioning on F_{-i}. Define a *pure-strategy dominance (PSD) model* $\mathcal{P}^{PSD}(\cdot)$: For each player $i \in N$, $\mathcal{P}_i^{PSD}(\cdot)$ is the set of player i's preferences defined by the conditional pure-strategy dominance. Börgers (1993) shows an equivalence between $\mathcal{P}^{PSD}$ and $\mathcal{P}^{OEU}$.

Li (2017) introduces the notion of "obviously dominant strategy": For any deviation, at any decision node where both strategies first diverge, the best outcome under the deviation is no better than the worst outcome under the dominant strategy. In the same vein, we define a weak form of obvious dominance for comparing with pure strategies in game Γ. Formally, $s_i \in E_i$ is said to be *conditionally obviously dominated* on E_{-i} if there exists $s_i' \in E_i$ such that

$$\max_{s_{-i}\in E_{-i}} u_i(\mathbf{z}(s_i, s_{-i})) \leq \min_{s_{-i}\in E_{-i}} u_i\left(\mathbf{z}(s_i', s_{-i})\right) \text{ and } u_i(\mathbf{z}(s_i, s_{-i})) < u_i\left(\mathbf{z}(s_i', s_{-i})\right) \text{ for some } s_{-i} \in E_{-i}.$$

That is, the obvious dominance relation requires that conditioning on E_{-i}, the best payoff outcome under the obviously dominated strategy (s_i) is no better than the worst payoff outcome under the obvious dominator (s_i'). (We note that a strategy with non-constant payoffs can never be constantly monotone dominated, even if this strategy is strictly dominated and obviously dominated.) Define an *obvious dominance (OD) model* $\mathcal{P}^{OD}(\cdot)$: For each player $i \in N$, $\mathcal{P}_i^{OD}(\cdot)$ is the set of player i's preferences defined by the conditional obvious dominance.

For all dominance models $\mathcal{P}(\cdot)$, the notion of $\mathcal{P}_i$-best reply can be presented by the dominance in $\mathcal{P}_i(\cdot)$ conditioning on decision nodes $h \in H_i$. Consider the pure-strategy dominance model $\mathcal{P}^{PSD}(\cdot)$, for example. A strategy $s_i \in E_i$ is a never $\mathcal{P}_i^{PSD}$-best reply on E iff there exists $h \in H_i$ such that for each $F_{-i}(h) \subseteq E_{-i}(h)$ there is $s_i' \in E_i$ such that[9]

(1) s_i and s_i' can both reach h via E_{-i}, and

(2) $u_i(\mathbf{z}(s_i, s_{-i})) \leq u_i(\mathbf{z}(s_i', s_{-i})), \forall s_{-i} \in F_{-i}(h)$, and the inequality is strict for some $s_{-i} \in F_{-i}(h)$.

6 Concluding remarks

The Ellsberg paradox and vast experimental evidence demonstrate that a decision maker may violate some basic tenets of the SEU theory and thereby motivate generalizations of the SEU model. Various preference models have been developed for the purpose of the descriptive validity of the actual behavior. In this paper, we study the solution concept of EFR under a fairly broad range of

[9] See Guarino (2020) for extensive discussion of conditional pure-strategy dominance.

general preferences—namely, constantly monotone (CM) preferences, in a generic class of perfect-information games with relevant ties. More specifically, in generic games, we have shown that any arbitrary preference model that admits SEU preferences and satisfies the constant monotonicity property must yield a unique BI outcome, even though the EFR strategy profile and the BI strategy profile might be distinct (Theorem 1). Our result extends Battigalli's (1997) outcome equivalence between BI and EFR to a wide variety of preference models. It provides an outcome-indistinguishability result for EFR: Relaxation of the SEU model does not change the observed EFR outcome, and thus has no empirical significance in generic games. Moreover, our outcome-equivalence result is robust against any order of EFR elimination procedure. As a by-product, Theorem 1 extends Chen and Micali's (2013) outcome order independence result to a variety of preference models in the class of perfect-information games with relevant ties. In a rich model that contains all CM preferences, we have shown that EFR gives rise to the unique BI plan of action for all players in a generic game without relevant ties in the sense of Heifetz and Perea (2015) (Theorem 2).

An important feature of this paper is that the framework allows players to have broadly general preferences that include SEU preferences as a special case. In light of our analysis, we allow admissible preferences to reside in a broad domain of CM preferences; in particular, we do not require that preferences have utility function representations or that they are dynamically consistent in dynamic strategic contexts. Our analysis is applicable to various preference models discussed in the literature; e.g., the probabilistic sophistication model, ordinal expected utility model, maxmin expected utility model, Choquet expected utility model, smooth ambiguity model, lexicographic preference model, and obvious dominance preference model. Several papers discuss observable implications for rationalizability in strategic games under various models of preference. For example, in finite strategic games, Epstein (1997, pp. 12-13) points out that rationalizable strategic behavior in the ordinal expected utility model (Börgers 1993) is observationally indistinguishable from that in the probabilistic sophistication model (Machina and Schmeidler 1992). Lo (2000) obtains an indistinguishability result for all models of preference that satisfy Savage's axiom P3. Chen and Luo (2012) show an indistinguishability result for compact Hausdoff strategic games under "concave-like" condition. A key difference is that this paper studies the solution concept of EFR in dynamic games, whereas the aforementioned papers focus exclusively on the solution concept of normal-form rationalizability in static games. Indistinguishability in static environments relies on a "monotonicity" property of normal-form rationalizability: The more inclusive a preference model, the more inclusive the rationalizable solution set w.r.t. the preference model. By contrast, the illustrative example in Figure 1 shows the failure of the monotonicity property for EFR in dynamic environments; thus the argument for the outcome-indistinguishability of EFR differs substantially from that for normal-form rationalizability.

Our paper is also related to the literature on the robustness of rationalizability. In the context of strategic games, Hu (2007) shows that the rationalizable set is robust to small deviations from rationality—i.e., the strategies caused by "common p-belief of rationality" are close to the rationalizable set when $p \to 1$. Ely (2005) offers a robustness result of rationalizable sets against small uncertainty of payoffs in incomplete information games. Bergemann and Morris (2007, 2009) and Bergemann et al. (2017) study the "(rationalizable) strategic distinguishability" in static environments in which payoff-relevant types may not be observable. In this respect, this paper offers a (rationalizable) outcome indistinguishability result for general preferences in dynamic environments with perfect information.

7 Appendix: Proofs

To prove our results, we need to introduce some notation and definitions. Consider a (finite) perfect information game Γ with a unique BI strategy profile s^{BI}. We say the *degree* of node $h \in H$ is k if the longest path from h to some terminal node consists of k actions. (The initial node h^0 has the highest degree K_Γ; a terminal node has degree 0.) Let $H^k \subseteq H$ be the set of all nodes that have degree k. Let $H_i(s_i)$ denote the set of i's reachable nodes via $s_i \in S_i$; i.e.,

$$H_i(s_i) = \{h \in H_i : (s_i, s_{-i}) \in S(h) \text{ for some } s_{-i} \in S_{-i}\}.$$

We define a natural procedure to find out BI strategies; cf. Osborne and Rubinstein (1995, Section 6.6).

Definition 3. *The backward iterated dominance procedure (BIDP) is defined as* $\{BI^k\}_{k=0}^{K_\Gamma}$ *such that* $BI^0 \equiv S$ *and for* $k = 1, 2, \ldots, K_\Gamma$*,*

$$BI^k \equiv \times_{i \in N} \left\{ s_i \in BI_i^{k-1} : s_i(h) = s_i^{BI}(h),\ \forall h \in H^k \cap H_i(s_i) \right\}.$$

Obviously, BIDP provides an algorithm to determine the unique BI plan of action (induced by s^{BI}) for all players in Γ. That is, for all $s_i \in BI_i^{K_\Gamma}$, $s_i(h) = s_i^{BI}(h)$, $\forall h \in H_i(s_i^{BI})$; hence, $\mathbf{z}(s) = \mathbf{z}\left(s^{BI}\right)$, $\forall s \in BI^{K_\Gamma}$.

The notion of $\mathcal{P}$-EFR in Definition 2 implicitly requires that all "inferior" strategies be eliminated in every round of elimination. We can consider a more flexible form of elimination procedure that allows us to eliminate some inferior strategies, rather than all inferior ones, in each round of elimination.

Definition 4. *(Luo, Qian and Qu (2020)) An* elimination order *of $\mathcal{P}$-EFR is defined as a reduction sequence of product sets of strategies $\{D^k\}_{k\geq 0}$ in S such that (i) $D^0 = S$; (ii) for all $k \geq 0$, $\mathbb{BR}(\mathcal{P}, D^k) \subseteq D^{k+1} \subseteq D^k$; (iii) $D^\infty \equiv \cap_{k\geq 0} D^k = \mathbb{BR}(\mathcal{P}, D^\infty)$.*

Because Γ is finite, w.l.o.g., we only consider elimination orders of $\mathcal{P}$-EFR that stop after finitely many rounds—i.e., $\{D^k\}_{k\geq 0}$ in Definition 4 satisfies a "stopping" property $D^K = \mathbb{BR}(\mathcal{P}, D^K)$ for some positive integer K. We say $\mathcal{P}$-EFR is *outcome-order independent* if every elimination order of $\mathcal{P}$-EFR yields the same outcome set of terminal nodes. To prove Theorem 1, we need the following Lemmas 1-2.

Lemma 1. *(Chen and Micali (2013)) $\mathcal{P}^{\mathrm{SEU}}$-EFR is outcome-order independent.*

Lemma 2. *Suppose Γ is a (finite) perfect-information game without relevant ties. Let $\{D^k\}_{k\geq 0}$ be an elimination order of $\mathcal{P}$-EFR. Then, D^∞ yields a unique terminal node if $D^\infty \neq \emptyset$.*

Proof of Lemma 2. Suppose, on the contrary, that D^∞ yields multiple terminal nodes. Let $H^\infty \equiv \{h \in H \backslash Z : D^\infty(h) \text{ yields multiple terminal nodes}\}$. Then $h_0 \in H^\infty \neq \emptyset$. Let $\underline{h}$ be a "minimal" element in H^∞. That is, (i) $D^\infty(\underline{h})$ yields multiple terminal nodes and (ii) $D^\infty(h)$ yields a unique terminal node for all nodes h that follow $\underline{h}$. Let $s^1, s^2 \in D^\infty(\underline{h})$ reach terminal nodes $z_1 \neq z_2$, respectively. Assume $\underline{h} \in H_i$. Let h^1 and h^2 be two nodes that immediately follow from actions $s_i^1(\underline{h})$ and $s_i^2(\underline{h})$, respectively. Now consider an arbitrary $s_{-i} \in D_{-i}^\infty(\underline{h})$. Thus, $(s_i^1, s_{-i}) \in D^\infty(h^1)$. By (ii), $\mathbf{z}(s_i^1, s_{-i}) = \mathbf{z}(s^1) = z_1$. Similarly, $\mathbf{z}(s_i^2, s_{-i}) = z_2$. Therefore, s_i^1 and s_i^2 are two constant acts conditioning on $D_{-i}^\infty(\underline{h})$. Since the game has no relevant ties, $u_i(z_1) \neq u_i(z_2)$. Assume $u_i(z_1) < u_i(z_2)$. By constant monotonicity, $s_i^2 \succ^{\wp}_{D_{-i}^\infty(\underline{h})} s_i^1$ for all $\wp \in \mathcal{P}_i(\cdot)$. Hence, $s_i^1 \in D_i^\infty \backslash \mathbb{BR}_i(\mathcal{P}_i, D^\infty)$, contradicting $D^\infty = \mathbb{BR}(\mathcal{P}, D^\infty)$. $\square$

Proof of Theorem 1. Since Γ is a (finite) perfect-information game without relevant ties, Γ must have a unique BI strategy profile s^{BI}. Let $z^{BI} = \mathbf{z}(s^{BI})$ denote the BI outcome in Γ. Because BIDP is an elimination order of $\mathcal{P}^{SEU}$-EFR, by Lemma 1, all elimination orders of $\mathcal{P}^{SEU}$-EFR yield z^{BI}. Now consider an arbitrary elimination order of $\mathcal{P}$-EFR $\{D^k\}_{k\geq 0}$ such that $D^K = \mathbb{BR}(\mathcal{P}, D^K)$ for some positive integer K. Because $\mathcal{P}(\cdot) \supseteq \mathcal{P}^{SEU}(\cdot)$, every never $\mathcal{P}_i$-best reply on E is a never $\mathcal{P}_i^{SEU}$-best reply on E for every player i in Γ. Therefore, we can construct an elimination order of $\mathcal{P}^{SEU}$-EFR $\{\widetilde{D}^k\}_{k\geq 0}$ such that

$$\widetilde{D}^k = \begin{cases} D^k, & \text{if } k \leq K \\ \mathbb{BR}(\mathcal{P}^{SEU}, \widetilde{D}^{k-1}), & \text{if } k > K \end{cases}.$$

That is, $\{D^k\}_{k\geq 0}^K$ is the first few steps in an elimination order of $\mathcal{P}^{SEU}$-EFR $\{\widetilde{D}^k\}_{k\geq 0}$. Again by Lemma 1, $z^{BI} \in \mathbf{z}(D^K)$. By Lemma 2, $\mathbf{z}(D^\infty) = \mathbf{z}(D^K)$ is a singleton and hence $\mathbf{z}(D^\infty) = z^{BI}$. $\square$

To prove Theorem 2, we need the following Lemmas 3-4. Lemma 3 provides a characterization of the $\mathcal{P}_i^{CM}$-best replies by a notion of conditional constant-strategy dominance.

Lemma 3. *Let Γ be a (finite) perfect-information game and $E = E_i \times E_{-i} \subseteq S$. $s_i \notin \mathbb{BR}_i(\mathcal{P}_i^{\mathrm{CM}}, E)$ if and only if s_i is conditionally constantly dominated by some $s_i' \in E_i$ at some $h \in H_i$ on E in the following sense: h is reached by both s_i and s_i' via E_{-i} such that*

$$u_i\left(\mathbf{z}(s_i, s_{-i})\right) = r < r' = u_i\left(\mathbf{z}(s_i', s_{-i})\right), \; \forall s_{-i} \in E_{-i}\left(h\right).$$

Proof of Lemma 3. The "if" part: Suppose s_i is conditionally constantly dominated by $s_i' \in E_i$ at $h \in H_i$ on E. By constant monotonicity, $s_i' \succ_{E_{-i}(h)}^{\wp} s_i$ for all $\wp \in \mathcal{P}_i^{CM}(\cdot)$. Thus, $s_i \notin \mathbb{BR}_i(\mathcal{P}_i^{CM}, E)$.

The "only if" part: Suppose s_i is not conditionally constantly dominated at $h \in H_i$ on E. Consider an arbitrary $h \in H_i$ and $s_i' \in E_i$ such that h is reached by s_i and s_i' via E_{-i}. Pick $\succeq_{E_{-i}(h)}^{*} \in \mathcal{P}_i^{CM}\left(S_{-i}|E_{-i}\left(h\right)\right)$ that specifies only a preference ordering over constant acts conditioning on $E_{-i}\left(h\right)$. Therefore, $s_i' \not\succ_{E_{-i}(h)}^{*} s_i$, $\forall s_i' \in S_i$. Thus, $s_i \in \mathbb{BR}_i(\mathcal{P}_i^{CM}, E)$ by choosing $\wp \in \mathcal{P}_i^{CM}(\cdot)$ such that $\succeq_{E_{-i}(h)}^{\wp} = \succeq_{E_{-i}(h)}^{*}$. $\square$

We next provide an algorithm for the $\mathcal{P}^{CM}$-EFR procedure. Consider a (finite) perfect information game Γ without relevant ties in the sense of Heifetz and Perea (2015). For expositional simplicity, we will assume Γ has no trivial moves in the following sense: For every $i \in N$ and $h \in H_i$, player i has at least two moves at decision node h and no consecutive moves at h—i.e., $h' \notin H_i$ if h' immediately follows h. We say action a has *degree* $k+1$ if it leads to a node that has degree k. (An action with degree 1 leads to a terminal node; it is thus a "constant" act in this situation. Under the assumption that Γ has no trivial moves, the largest degree of action is K_Γ.) Let $A_h^k = \{a \in A_h : a \text{ has degree } k\}$. Let $z^{BI}(h)$ denote the BI terminal node of the subgame starting from $h \in H$, and let $z^{BI}(a) \equiv z^{BI}(h_a)$ where h_a is the node that immediately follows from action a. We define an algorithm for $\mathcal{P}^{CM}$-EFR by iteratively removing inferior strategies, in terms of constantly dominated actions, from surviving actions that have degree less than or equal to the number of elimination rounds.

Definition 5. *The algorithm for $\mathcal{P}^{\mathrm{CM}}$-EFR is defined as $\left\{\Sigma^k\right\}_{k=0}^{K_\Gamma}$ such that $\Sigma^0 \equiv S$ and for $k = 1, 2, \ldots, K_\Gamma$,*

$$\Sigma^k \equiv \times_{i\in N}\left\{s_i \in \Sigma^{k-1} : \forall h \in H_i(s_i),\, \forall s_i(h) \in A_h^{\leq k},\, s_i(h) = \arg\max_{a \in A_h^{\leq k}} u_i(z^{BI}(a))\right\},$$

where $A_h^{\leq k} = \cup_{\kappa \leq k} A_h^{\kappa}$ denotes the set of actions in A_h with degree less than or equal to k.

Lemma 4. *Suppose Γ is a (finite) perfect-information game without relevant ties in the sense of Heifetz and Perea (2015). Then, for all $k \geq 0$, (i) for all $h \in H^{\kappa}$ (where $\kappa > k$), there exist $s, s' \in \Sigma^k(h)$ and $s(h) \neq s'(h)$; (ii) $\Sigma^k \subseteq BI^k$ and $\mathbf{z}(\Sigma^k(h)) = \left\{z^{BI}(h)\right\}$, $\forall h \in H^k$; and (iii) $\mathbb{BR}^k(\mathcal{P}^{\mathrm{CM}}, S) = \Sigma^k$.*

Proof of Lemma 4. (i) Let $h \in H^{\kappa}$, where $\kappa > k$. Then, there is $b \in A_h^{\kappa}$. If cardinality $|A_h^{\kappa}| > 1$, we find another $b' \in A_h^{\kappa}$; if cardinality $|A_h^{\kappa}| = 1$, by no trivial moves, we can find another $b' = \arg\max_{a \in A_h^{\leq k}} u_i(z^{BI}(a))$. By Definition 5,

$$\Sigma^k \equiv \times_{i \in N} \left\{ s_i \in S_i : \forall h \in H_i(s_i), \, \forall s_i(h) \in A_h^{\leq k}, \, s_i(h) = \arg\max_{a \in A_h^{\leq k}} u_i(z^{BI}(a)) \right\}.$$

Thus, we can construct strategy profiles s and s' in Σ^k as follows:

1. $s, s' \in S(h)$, $s(h) = b$ and $s'(h) = b'$;
2. $s(h') = s'(h') = \arg\max_{a \in A_{h'}^{\leq k}} u_i(z^{BI}(a))$ for every $h' \in H$ that has degree less than or equal to k;
3. $s(h'), s'(h') \in A_{h'}^{>k}$ for every $h' \in H$ that has degree greater than k.

(ii) Let $i \in N$ and $k = 0, 1, ..., K_{\Gamma}$. By Definition 3,

$$BI_i^k = \left\{ s_i \in S_i : s_i(h) = s_i^{BI}(h), \ \forall h \in H^{\leq k} \cap H_i(s_i) \right\},$$

where $H^{\leq k} = \{h \in H : h \text{ has degree less than or equal to } k\}$. Let $s_i \in \Sigma_i^k$ and $h \in H^{\leq k} \cap H_i(s_i)$. Then, $A_h^{\leq k} = A_h$ and $s_i(h) = \arg\max_{a \in A_h} u_i(z^{BI}(a)) = s_i^{BI}(h)$. Thus, $s_i \in BI_i^k$. Hence, $\Sigma^k \subseteq BI^k$. Now, consider $h \in H^k$. Let $\hbar$ be the immediate predecessor of h such that $a \in A_{\hbar}$ leads to h. Then $\hbar \in \cup_{\kappa > k} H^{\kappa}$. By (i), $\Sigma^k(h) \neq \emptyset$. Since $\Sigma^k(h) \subseteq \Sigma^k \subseteq BI^k$, $\mathbf{z}(\Sigma^k(h)) = \left\{z^{BI}(h)\right\}$.

(iii) Denote $\mathbb{BR}^k(\mathcal{P}^{CM}, S) = S^k$. We prove this result by induction. For $\kappa = 0$, the result is true because $S^0 = S = \Sigma^0$. Suppose $S^{\kappa} = \Sigma^{\kappa}$ for all $\kappa \leq k$. We proceed to show that $S^{k+1} = \Sigma^{k+1}$. Consider any arbitrary $i \in N$.

Let $s_i \in \Sigma_i^k \backslash \Sigma_i^{k+1}$. Then there is $h \in H_i(s_i)$ such that

$$s_i(h) \in A_h^{\leq k+1} \text{ and } s_i(h) \neq \arg\max_{a \in A_h^{\leq k+1}} u_i(z^{BI}(a)) \equiv a_h^*.$$

Thus, $u_i(z^{BI}(a_h^*)) > u_i(z^{BI}(s_i(h)))$. Let s' be the constructed strategy profile in the proof of (i) such that $s'(h) = a_h^*$. Let h^1 and h^2 be nodes that immediately follow from $s_i(h)$ and a_h^*. Let

$(s_i, s_{-i}) \in \Sigma^k\left(h^1\right)$ and $(s_i', s_{-i}) \in \Sigma^k\left(h^2\right)$. Since $s_i(h)$ and a_h^* have degree less than or equal to k, $h^1, h^2 \in H^{\leq k}$. By (ii), $\mathbf{z}(s_i, s_{-i}) = z^{BI}(h^1) = z^{BI}(s_i(h))$ and $\mathbf{z}(s_i', s_{-i}) = z^{BI}(h^2) = z^{BI}(a_h^*)$ for all $s_{-i} \in \Sigma_{-i}^k(h)$. Therefore, s_i is conditionally constantly dominated by s_i' at h on $\Sigma^k = S^k$. By Lemma 3, $s_i \in S_i^k \backslash S_i^{k+1}$. That is, $S_i^{k+1} \subseteq \Sigma_i^{k+1}$.

Let $s_i \in \Sigma_i^{k+1}$ and $h \in H_i(s_i)$. By Definition 5, we have (1) $s_i(h) \in A_h^{>k+1}$ or (2) $s_i(h) = \arg\max_{a \in A_h^{\leq k+1}} u_i(z^{BI}(a))$. In case (1), $s_i(h)$ leads to some $h' \in H^{>k}$; thus, $s_i \in \Sigma_i^k(h')$. Since Γ has no trivial moves, $j \neq i$ moves at h'. By (i), we have $s^1, s^2 \in \Sigma^k(h')$ such that $s_j^1(h') \neq s_j^2(h')$ and $s_i^1 = s_i = s_i^2$. That is, s_i can lead to at least two different terminal nodes via $\Sigma_{-i}^k(h)$. Because Γ is without relevant ties in the sense of Heifetz and Perea (2015), s_i is not a constant-payoff strategy and thus it is not conditionally constantly dominated at h on Σ^k. Now consider case (2). Let $s' \in \Sigma^k(h)$. Based on the argument in case (1), it suffices to consider $s_i'(h) \in A_h^{\leq k+1}$. Since $s_i(h) = \arg\max_{a \in A_h^{\leq k+1}} u_i(z^{BI}(a))$, by (ii), we have

$$u_i(\mathbf{z}\left(s_i, s_{-i}'\right)) = u_i\left(z^{BI}(s_i(h)\right) \geq u_i\left(z^{BI}(s_i'(h)\right) = \mathbf{z}(s'),\ \forall s' \in \Sigma^k(h).$$

Thus, s_i is not constantly dominated by $s_i' \in \Sigma_i^k(h)$ at h on Σ^k. Therefore, s_i is not conditionally constantly dominated on Σ^k. By Lemma 3, $s_i \in \mathbb{BR}_i(\mathcal{P}^{CM}, \Sigma^k)$. By induction hypothesis, $\mathbb{BR}(\mathcal{P}^{CM}, \Sigma^k) = \mathbb{BR}(\mathcal{P}^{CM}, S^k) = S^{k+1}$. Thus, $s_i \in S_i^{k+1}$. That is, $\Sigma_i^{k+1} \subseteq S_i^{k+1}$. $\square$

Proof of Theorem 2. Let $z^{BI} = \mathbf{z}(s^{BI})$ denote the outcome resulting from the unique BI strategy profile s^{BI} in Γ. By Lemma 4(ii) and (iii), $\mathbb{BR}^{K_\Gamma}(\mathcal{P}^{CM}, S) = \Sigma^{K_\Gamma} \subseteq BI^{K_\Gamma}$ and $\mathbf{z}\left(\Sigma^{K_\Gamma}\right) = \left\{z^{BI}\right\}$. Therefore, there exists $s \in \mathbb{BR}^{K_\Gamma}(\mathcal{P}^{CM}, S) \cap BI^{K_\Gamma}$. Since "equivalent" strategies in $BI_i^{K_\Gamma}$ are indifferent to player i, $BI_i^{K_\Gamma} \subseteq \mathbb{BR}_i^{K_\Gamma}(\mathcal{P}^{CM}, S)$ must survive the $\mathcal{P}^{CM}$-EFR procedure for every $i \in N$. Therefore, $\Sigma^{K_\Gamma} = BI^{K_\Gamma}$. $\square$

Proof of Corollary 1. (i) Let $i \in N$. For all $z, z' \in Z$, define

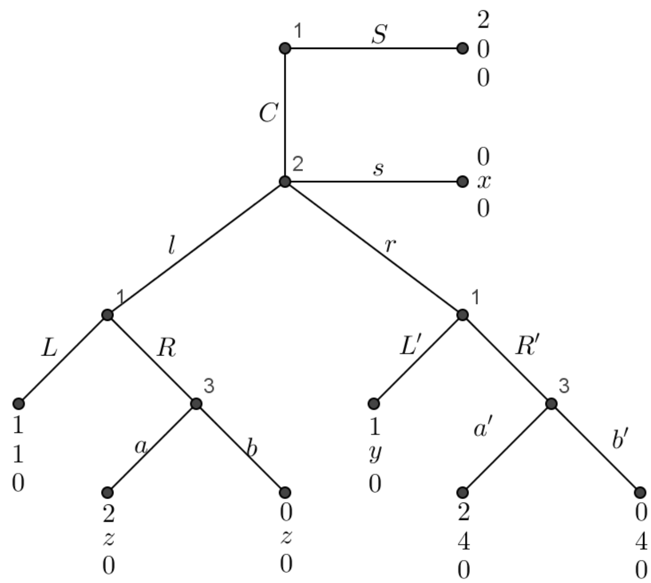
$$\delta_i\left(z, z'\right) = \begin{cases} 1, & \text{if } u_i(z) = u_i(z') \\ \left|u_i(z) - u_i(z')\right|, & \text{if } u_i(z) \neq u_i(z') \end{cases}.$$

Define $\delta = \frac{1}{2}\min_{i \in N}\min_{z, z' \in Z}\left|\delta_i(z, z')\right|$. Let Γ^ε be an ε-perturbation of Γ for sufficiently small positive $\varepsilon \in (0, \delta)$. Then, if $\|\mathbf{u}_i^\varepsilon - \mathbf{u}_i\|_\infty \leq \varepsilon$, then $u_i(z) < u_i(z')$ implies $u_i^\varepsilon(z) < u_i^\varepsilon(z')$, $\forall z, z' \in Z$. Since Γ is without relevant ties, Γ^ε is without relevant ties. Moreover, Γ^ε has the same BI strategy profile of Γ. The result of Corollary 1(i) follows directly from Theorem 1.

(ii) The proof of Corollary 1(ii) is similar to the proof of Corollary 1(i). We thus omit it. $\square$

8 Appendix$^+$

Theorem 1 shows that EFR is outcome-indistinguishable under certain preference models in the class of perfect-information games without relevant ties. As demonstrated by the example below, EFR might generate different outcome sets under different preference models (possibly with non-Bayesian beliefs) if there are relevant ties. Consider the parameterized game $\Gamma(x, y, z)$, where parameters $x, y, z \in \mathbb{R}$ are payoffs for player 2; player 3's payoffs are identical on all terminal nodes.



A parameterized game $\Gamma(x, y, z)$, where $x, y, z \in \mathbb{R}$

We consider four models of preference: $\mathcal{P}^{SEU}(\cdot)$, $\mathcal{P}^{OEU}(\cdot)$, $\mathcal{P}^{MEU}(\cdot)$ and $\mathcal{P}^{SD}(\cdot)$ (cf. Section 5). We will show the following:

- $\mathcal{P}^{SEU}(\cdot)$-EFR and $\mathcal{P}^{OEU}(\cdot)$-EFR are outcome-distinguishable in $\Gamma(2, 1, 4)$.
- $\mathcal{P}^{OEU}(\cdot)$-EFR and $\mathcal{P}^{MEU}(\cdot)$-EFR are outcome-distinguishable in $\Gamma(1, 1, 4)$.
- $\mathcal{P}^{MEU}(\cdot)$-EFR and $\mathcal{P}^{SD}(\cdot)$-EFR are outcome-distinguishable in $\Gamma(1, 0, 0)$.

The EFR procedures for all of the above games and preference models involve at most 2 rounds of elimination.

Round 1: Because CLL' is strictly dominated by S regardless of $x, y, z \in \mathbb{R}$, it should be eliminated for all of the preference models and games $\Gamma(x, y, z)$.

Round 2: Conditioning on player 2's node, player 2 believes that player 1 should play CLR', CRL' or CRR'; player 2's payoffs can be summarized in the following table. (Note: Player 3's choices do not affect payoffs for player 2.)

	CLR'	CRL'	CRR'
s	x	x	x
l	1	z	z
r	4	y	4

- $\Gamma(2,1,4)$: s is eliminated in $\mathcal{P}^{SEU}(\cdot)$ because it is strictly dominated by $0.5s + 0.5r$. However, s cannot be eliminated in $\mathcal{P}^{OEU}(\cdot)$ because it is not pure-strategy dominated in Börgers' sense.
- $\Gamma(1,1,4)$: s is pure-strategy dominated in Börgers' sense so that it is eliminated in $\mathcal{P}^{OEU}(\cdot)$. However, s cannot be eliminated in $\mathcal{P}^{MEU}(\cdot)$ because s, l, r are indifferent if player 2 holds a MEU belief that admits the set of all distributions over $\{CLR', CRL', CRR'\}$.
- $\Gamma(1,0,0)$: l is eliminated in $\mathcal{P}^{MEU}(\cdot)$ because (i) l is no worse than s if and only if player 2 holds a single prior belief that assigns probability 1 to CLR' and (ii) r is strictly better than l under such belief. However, l is not strictly dominated by s or r and hence is not eliminated in $\mathcal{P}^{SD}(\cdot)$.

In this example, BI and EFR are also outcome-distinguishable: $CrR'b'$ is not a BI outcome, but it is an EFR outcome in all of the above models and games. (The possible BI outcomes are S, $ClRa$ or $CrR'a'$, in which player 1 achieves the highest payoff 2.)

References

1. Ahn, D., Hierarchies of ambiguous beliefs, JET 136(2007), 286-201.
2. Arieli, I., Aumann, R.J., The logic of backward induction, JET 159(2015), 443-464.
3. Asheim, G.B., On the epistemic foundation for backward induction, Math. Soc. Sci. 44(2002), 121-144.
4. Aumann, R.J., Backward induction and common knowledge of rationality, GEB 8(1995), 6-19.
5. Battigalli, P., On rationalizability in extensive games, JET 74(1997), 40-61.
6. Battigalli, P., Siniscalchi, M., Strong belief and forward induction reasoning, JET 106(2002), 356-391.
7. Bergemann, D., Morris, S., Strategic Distinguishability with an Application to Robust Virtual Implementation. Yale University, Mimeo (2007).

8. Bergemann, D., Morris, S., Robust virtual implementation, TE 4(2009), 45-88.

9. Bergemann, D., Morris, M., Takahashi, S., Strategic distinguishability and interdependent preferences, JET 168(2017), 329-371.

10. Bernheim, B.D., Rationalizable strategic behavior, Econometrica 52(1984), 1007-1028.

11. Blume, L., Brandenburger, A., Dekel, E., Lexicographic probabilities and choice under uncertainty, Econometrica 59(1991), 61-79.

12. Börgers, T., Pure strategy dominance, Econometrica 61(1993), 423-430.

13. Camerer, C., Weber, M., Recent developments in modeling preferences: Uncertainty and ambiguity, J. Risk Uncertainty 5(1992), 325-370.

14. Catonini, E., On non-monotonic strategic reasoning, GEB 120(2020), 209-224.

15. Chen, J., Micali, S., The order independence of iterated dominance in extensive games, TE 8(2013), 125-163.

16. Chen, Y.-C., Luo, X., An indistinguishability result on rationalizability under general preferences, ET 51(2012), 1-12.

17. Chen, Y.-C., Luo, X., Qu, C., Rationalizability in general situations, ET 61(2016), 147-167

18. Epstein, L.G., Preference, rationalizability and equilibrium, JET 73(1997), 1-29.

19. Epstein, L.G. , Breton, M.L., Dynamically consistent beliefs must be bayesian, JET 61(1993), 1-22.

20. Epstein, L., Wang, T., "beliefs about beliefs" without probabilities, Econometrica 64(1996), 1343-1373.

21. Etner, J., Jeleva, M., Tallon, J.M., Decision theory under ambiguity, Journal of Economic Surveys 26 (2012), 234-270.

22. Gilboa, I., Marinacci, M., Ambiguity and the bayesian paradigm. In: Acemoglu, D., Arellano, M., Dekel, E. (eds.), Advances in Economics and Econometrics: Tenth World Congress, Vol. 1, pp.179-242. Cambridge University Press, Cambridge (2013).

23. Ghirardato, P., Revisiting Savage in a conditional world, ET 20(2002), 83-92.

24. Gilboa, I., Schmeidler, D., Maxmin expected utility with non-unique prior, JMathE 18(1989), 141-153.

25. Guarino, P., Conditional Pure Strategy Dominance, Alpen-Adria-Universität Klagenfurt, Mimeo (2020).

26. Heifetz, A., Perea, A., On the outcome equivalence of backward induction and extensive form rationalizability, IJGT 44(2015), 37-59.

27. Hu, T.W., On p-rationalizability and approximate common certainty of rationality, JET 136(2007), 379-391.

28. Klibanoff, P., Marinacci, M., Mukerji, S., A smooth model of decision making under ambiguity, Econometrica 73(2005), 1849-1892.

29. Li, S., Obviously strategy-proof mechanisms, AER 107(2017), 3257-87.

30. Lo, K.C., Rationalizability and the savage axioms, ET 15(2000), 727-733.

31. Luo, X., Qian, X., Qu, C., Iterated elimination procedures, ET 70(2020), 437-465.

32. Machina, M., Schmeidler, D., A more robust definition of subjective probability, Econometrica 60(1992), 745-780.

33. Morris, M., The logic of beliefs and beliefs change: A decision theoretic approach, JET 69(1996), 1-23.

34. Myerson, M., Multistage games with communication, Econometrica 54 (1986), 323-358.

35. Osborne, M.J., Rubinstein, A., A Course in Game Theory. MIT Press, MA (1994).

36. Pearce, D., Rationalizable strategic behavior and the problem of perfection, Econometrica 52(1984), 1029-1050.

37. Perea, A., Belief in the opponents' future rationality, GEB 83(2014), 231-254.

38. Perea, A., Why forward induction leads to the backward induction outcome: A new proof for Battigalli's theorem, GEB 110(2019), 120-138.

39. Reny, P., Backward induction, normal-form perfection and explicable equilibria, Econometrica 60(1992), 626-649.

40. Rubinstein, A., Comments on the interpretation of game theory, Econometrica 59(1991), 909-924.

41. Savage, L., The Foundations of Statistics. NY: Wiley (1954).

42. Schmeidler, D., Subjective probability and expected utility without additivity, Econometrica 57(1989), 571-587.

43. Stalnaker, R., Belief revision in games: forward and backward induction, Mathematical Social Sciences 36(1998), 31-56.

44. Shimoji, M., Watson, J., Conditional dominance, rationalizability, and game forms, JET 83(1998), 161-195.